BV flag in Crab-Cavities

Andrea Latina

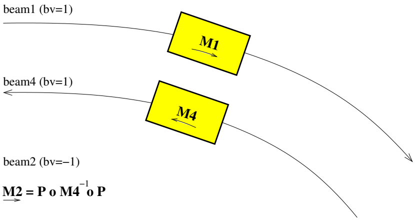
andrea.latina@cern.ch

HSS meeting - Mar 18, 2015

Table of contents

MAD-X: beam1, beam2, and beam4

- beam1: forward beam (clockwise)
- beam4: backward beam (counterclockwise)
- beam2: (bv=-1) backward beam seen in the forward direction



- map for beam1: e.g. $\mathbf{M1}\,(K,\ V,\ LAG,\ \ldots\ )$
- map for beam4: e.g. $\mathbf{M4} = \mathbf{M1}\,(f\,(K,\ V,\ LAG,\ \ldots\ ))$, change of reference frame for **M1**
- with bv=-1, **M2** is *reflected map* for beam2, to track beam2 in the frame of beam1:

$$\mathbf{M2} \;=\; P \circ \mathbf{M4}^{-1} \circ P \;=\; \mathbf{M1}\,(g \circ f\,(K,\ V,\ LAG,\ \ldots\ ))$$

P is the coordinate change from beam4 to beam2

Purpose of BV-flag in MAD-X

- The goal of the BV-flag is to express **M1**, **M2**, and **M4** through a single map, **M**, accepting an additional argument, bv:
  - beam1:

$$\text{map:}\ \mathbf{M1} = \mathbf{M}\,(K,\ V,\ LAG,\ \dots\ ,\ \text{bv} = +1)$$

  - beam2:

$$\text{map:}\ \mathbf{M2} = \mathbf{M}\,(K,\ V,\ LAG,\ \dots\ ,\ \text{bv} = -1)$$

  with:

$$\mathbf{M} :\overset{\text{def}}{=} \begin{cases} \mathbf{M1}\,(K,\ V,\ LAG,\ \dots\ ) & \text{if bv} = +1 \\ \mathbf{M1}\,(g \circ f\,(K,\ V,\ LAG,\ \dots\ )) = \mathbf{M2} = \ P \circ \mathbf{M4}^{-1} \circ P & \text{if bv} = -1 \end{cases}$$

  Note that the actual expressions of $f\,(\dots)$ and $g\,(\dots)$ (and of $g\ \circ\ f\,(\dots)$) depend on the type of element and physics needed.

- Repeat: purpose of BV-flag is to help providing a unified expression for $\mathbf{M1}\,(\dots)$ and $\mathbf{M2}\,(g\ \circ\ f\,(\dots))$ (passing through **M4**)

The sequences LHCB1 and LHCB2 (and beam_four)

Two "degrees of freedom" exist:

- the definition of $f(\ldots)$ and $g(\ldots)$

from which depends the implementation in MAD-X of $g \circ f(\ldots)$

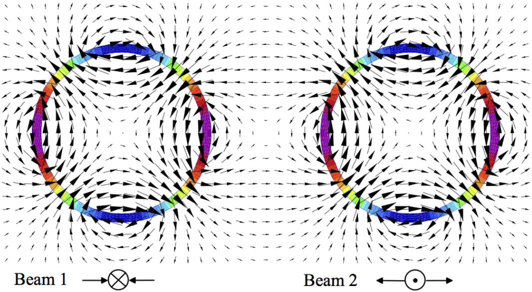
The "LHC Optics Web Home" came to help, providing the following sequences:

- LHCB1: clockwise sequence for beam1
- LHCB2: clockwise sequence for beam2
- beam_four.seq: counterclockwise sequence for beam4

LHCB2 defines $f(\ldots)$

Elements LHCB1 and LHCB2 - 1/3

- Twin-aperture quadrupoles, e.g.:



(S. Russenschuck, Field computation for accelerator magnets; here beam2 is beam4)

- MQ.15R1.B1, K1 := kqf.a12, polarity=+1; ! beam1 sees a focusing quad (with bv=1)
- MQ.15R1.B2, K1 := -kqd.a12, polarity=-1; ! beam4 sees a defocusing quad; but LHCB2 reports a focusing quad, with bv=-1 with
  - kqd.a12 := kqd ; kqd := -0.008600955656 ;
  - kqf.a12 := kqf ; kqf := 0.008990100753 ;
- [beam_four.seq: MQ.15R1.B2, K1 := kqd.a12, polarity=-1;]

Elements LHCB1 and LHCB2 - 2/3

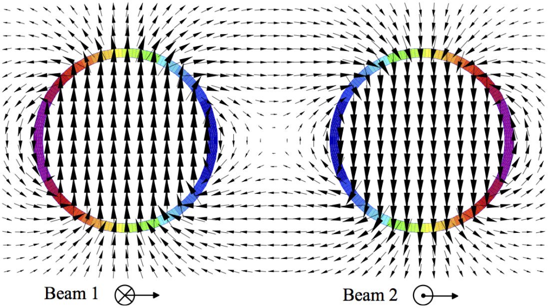
- Single-aperture quadrupoles for low-β triplets: MQXA, MQXB
  - LHCB1: MQXA.1R1, K1 := kqx.r1+ktqx1.r1, polarity=+1;
  - LHCB2: MQXA.1R1, K1 := kqx.r1+ktqx1.r1, polarity=+1;
    that is
    - they have the same strength

- ACSCA Cavities
  - V6.503
    - LHCB1: ACSCA.D5L4.B1, VOLT := VRF400/8, LAG := LAGRF400.B1;
    - LHCB2: ACSCA.D5L4.B2, VOLT := VRF400/8, LAG := LAGRF400.B2;
  - Browsing the files I found LAGRF400.B2=0.0, or
    - ACSCA : RFCAVITY, L := l.ACSCA, VOLT := VRF400/8, LAG := 0.5, HARMON := HRF400;
      which is used by both structures
  - beam_four.seq:
    ACSCA.D5L4.B2, VOLT := VRF400/8, LAG := 0.5 − (LAGRF400.B2);
    - Phase dependence: $\sin(\phi - kz) \quad \rightarrow \quad \sin(\phi + kz) \equiv \sin(\pi - \phi - kz)$

$$\phi \rightarrow \pi - \phi$$

Elements LHCB1 and LHCB2 - 3/3

- Main dipoles
  - MB.B19R8.B1, ANGLE := ab.a81, K0 := kb.a81, polarity=+1;
  - MB.B19R8.B2, ANGLE := -ab.a81, K0 := -kb.a81, polarity=+1;
  - beam_four.seq: MB.B19R8.B2, ANGLE := -ab.a81, K0 := -kb.a81, polarity=-1;



(S. Russenschuck, Field computation for accelerator magnets; here beam2 is beam4)

- Here $B_y \rightarrow -B_y$ for obvious reasons

Fields, as seen in a $(-x,\ y,\ -z)$ reference frame

- Multipolar expansion of the magnetic field:

beam2 (x, y, z)		beam4 $(-x, y, -z)$	
$k0$	$\frac{q}{P_0 c} B_y$	$\frac{q}{P_0 c} B_y$	$k0$
$\mathbf{k1}$	$\frac{q}{P_0 c} \frac{\partial B_y}{\partial x}$	$\frac{q}{P_0 c} \frac{\partial B_y}{\partial -x}$	$-\mathbf{k1}$
$k2$	$\frac{q}{P_0 c} \frac{\partial^2 B_y}{\partial x^2}$	$\frac{q}{P_0 c} \frac{\partial^2 B_y}{\partial (-x)^2}$	$k2$
$\mathbf{k3}$	$\frac{q}{P_0 c} \frac{\partial^3 B_y}{\partial x^3}$	$\frac{q}{P_0 c} \frac{\partial^3 B_y}{\partial (-x)^3}$	$-\mathbf{k3}$

(normal components)

beam2 (x, y, z)		beam4 $(-x, y, -z)$	
$\mathbf{k0_s}$	$\frac{q}{P_0 c} B_x$	$\frac{q}{P_0 c} (-B_x)$	$-\mathbf{k0_s}$
$k1_s$	$\frac{q}{P_0 c} \frac{\partial B_x}{\partial x}$	$\frac{q}{P_0 c} \frac{\partial - B_x}{\partial -x}$	$k1_s$
$\mathbf{k2_s}$	$\frac{q}{P_0 c} \frac{\partial^2 B_x}{\partial x^2}$	$\frac{q}{P_0 c} \frac{\partial^2 - B_x}{\partial (-x)^2}$	$-\mathbf{k2_s}$
$k3_s$	$\frac{q}{P_0 c} \frac{\partial^3 B_x}{\partial x^3}$	$\frac{q}{P_0 c} \frac{\partial^3 - B_x}{\partial (-x)^3}$	$k3_s$

(skew components)

  - *k0 = dipole, k1 = quadrupole, k2 = sextupole, ...*

- Electric field:

beam2 (x, y, z)	beam4 $(-x, y, -z)$
$(E_x,\ E_y,\ E_z)$	$(-E_x,\ E_y,\ -E_z)$

  - notice that the voltage changes sign: $V = \int E_z dz \rightarrow -V = -\int E_z dz$

Summary of the transformation rules

A summary of these transformation rules is given in beam2-beam4.doc by S. Fartoukh

SBEND: put a minus sign in front of the definition of the parameters, TILT, K0S, E1, E2, K1 and K3, and swap the parameter definitions of (FINT, FINTX), (E1,E2), (H1,H2)

- beam2: SBEND,L=l,ANGLE=angle,TILT=tilt,K0=k0,K0S=k0s,K1=k1,E1=e1, E2=e2, ...
- beam4: SBEND,L=l,ANGLE=angle,TILT=-tilt,K0=k0,K0S=-k0s,K1=-k1,E1=-e2,E2=-e1

QUADRUPOLE: put a minus sign in front of the definition of the parameters K1 and TILT

- beam2: QUADRUPOLE,L=l,TILT=tilt,K1=k1,K1s=k1s;
- beam4: QUADRUPOLE,L=l,TILT=-tilt,K1=-k1,K1s=k1s;

SEXTUPOLE: put a minus sign in front of the definition of the parameters K2S and TILT

- beam2: SEXTUPOLE,L=l,TILT=tilt,K2=k2,K2s=k2s;
- beam4: SEXTUPOLE,L=l,TILT=-tilt,K2=k2,K2s=-k2s;

OCTUPOLE: put a minus sign in front of the definition of the parameters K3 and TILT

- beam2: OCTUPOLE,L=l,TILT=tilt,K3=k3,K3s=k3s;
- beam4: OCTUPOLE,L=l,TILT=-tilt,K3=-k3,K3s=k3s;

MULTIPOLE: put a minus sign in front of the definition of the tilt and of the parameters related to the odd skew and even normal multipoles

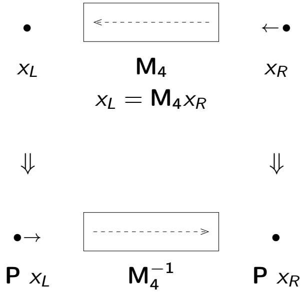
- beam2: MULTIPOLE,TILT=tilt,KNL:={k0,k1,k2,k3,..}, KSL:={k0s,k1s,k2s,k3s,..};
- beam4: MULTIPOLE,TILT=-tilt,KNL:={k0,-k1,k2,-k3,..}, KSL:={-k0s,k1s,-k2s,k3s,..};

The reflected map $\mathbf{M}_2$

Take two particles, x_L (left) and x_R (right) going through the same element:

beam4 (forward tracking):

beam2 (backward tracking):



that is

$$\mathbf{P}\ x_R = \mathbf{M}_4^{-1} \circ \mathbf{P}\ x_L$$

$$x_R = \underbrace{\mathbf{P} \circ \mathbf{M}_4^{-1} \circ \mathbf{P}}_{\mathbf{M}_2}\ x_L$$

$\mathbf{M}_2$ is the *reflected map*:

$$\mathbf{M}_2 = \mathbf{P} \circ \mathbf{M}_4^{-1} \circ \mathbf{P}$$

and

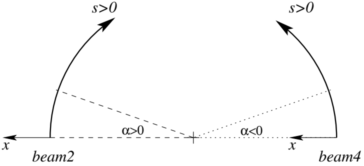
$$x_R = \mathbf{M}_2\ x_L$$

Change of coordinates, P

Two change of coordinates are necessary for going from beam4 to beam2:

1. Change of reference frame:

$$(x,\ y,\ s) \rightarrow (-x,\ y,\ -s)$$



2. Direction of motion is reverted to perform back-tracking:

$$v \rightarrow -v$$

That is, P:

$$\underset{\textbf{beam2}}{\begin{pmatrix} -x \\ p_x \\ y \\ -p_y \\ -z \\ p_z \end{pmatrix}} \leftarrow \underbrace{\begin{pmatrix} \mathbf{1} & & & & & \\ & -\mathbf{1} & & & & \\ & & \mathbf{1} & & & \\ & & & -\mathbf{1} & & \\ & & & & \mathbf{1} & \\ & & & & & -\mathbf{1} \end{pmatrix}}_{v \rightarrow -v} \underbrace{\begin{pmatrix} -\mathbf{1} & & & & & \\ & -\mathbf{1} & & & & \\ & & \mathbf{1} & & & \\ & & & \mathbf{1} & & \\ & & & & -\mathbf{1} & \\ & & & & & -\mathbf{1} \end{pmatrix}}_{(\mathrm{x},\ \mathrm{y},\ \mathrm{s}) \rightarrow (-\mathrm{x},\ \mathrm{y},\ -\mathrm{s})} \underset{\textbf{beam4}}{\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ p_z \end{pmatrix}}$$

- The transformation for the MAD-X longitudinal canonical variables is: $(t,\ p_t) \rightarrow (-t,\ p_t)$, such that:

$$P: \quad (x,\ p_x,\ y,\ p_y,\ t,\ p_t) \quad \rightarrow \quad (-x,\ p_x,\ y,\ -p_y,\ -t,\ p_t)$$

Properties: $P^{\mathbf{2}} = I$; $P^{-\mathbf{1}} = P$.

Reflected map and reflected sequence

For maps **M4**, where the diagonal 2×2 blocks can be written as 2×2 matrices:

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

with $\det(\mathbf{M}) = 1$ (i.e. symplectic; e.g. drifts, quadrupoles, sbends) , it can be proven that the reflected map, **M2** is equivalent to **M4**:

$$\mathbf{M2} = P \circ \mathbf{M4}^{-1} \circ P \equiv \mathbf{M4}$$

$$P \circ \mathbf{M}^{-1} \circ P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & a \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \mathbf{M}$$

it can be shown also that **M2** is the transfer matrix for the mirror image of the beamline.

For instance, if **M** is the matrix for a bend followed by a drift and then a quadrupole

$$\mathbf{M4} = \mathbf{M}_q \mathbf{M}_d \mathbf{M}_b$$

then

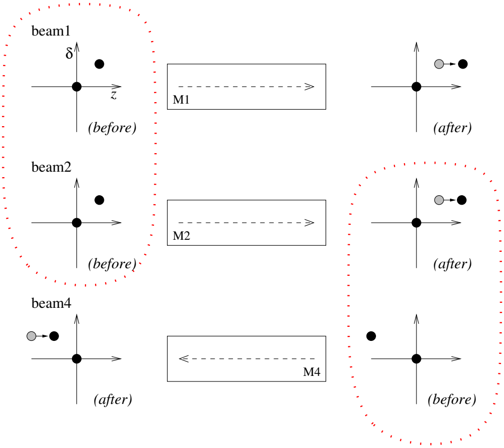
$$\mathbf{M2} = \mathbf{M}_b \mathbf{M}_d \mathbf{M}_q$$

This is a property of the Hamilton equations: they are invariant for time reversal.

Example: BV-flag in drifts

In drifts $g \circ f$ is the identity, and **M2** $\equiv$ **M4** $\equiv$ **M1**

- E.g. two particles $(t,\, p_t)_1 = (0,\, 0)$ and $(t,\, p_t)_2 = (1,\, 1)$:



- $t > 0$ is the head for beam1, but the tail for beam4.
- Remark: in a drift the matrix element $R_{56} = \frac{L}{\beta_0^2 \gamma_0^2}$ is > 0.

Example: BV-flag in a quadrupoles

- In MAD-X we have (this is $g \circ f$):

$$\boxed{\text{g}\circ\text{f}: \qquad K_1 \leftarrow \texttt{bv} \cdot K_1}$$

- **M4**:

$$\mathbf{M4} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) & & & & 0 \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) & & & & \\ & & \cos\left(\sqrt{-K}L\right) & \frac{1}{\sqrt{-K}}\sin\left(\sqrt{-K}L\right) & & \\ & & -\sqrt{-K}\sin\left(\sqrt{-K}L\right) & \cos\left(\sqrt{-K}L\right) & & \\ & & & & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & & & & & 1 \end{pmatrix}$$

- inverse (**M4**):

$$(\mathbf{M4})^{-1} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & -\frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) & & & & 0 \\ \sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) & & & & \\ & & \cos\left(\sqrt{-K}L\right) & -\frac{1}{\sqrt{-K}}\sin\left(\sqrt{-K}L\right) & & \\ & & \sqrt{-K}\sin\left(\sqrt{-K}L\right) & \cos\left(\sqrt{-K}L\right) & & \\ & & & & 1 & \frac{-L}{\beta_0^2\gamma_0^2} \\ 0 & & & & & 1 \end{pmatrix}$$

which is just like $L \to -L$

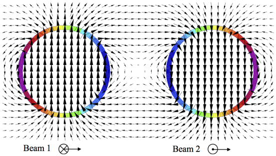
- One can verify that:

$$\mathbf{M2} = P \circ \mathbf{M4}^{-1} \circ P \;\equiv\; \mathbf{M4}$$

Example: BV-flag in sector bends

- The dipoles for beam1 and beam4 must be such that two beams are bent to follow the same circumference. This is provided by:

beam1	*beam4*
$\vec{B} = \begin{pmatrix} 0 \\ B_y \\ 0 \end{pmatrix}$	$\vec{B} = \begin{pmatrix} 0 \\ -B_y \\ 0 \end{pmatrix}$



- beam2 $\leftrightarrow$ beam4
  - beam2: SBEND,L=l,ANGLE=angle,TILT=tilt,K0=k0,K0S=k0s,K1=k1,...
  - beam4: SBEND,L=l,ANGLE=angle,TILT=-tilt,K0=k0,K0S=-k0s,K1=-k1,...
- in MAD-X code (PROBLEM?):

$$\text{gof}: \quad \texttt{angle} \leftarrow \texttt{bv} \cdot \texttt{angle};$$

- MB.B19R8.B1, ANGLE := ab.a81, K0 := kb.a81, polarity=+1; ! beam1 sees a force bending right
- MB.B19R8.B2, ANGLE := -ab.a81, K0 := -kb.a81, polarity=+1; ! LHCB2 reports a magnet bending left (correct). With BV in the previous formula, MAD-X beam2 and beam4 see a force bending right (shouldn't it be left? see below)
- beam_four.seq: MB.B19R8.B2, ANGLE := -ab.a81, K0 := -kb.a81, polarity=-1; ! beam4 is bent left

Example: BV-flag in RF cavities

If the LHC RF cavities are defined like this (as it is now):

- for LHCB1: CRAB.B1: RFCAVITY, l=0, volt:= Vrf, LAG=LAG,
- for LHCB2: CRAB.B2: RFCAVITY, l=0, volt:= Vrf, LAG=LAG,
- beam_four.sew: CRAB.B2: RFCAVITY, l=0, volt:= Vrf, LAG=0.5-LAG

then the map must be:4

$$\boxed{\Delta \mathrm{E} = \mathrm{bv}\, \mathrm{V} \sin\left(\phi_0 - k_0\, z\right)}$$

See demostration in the Extra slides

This is different from what is currently implemented in MAD-X.

Example: BV-flag in RF cavities (alternative)

If, alternatively, the LHC RF cavities are defined like this:

- for LHCB1: CRAB.B1: RFCAVITY, l=0, volt:= Vrf, LAG=LAG,
- for LHCB2: CRAB.B2: RFCAVITY, l=0, volt:= Vrf, LAG=0.5-LAG,
- beam_four.sew: CRAB.B2: RFCAVITY, l=0, volt:= Vrf, LAG=LAG

then the map must be:

$$\boxed{\Delta\text{E}=\text{bv}\,\text{V}\sin\left(\phi_{\mathbf{0}}-\text{bv}\,k_{\mathbf{0}}z\right)}$$

that is:

$$\Delta E=\text{bv}\,V\,\sin\left(\phi_{\mathbf{RF}}-\text{bv}\,k_{\mathbf{RF}}z\right)\qquad\Rightarrow\begin{cases}\Delta E=V\,\sin\left(\phi_{\mathbf{0}}-k_{\mathbf{0}}z\right) & \text{in case bv=+1}\\ \Delta E=-V\,\sin\left(\phi_{\mathbf{0}}+k_{\mathbf{0}}z\right) & \text{in case bv=-1}\end{cases}$$

Example:

- beam1: bv=1, $(0,0,0,0,z,0)$, $\phi_{\mathbf{0}}=0$

$$\Delta E=V\sin\left(0-k_{\mathbf{0}}\,z\right)=\begin{cases}\Delta E>0 & \text{if } z<0 \quad \textit{(tail accelerated)}\\ \Delta E=0 & \text{if } z=0\\ \Delta E<0 & \text{if } z>0 \quad \textit{(head decelerated)}\end{cases}$$

- beam2: bv=-1, $(0,0,0,0,z,0)$, $\phi_{\mathbf{0}}=0$

$$\Delta E=-V\,\sin\left(0+k_{\mathbf{0}}\,z\right)=V\,\sin\left(-k_{\mathbf{0}}z\right)=\begin{cases}\Delta E>0 & \text{if } z<0 \quad \textit{(head accelerated)}\\ \Delta E=0 & \text{if } z=0\\ \Delta E<0 & \text{if } z>0 \quad \textit{(tail decelerated)}\end{cases}$$

beam4 is as expected: head decelerated, tail accelerated.

This is different from what is currently implemented in MAD-X.

Case of a crab cavity

Map of a crab cavity for beam1, as it is in the code:

$$\Delta p_x = \frac{V_{\text{crab}}}{P_0} \cdot \sin\left(\phi_0 - k\,z\right)$$
$$\Delta p_t = -k\,x\,\frac{V_{\text{crab}}}{P_0}\,\cos\left(\phi_0 - k\,z\right)$$

Proposed definition:

- LHCB1: CRAB.B1: crabcavity, l=0, volt:= Vcrab,, LAG=PHI
- LHCB2: CRAB.B2: crabcavity, l=0, volt:= Vcrab, LAG=PHI,
- beam_four.seq: CRAB.B2: crabcavity, l=0, volt:= Vcrab, LAG=0.5-PHI

Map of the crab cavity including the BV-flag:

$$\boxed{\begin{aligned}\Delta p_x &= \frac{\mathtt{bv}\cdot V_{\text{crab}}}{P_0} \cdot \sin\left(\phi_0 - k\,z\right)\\ \Delta p_t &= -k\,x\,\frac{\mathtt{bv}\cdot V_{\text{crab}}}{P_0}\,\cos\left(\phi_0 - k\,z\right)\end{aligned}}$$

This is different from what is currently implemented in MAD-X.

Ultimate test

1. Given two elements
   - for LHCB1
   - for LHCB2

2. Reflect the sequence LHCB2 with the rules prescribed in slides 9 and 10 ($k_0 \rightarrow k_0,\ k_1 \rightarrow -k_1,\ k_2 \rightarrow k_2,\ \ldots$ LAG=$\pi-$LAG)
   - Compute the map **M2** with bv=-1
   - Compute the map **M4** with bv=+1

3. Verify that

$$\mathbf{M}_2 = \mathbf{P} \circ \mathbf{M}_4^{-1} \circ \mathbf{P}$$

Ultimate test on a crab cavity

- for LHCB1: CRAB.B1: crabcavity, l=0, volt:= Vcrab,, LAG=PHI
- for LHCB2: CRAB.B2: crabcavity, l=0, volt:= Vcrab, LAG=PHI,
- beam_four.seq: CRAB.B2: crabcavity, l=0, volt:= Vcrab, LAG=0.5-PHI

Map:

$$\Delta p_x = \frac{\mathtt{bv} \cdot V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \sin\left(\phi_{\mathbf{0}} - k\,z\right)$$

$$\Delta p_t = -k\,x\,\frac{\mathtt{bv} \cdot V_{\mathbf{crab}}}{P_{\mathbf{0}}}\,\cos\left(\phi_{\mathbf{0}} - k\,z\right)$$

TEST:

$$\mathbf{M}_2 \stackrel{?}{=} \mathbf{P} \circ \mathbf{M}_4^{-1} \circ \mathbf{P}$$

1) **M2**: bv=-1:

$$\Delta p_x = \frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \sin\left(\phi_{\mathbf{0}} - k\,z\right)$$

$$\Delta p_t = -k\,x\,\frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \cos\left(\phi_{\mathbf{0}} - k\,z\right)$$

2) $P \circ \mathbf{M4}^{-1} \circ P$:

1. Apply P:

$$\begin{pmatrix} z \\ p_t \end{pmatrix} \rightarrow \begin{pmatrix} -z \\ p_t \end{pmatrix}; \qquad \begin{pmatrix} x \\ p_x \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ p_x \end{pmatrix}$$

2. $\mathbf{M4} \circ P$: bv=+1, $\phi_{\mathbf{0}} \rightarrow \pi - \phi_{\mathbf{0}}$

$$\underbrace{\begin{aligned} \Delta p_x &= \frac{V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \sin\left(\pi\text{-}\phi_{\mathbf{0}} + k\,z\right) \\ \Delta p_t &= k\,x\,\frac{V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \cos\left(\pi\text{-}\phi_{\mathbf{0}} + k\,z\right) \end{aligned}}_{\mathbf{M4} \circ P}$$

3. $\mathbf{M4}^{-1} \circ P$: $V \rightarrow -V$

$$\underbrace{\begin{aligned} \Delta p_x &= \frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \sin\left(\pi\text{-}\phi_{\mathbf{0}} + k\,z\right) \\ \Delta p_t &= k\,x\,\frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \cos\left(\pi\text{-}\phi_{\mathbf{0}} + k\,z\right) \end{aligned}}_{\mathbf{M4}^{-1} \circ P}$$

4. $P \circ \mathbf{M4}^{-1} \circ P$: apply P again:

$$\begin{pmatrix} z \\ p_t \end{pmatrix} \rightarrow \begin{pmatrix} -z \\ p_t \end{pmatrix}; \qquad \begin{pmatrix} x \\ p_x \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ p_x \end{pmatrix}$$

(no changes for the kick)

$$\underbrace{\begin{aligned} \Delta p_x &= \frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \sin\left(\pi\text{-}\phi_{\mathbf{0}} + k\,z\right) \\ \Delta p_t &= k\,x\,\frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \cos\left(\pi\text{-}\phi_{\mathbf{0}} + k\,z\right) \end{aligned}}_{P \circ \mathbf{M4}^{-1} \circ P}$$

- Indeed it $\mathbf{M_2} = \mathbf{P} \circ \mathbf{M_4^{-1}} \circ \mathbf{P}$:

$$\underbrace{\boxed{\begin{aligned} \Delta p_x &= \frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \sin\left(\phi_{\mathbf{0}} - k\,z\right) \\ \Delta p_t &= -k\,x\,\frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \cos\left(\phi_{\mathbf{0}} - k\,z\right) \end{aligned}}}_{P \circ \mathbf{M4}^{-1} \circ P \equiv \mathbf{M2}}$$

- Additional test:

$$\mathbf{M4}^{-1}\mathbf{M4} = \mathbf{I}$$

$$\underbrace{\begin{aligned} \Delta p_x &= \frac{V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \sin\left(\pi\text{-}\phi_{\mathbf{0}} - k\,z\right) \\ \Delta p_t &= -k\,x\,\frac{V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \cos\left(\pi\text{-}\phi_{\mathbf{0}} - k\,z\right) \end{aligned}}_{\mathbf{M4}}$$

$$\underbrace{\begin{aligned} \Delta p_x &= \frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \sin\left(\pi\text{-}\phi_{\mathbf{0}} - k\,z\right) \\ \Delta p_t &= -k\,x\,\frac{-V_{\mathbf{crab}}}{P_{\mathbf{0}}} \cdot \cos\left(\pi\text{-}\phi_{\mathbf{0}} - k\,z\right) \end{aligned}}_{\mathbf{M4}^{-1}}$$

$$\mathbf{M4}^{-1}\mathbf{M4} = \mathbf{I} : \qquad \begin{cases} \Delta p_x &= 0 \\ \Delta p_t &= 0 \end{cases}$$

Conclusions

- BV-flag has been understood
- Reflected maps have been introduced and commented
- The derived reflected maps for RF cavity and a crab cavity have been proposed and tested
- MAD-X code needs to be updated

References

- Mario Conte, Introduction to Accelerator Physics
- Andrzej Wolski, Beam Dynamics in High Energy Particle Accelerators
- Stephane Fartoukh, document beam2-beam4.doc
- Yi-Peng Sun et al., Beam dynamics aspects of crab cavities in the CERN Large Hadron Collider, PhysRevSTAB.12.101002
- Hung Jin Kim, Symplectic map of crab cavity, fermilab-tm-2523-apc.pdf
- Sequences LHCB1, LHCB2, and beam_four.seq

Extra: Ultimate test on an RF cavity

- for LHCB1: CRAB.B1: RFCAVITY, l=0, volt:= Vrf, LAG=LAG,
- for LHCB2: CRAB.B2: RFCAVITY, l=0, volt:= Vrf, LAG=LAG,
- beam_four.sew: CRAB.B2: RFCAVITY, l=0, volt:= Vrf, LAG=0.5-LAG

Map:

$$\Delta E = \mathrm{bv} \cdot V \sin\left(\phi_{\mathrm{RF}} - k_{\mathrm{RF}} z\right)$$

TEST:

$$\mathbf{M}_2 \stackrel{?}{=} \mathbf{P} \circ \mathbf{M}_4^{-1} \circ \mathbf{P}$$

1) **M2**: bv=-1:

$$\boxed{\Delta\mathrm{E}=\text{-}\mathrm{V}\sin\left(\phi_{\mathbf{RF}} - k_{\mathbf{RF}} z\right)}$$

2) $P \circ \mathbf{M4}^{-1} \circ P$:

1. Apply P:

$$\begin{pmatrix} z \\ p_t \end{pmatrix} \rightarrow \begin{pmatrix} -z \\ p_t \end{pmatrix}$$

2. $\mathbf{M4} \circ P$: bv=+1, $\phi_{\mathbf{0}} \rightarrow \pi - \phi_{\mathbf{0}}$

$$\underbrace{\Delta E = V \sin\left(\pi - \phi_{\mathbf{RF}} + k_{\mathbf{RF}} z\right)}_{\mathbf{M4} \circ P}$$

3. $\mathbf{M4}^{-1} \circ P$: $V \rightarrow -V$

$$\underbrace{\Delta E = -V \sin\left(\pi - \phi_{\mathbf{RF}} + k_{\mathbf{RF}} z\right)}_{\mathbf{M4}^{-1} \circ P}$$

4. $P \circ \mathbf{M4}^{-1} \circ P$: apply P again:

$$\begin{pmatrix} z \\ p_t \end{pmatrix} \rightarrow \begin{pmatrix} -z \\ p_t \end{pmatrix}$$

(no changes for the kick)

$$\Delta E = \underbrace{-V \sin\left(\pi - \phi_{\mathbf{RF}} + k_{\mathbf{RF}} z\right)}_{P \circ \mathbf{M4}^{-1} \circ P}$$

5. Indeed $\mathbf{M_2} = \mathbf{P} \circ \mathbf{M_4^{-1}} \circ \mathbf{P}$:

$$\Delta E = \underbrace{-V \sin\left(\pi - \phi_{\mathbf{RF}} + k_{\mathbf{RF}} z\right)}_{P \circ \mathbf{M4}^{-1} \circ P} = \underbrace{\boxed{\text{-V} \sin\left(\phi_{\mathbf{RF}} - k_{\mathbf{RF}} z\right)}}_{P \circ \mathbf{M4}^{-1} \circ P \equiv \mathbf{M2}}$$

- Aditional test:

$$\mathbf{M4}^{-1}\mathbf{M4} = \mathbf{I}$$

$$\underbrace{\Delta E = V \sin\left(\pi - \phi_{\mathbf{RF}} - k_{\mathbf{RF}} z\right)}_{\mathbf{M4}}$$

$$\underbrace{\Delta E = -V \sin\left(\pi - \phi_{\mathbf{RF}} - k_{\mathbf{RF}} z\right)}_{\mathbf{M4}^{-1}}$$

$$\mathbf{M4}^{-1}\mathbf{M4} = \mathbf{I}: \qquad \begin{cases} \Delta p_x & = 0 \\ \Delta p_t & = 0 \end{cases}$$

Extra: Ultimate test on a drift

for LHCB1: DRIFT.B1: DRIFT, L=L0;
for LHCB2: DRIFT.B2: DRIFT, L=L0;
beam_four.sew: DRIFT.B2: DRIFT, L=L0;

Map:

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

TEST:

$$\mathbf{M}_2 \stackrel{?}{=} \mathbf{P} \circ \mathbf{M}_4^{-1} \circ \mathbf{P}$$

1) **M2**: bv=-1:

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

2) $P \circ \mathbf{M4}^{-1} \circ P$:

1. Apply P:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \leftarrow \begin{pmatrix} -x \\ p_x \end{pmatrix}$$

2. $\mathbf{M4} \circ P$: bv=+1, $\phi_0 \to \pi - \phi_0$

$$\underbrace{\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x \\ p_x \end{pmatrix} = \begin{pmatrix} -x + p_x L \\ p_x \end{pmatrix}}_{\mathbf{M4} \circ P}$$

3. $\mathbf{M4}^{-1} \circ P$: $L \to -L$

$$\underbrace{\begin{pmatrix} 1 & -L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -x \\ p_x \end{pmatrix} = \begin{pmatrix} -x - p_x L \\ p_x \end{pmatrix}}_{\mathbf{M4}^{-1} \circ P}$$

4. $P \circ \mathbf{M4}^{-1} \circ P$: apply P again:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \to \begin{pmatrix} -x \\ p_x \end{pmatrix}$$

   (no changes for the kick)

5. Indeed $\mathbf{M}_2 = \mathbf{P} \circ \mathbf{M}_4^{-1} \circ \mathbf{P}$:

$$P \circ \underbrace{\begin{pmatrix} -x - p_x L \\ p_x \end{pmatrix}}_{\mathbf{M4}^{-1} \circ P} \to \underbrace{\begin{pmatrix} x + p_x L \\ p_x \end{pmatrix}}_{P \circ \mathbf{M4}^{-1} \circ P \equiv \mathbf{M2}}$$

- Aditional test:

$$\mathbf{M4}^{-1}\mathbf{M4} = \mathbf{I}$$

$$\begin{pmatrix} 1 & -L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$