Dual parametrization of GPDs v.s. the Mellin-Barnes transform approach and the $J=0$ fixed pole issue

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Outline

- Introduction
- Conformal PW expansion: Mellin-Barnes techniques and the dual parametrization
- Equivalence of the dual parametrization and SO(3) PW expansion within MB techniques.
- Abel transform tomography, Froissart-Gribov projection and the $J = 0$ “fixed pole” contribution.
- Conclusions and outlook

D. Müller, M. Polyakov and K.S., JHEP 1503, 052 (2015).

D. Müller and K.S., arXiv:1507.02164 [hep-ph] (2015).

A note on GPD representations

- GPD modelling can be done in various representations: (DD representation, conformal PW expansions, expansions over orthogonal polynomials,...)

List of non-trivial requirements:

- polynomiality
- hermiticity
- T-invariance
- positivity

Other sources of inspiration:

- evolution properties
- relation to PDFs and FFs
- analyticity
- Regge theory

- Should be possible to map one representation to another (as long as basic properties are satisfied).
- "Which representation is better is not a meaningful question!" (see K. Kumerički & D. Müller'09).
- The hope: get more insight from considering various GPD properties within different representations.

Conformal PW expansion for quark GPDs

- Idea: expand GPDs over the conformal basis $c_n(x,\eta) = N_n \times \eta^n C_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right)$
- Main advantage: trivial solution of the LO evolution equations.

- Conformal moments of quark GPDs

$$H_n(\eta,t) = \int_{-1}^{1} dx\, c_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right) H(x,\eta,t)\,.$$

- $c_n(x,\eta)$ form a complete basis on $[-\eta,\ \eta]$ with the weight $\left(1-\frac{x^2}{\eta^2}\right)$.
- $p_n(x,\eta)$ include the weight and θ-function to ensure the support.
- Orthogonality of the basis: $\int_{-1}^{1} dx\, p_n(x,\eta) c_n(x,\eta) = (-1)^n \delta_{mn}$

Conformal PW expansion for GPDs:

$$H(x,\eta,t) = \sum_{n=0}^{\infty} p_n(x,\eta) H_n(\eta,t)\,.$$

- Conformal moments are reproduced by this series.
- Restricted support property $\not\Rightarrow$ GPD vanishes in the outer region.
- The expansion is to be understand as an ill-defined sum of generalized functions.

Ways to assign meaning to conformal PW expansion I

- Sommerfeld-Watson transform + Mellin-Barnes integral techniques D. Müller and A. Schäfer'05; A. Manashov, M. Kirch and A. Schafer'05;

Idea: Inverse Mellin transform

$$M(n) = \int_0^\infty dx x^n f(x); \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds x^{-j-1} M(j).$$

- Mellin-Barnes integral representation for GPDs:

$$H(x,\eta,t) = \frac{i}{2} \int_{c-i\infty}^{c+i\infty} dj \frac{1}{\sin \pi j} p_j(x,\eta) H_j(\eta,t)\,.$$

Ways to assign meaning to conformal PW expansion II

- 1. Shuvaev transform A. Shuvaev'99, J. Noritzsch'00;
  2. Dual parametrization of GPDs M. Polyakov and A. Shuvaev'02;

- How to restore $f(x)$ from its Mellin moments $M_n = \int dx x^n f(x)$?

- Formal solution:

$$f(x) = \sum_{n=0}^{\infty} M_n \delta^{(n)}(x) \frac{(-1)^n}{n!} .$$

✓ A trick: $\delta^{(n)}(x) = \frac{(-1)^n n!}{2\pi i} \left[\frac{1}{(x - i\epsilon)^{n+1}} - \frac{1}{(x + i\epsilon)^{n+1}} \right] .$

Define $F(z) = \sum_{n=0}^{\infty} \frac{M_n}{z^{n+1}}$; compute discontinuity $f(x) = \frac{1}{2\pi i} \left[F(x - i\epsilon) - F(x + i\epsilon) \right]$.

- Introduce $f_\eta(y)$ whose Mellin moments generate Gegenbauer moments of GPD:

$$\int_0^1 dy y^n f_\eta(y) = H_n(\eta)$$

- Explicitly construct the kernel $K(x, \eta; y)$ as discontinuity of a certain function. Then

$$H(x, \eta) = \int_0^1 dy \, K(x, \eta; y) \, f_\eta(y) .$$

Dual Parametrization: basic facts

Dual Parametrization (M. Polyakov, A. Shuvaev'02):

- Mellin moments expanded in partial waves of the t-channel (t-channel refers to $\bar{h}h \to \gamma^*\gamma$):

$$N_n^{-1}\frac{(n+1)(n+2)}{2n+3}H_n(\eta,t) = \eta^{n+1}\sum_{l=0}^{n+1} B_{nl}(t)P_l\left(\frac{1}{\eta}\right)$$

Conformal PW expansion reads:

$$H(x,\eta,t) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{l=0\\ \text{even}}}^{n+1} B_{nl}(t)\,\theta\left(1-\frac{x^2}{\eta^2}\right)\left(1-\frac{x^2}{\eta^2}\right)C_n^{\frac{3}{2}}\left(\frac{x}{\eta}\right)P_l\left(\frac{1}{\eta}\right)$$

- Same idea as the Shuvaev transform: Mellin moments of $Q_{2\nu}(y,t)$ generate the generalized F.Fs. B_{nl}.

$$B_{n\ n+1-2\nu}(t) = \int_0^1 dy y^n Q_{2\nu}(y,t)\,.$$

Then $\quad H(x>-\eta,\eta,t) = \sum_{\nu=0}^{\infty}\int_0^1 dy K_{2\nu}(x,\eta,y)y^{2\nu}Q_{2\nu}(y,t)\,.$

- Kernels $K_{2\nu}(x,\eta,y)$ are expressed through hypergeometric functions.

SO(3) PW expansion within MB integral approach

$$H_n(\eta,t) = \sum_{\nu=0}^{(n+1)/2} \eta^{2\nu} H_{n,n+1-2\nu}(t)\hat{d}_{00}^{n+1-2\nu}(\eta), \quad \text{for odd } n,$$

- $\hat{d}_{00}^{l}(\eta) = \frac{\Gamma(\frac{1}{2})\Gamma(1+J)}{2^J\Gamma(\frac{1}{2}+J)}\eta^l P_l\left(\frac{1}{\eta}\right)$ are the reduced Wigner functions.
- $H_{n,n+1-2\nu}(t)$ are called double partial wave amplitudes

Double PW expansion employed within MB approach

$$H(x \geq -\eta,\eta,t) = \sum_{\nu=0}^{\infty} \frac{1}{2i}\int_{c+2\nu-i\infty}^{c+2\nu+i\infty} dj\, \frac{p_j(x,\eta)}{\sin(\pi[j+1])}\, H_{j,j+1-2\nu}(t)\, \eta^{2\nu}\,\hat{d}_{00}^{j+1-2\nu}(\eta) - \sum_{\nu=1}^{\infty} \eta^{2\nu}\, p_{2\nu-1}(x,\eta) H_{2\nu-1,0}(t)\,.$$

Establishing equivalence and forming glossary

- Relation between dPWAs (MB approach) and generalized FFs (dual parametrization)

$$H_{n,n+1-2\nu}(t) = \frac{\Gamma(3+n)\Gamma\left(\frac{3}{2}+n-2\nu\right)}{2^{2\nu}\Gamma\left(\frac{5}{2}+n\right)\Gamma(2+n-2\nu)} B_{n,n+1-2\nu}(t)$$

- Forward-like functions from the dPWAs: inversion of the Mellin transform

$$y^{2\nu}Q_{2\nu}(y,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dj\, y^{-j-1}\, \frac{2^{2\nu}\Gamma(5/2+j+2\nu)\Gamma(2+j)}{\Gamma(3+j+2\nu)\Gamma(3/2+j)} H_{j+2\nu,j+1}(t)$$

- Main result of Müller, Polyakov and KS'14: MB representation for the kernel

$$K_{2\nu}(x,\eta|y) = K_{2\nu}^{J\neq 0}(x,\eta|y) - \eta^{2\nu}\, p_{2\nu-1}(x,\eta)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2+2\nu)}{2^{2\nu}\Gamma\left(\frac{3}{2}+2\nu\right)}\,\frac{1}{y} \quad ;$$

$$K_{2\nu}^{J\neq 0}(x,\eta|y) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} dj\, \eta^{2\nu}\, \frac{p_{j+2\nu}(x,\eta)}{\sin(\pi[1+j])}\, \frac{\Gamma(3+j+2\nu)\Gamma\left(\frac{3}{2}+j\right)}{2^{2\nu}\Gamma\left(\frac{5}{2}+j+2\nu\right)\Gamma(2+j)}\, y^j\, \hat{d}_{00}^{j+1}(\eta)\,.$$

- Straightforward calculation recovers the dual parametrization result.

Abel transform tomography

- LO DVCS amplitude $\mathcal{H}^{(+)}(\xi,t)=\int_0^1 dx H(x,\xi,t)\left[\frac{1}{\xi-x-i0}-\frac{1}{\xi+x-i0}\right]$:

$$\mathrm{Im}\mathcal{H}^{(+)}(\xi,t)=2\int_{\frac{1-\sqrt{1-\xi^2}}{\xi}}^{1}\frac{dx}{x}\,N(x,t)\frac{1}{\sqrt{\frac{2x}{\xi}-x^2-1}}\,.$$

- GPD quintessence: $N(x,t)=\underbrace{Q_0(x,t)}_{\text{PDFs}}+x^2\underbrace{Q_2(x,t)}_{\text{FFs of EMT tensor}}+x^4Q_4(x,t)+\ldots$
- M. Polyakov'07: using Joukowski conformal map the relation between $\mathrm{Im}\mathcal{H}(\xi)$ and GPD quintessence $N(x)$ can be presented in the form of the Abel integral equation.
- Projection property of GPD quintessence:

$$\int_0^1 dx x^{J-1}N(x,t)=B_{J-1\,J}(t)+B_{J+1\,J}(t)+B_{J+3\,J}(t)+\ldots\equiv F_J(t).$$

Froissart- Gribov projection I

Gribov'61, Froissart'61

DR for the elementary amplitude (analytically continued to the t-channel):

$$\mathcal{H}^{(+)}(\cos\theta_t, t) = \int_0^1 dx \frac{2x\cos^2\theta_t}{1 - x^2\cos^2\theta_t} H^{(+)}(x, x, t) + 4D(t)\,,$$

- Consider SO(3) PWAs

$$a_J(t) \equiv \frac{1}{2}\int_{-1}^{1} d(\cos\theta_t)\, P_J(\cos\theta_t)\mathcal{H}^{(+)}(\cos\theta_t, t)$$

Neumann's integral representation for the Legendre functions $\mathcal{Q}_J$

$$\frac{1}{2}\int_{-1}^{1} dz P_J(z)\frac{1}{z' - z} = \mathcal{Q}_J(z') \quad J \geq 0, \quad \text{integer.}$$

Froissart- Gribov projection II

- For even positive J

$$a_{J>0}(t) = 2\int_0^1 dx\, \frac{\mathcal{Q}_J(1/x)}{x^2} H^{(+)}(x,x,t)\,.$$

- For $J=0$ we get

$$a_{J=0}(t) = 2\int_0^1 dx\left[\frac{\mathcal{Q}_0(1/x)}{x^2} - \frac{1}{x}\right] H^{(+)}(x,x,t) + 4D(t)\,.$$

- N.B. $\frac{\mathcal{Q}_J(1/x)}{x^2} \sim x^{J-1}$ for small x.

Mellin moments of GPD quintessence $\Leftrightarrow$ Froissart- Gribov projection

$$\int_0^1 dy\, y^{J-1} N(y,t) = \int_0^1 dx\left[\frac{1}{\sqrt{x}}\frac{d}{dx} R_J(x)\right] H^{(+)}(x,x,t)\,,$$

where the auxiliary functions

$$\frac{1}{\sqrt{x}}\frac{d}{dx} R_J(x) = \left(\frac{1}{2}+J\right)\frac{\mathcal{Q}_J(1/x)}{x^2}\,.$$

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Compton Scattering and Fixed Poles in Parton Field-Theoretic Models*

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We extend a class of parton models to a fully gauge-invariant theory for the full Compton amplitude. The existence of local electromagnetic interactions is shown to always give rise to a constant real part in the high-energy behavior of the amplitude $T_1(\nu, q^2)$. In the language of Reggeization this is interpreted as a fixed pole at $J=0$ in T_1 and νT_2, with residue polynomial in the photon mass squared.

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Local two-photon couplings and the $J = 0$ fixed pole in real and virtual Compton scattering

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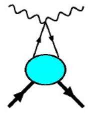
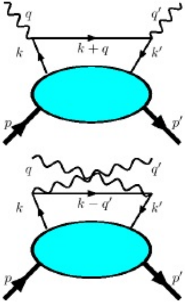
The local coupling of two photons to the fundamental quark currents of a hadron gives an energy-independent contribution to the Compton amplitude proportional to the charge squared of the struck quark, a contribution which has no analog in hadron scattering reactions. We show that this local contribution has a real phase and is universal, giving the same contribution for real or virtual Compton scattering for any photon virtuality and skewness at fixed momentum transfer squared t. The t dependence of this $J = 0$ fixed Regge pole is parameterized by a yet unmeasured even charge-conjugation form factor of the target nucleon. The $t = 0$ limit gives an important constraint on the dependence of the nucleon mass on the quark mass through the Weisberger relation. We discuss how this $1/x$ form factor can be extracted from high-energy deeply virtual Compton scattering and examine predictions given by models of the H generalized parton distribution.

$J = 0$ fixed pole manifestation in DVCS

- S. Brodsky, F. Estrada, A. Szczepaniak: local coupling of two photons to a quark in the high energy limit.





$$\frac{\hat{k}+\hat{q}+m}{(k+q)^2-m^2+i\epsilon} \to \frac{\gamma^+}{2p^+}\left(\frac{1}{x}+\frac{\xi}{x}\frac{1}{x-\xi++i\epsilon}\right)$$

$$\frac{\hat{k}-\hat{q}'+m}{(k-q')^2-m^2+i\epsilon} \to \frac{\gamma^+}{2p^+}\left(\frac{1}{x}-\frac{\xi}{x}\frac{1}{x+\xi++i\epsilon}\right)$$

Universal $J = 0$ fixed pole contribution into Compton amplitude

$$a^{\mathrm{f.p.}}_{J=0}(t|Q_1^2, Q_2^2) = -\frac{1}{\pi}\int_{\nu_{\mathrm{cut}}}^{(\infty)} d\nu \frac{2}{\nu}\,\mathrm{Im}\mathcal{H}(\nu, t|Q_1^2 = Q_2^2) \quad \textbf{(Conjecture)}.$$

Dispersive approach in the scaling regime I

$$\gamma^{(*)}(q_1) + N(p_1) \to \gamma^{(*)}(q_2) + N(p_2)$$

Scaling variables

$$\xi = \frac{Q^2}{P\cdot q} = \frac{Q^2}{2M\nu}; \qquad \eta = -\frac{\Delta\cdot q}{P\cdot q} = -\frac{\Delta\cdot q}{2M\nu}, \quad \text{where} \quad Q^2 = -q^2 \equiv -\frac{(q_1+q_2)^2}{4}.$$

Useful variable: $\vartheta \equiv \eta/\xi = \frac{q_1^2-q_2^2}{q_1^2+q_2^2} + \mathcal{O}(t/Q^2)$

- For $t=0$, the case $\vartheta = 0$ corresponds to the usual DIS kinematics.
- The case $\vartheta = 1$ corresponds to the DVCS kinematics.

LO Compton FF

$$\mathcal{H}(\xi, t|\vartheta) \stackrel{\text{LO}}{=} \int_0^1 dx\, \frac{2x}{\xi^2 - x^2 - i\epsilon} H^{(+)}(x, \eta = \vartheta\xi, t).$$

DRs within scaling variables

$$\mathcal{H}(\xi,t|\vartheta) = \frac{1}{\pi}\int_{(0)}^{1}\frac{d\xi'}{\xi'}\frac{2\xi^2}{\xi^2-\xi'^2-i\epsilon}\mathrm{Im}\mathcal{H}(\xi',t|\vartheta) + \underbrace{\mathcal{H}_\infty(t|\vartheta)}_{a^{\mathrm{f.p.}}_{J=0}(t|\vartheta)},$$

$$\mathcal{H}(\xi,t|\vartheta) = \frac{1}{\pi}\int_{0}^{1}d\xi'\frac{2\xi'}{\xi^2-\xi'^2-i\epsilon}\mathrm{Im}\mathcal{H}(\xi',t|\vartheta) + \underbrace{\mathcal{H}_0(t|\vartheta)}_{4D(t|\vartheta)}.$$

Relation between two constants

$$\mathcal{H}_\infty(t|\vartheta) = 4D(t|\vartheta) - \frac{2}{\pi}\int_{(0)}^{1}\frac{d\xi}{\xi}\,\mathrm{Im}\mathcal{H}(\xi,t|\vartheta).$$

GPD sum rule O. Teryaev'05

$$4D(t|\vartheta) \overset{\mathrm{LO}}{=} \int_0^1 dx\,\frac{2x}{x^2-\xi^2}\left[H^{(+)}(x,\vartheta x,t) - H^{(+)}(x,\vartheta\xi,t)\right].$$

Dispersive approach in the scaling regime III

- Caution! 'High energy' limit $\xi \to 0$ requires attention.
- Taken naively will miss $D^{\mathrm{f.p.}}(\eta|, t)$

$$H^{(+)}(x, \eta, t) = \mathbb{H}^{(+)}(x, \eta, t) + \theta(|\eta| - |x|) d^{\mathrm{f.p.}}(x/|\eta|, t)\,.$$

- Split $x \in [0, \vartheta\xi]$, and $x \in [\vartheta\xi, 1]$. Then

$$4D^{\mathrm{f.p.}}(t|\vartheta) \stackrel{\mathrm{LO}}{=} \lim_{\xi\to 0} \int_0^{\vartheta\xi} dx\, \frac{2x}{\xi^2 - x^2}\, d^{\mathrm{f.p.}}\left(\frac{x}{\vartheta\xi}, t\right) = \int_0^1 dx\, \frac{2x\vartheta^2}{1 - \vartheta^2 x^2}\, d^{\mathrm{f.p.}}(x, t)\,.$$

- No proof for $J = 0$ fixed pole universality conjecture! Back to the discussion of the D-term as inherent part of GPD (GPD holographic property) and presence/absence of $j = -1$ fixed poles.
- Counterexamples with auxiliary D-term exist. Such situation occurs in certain dynamical models. *E.g.* pion GPD in nonlocal chiral quark model. See K.S.'08

Conclusions

1. The dual parametrization approach is equivalent to the Mellin-Barnes type integral based techniques for GPDs.
2. Froissar-Gribov projection provides explanation for the properties of GPD quintessence function and Abel transform tomography.
3. There exists no proof for $J = 0$ pole universality for Compton scattering.
4. $J = 0$ pole universality is equivalent to GPD holographic property.
5. However, this is an additional "external principle". Hard to prove (or disprove).