Color and Kinematic Decomposition for QCD Amplitudes

based on work with Henrik Johansson
arXiv:1507.00332 [hep-ph]

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Review: KK relations

Kleiss, Kuijf (1988) [1]

Color ordering $\Rightarrow$ $(n-1)!/2$ primitive amplitudes:

$$\mathcal{A}_{n,0}^{\text{tree}} = \sum_{\sigma \in S_{n-1}(\{2,\ldots,n\})} \mathrm{Tr}\big(T^{a_1} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}\big) A(1, \sigma(2), \ldots, \sigma(n))$$

KK relations:

$$A(1, \beta, 2, \alpha) = (-1)^{|\beta|} \sum_{\sigma \in \alpha \sqcup\!\sqcup \beta^T} A(1, 2, \sigma)$$

$\Rightarrow$ KK basis of $(n-2)!$ primitives:

$$\big\{ A(1, 2, \sigma) \;\big|\; \sigma \in S_{n-2}(\{3, \ldots, n\}) \big\}$$

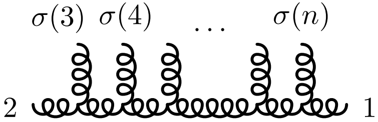
$\Rightarrow$ DDM decomposition:

Del Duca, Dixon, Maltoni (1999) [2, 3]

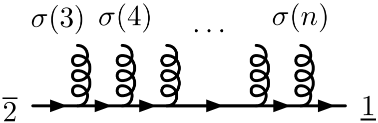
$$\mathcal{A}_{n,0}^{\text{tree}} = \sum_{\sigma \in S_{n-2}(\{3,\ldots,n\})} \tilde{f}^{\,a_2 a_{\sigma(3)} b_1} \tilde{f}^{\,b_1 a_{\sigma(4)} b_2} \ldots \tilde{f}^{\,b_{n-3} a_{\sigma(n)} a_1} \times A(1, 2, \sigma(3), \ldots, \sigma(n))$$

Review: DDM decomposition

Del Duca, Dixon, Maltoni (1999) [2, 3]



$$\mathcal{A}^{\text{tree}}_{n,0} = \sum_{\sigma \in S_{n-2}(\{3,\ldots,n\})} \tilde{f}^{\,a_2 a_{\sigma(3)} b_1} \tilde{f}^{\,b_1 a_{\sigma(4)} b_2} \ldots \tilde{f}^{\,b_{n-3} a_{\sigma(n)} a_1} \times A(1, 2, \sigma(3), \ldots, \sigma(n))$$



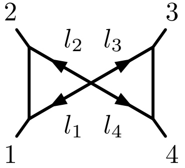
$$\mathcal{A}^{\text{tree}}_{n,1} = \sum_{\sigma \in S_{n-2}(\{3,\ldots,n\})} \left(T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}}\right)_{\bar{\jmath}_2 i_1} A(\underline{1}, \overline{2}, \sigma(3), \ldots, \sigma(n))$$

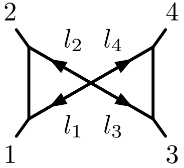
Invitation: Loop level applications

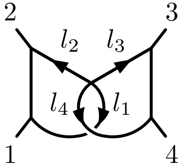
Badger, Mogull, AO, O'Connell (2015) [4]

$$A(1,2,3,4) + A(1,2,4,3) + A(1,4,2,3) = 0$$

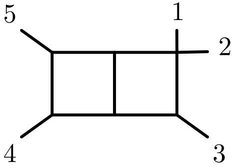
4 points, 2 loops:

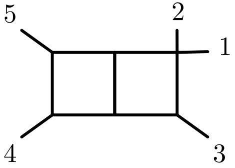


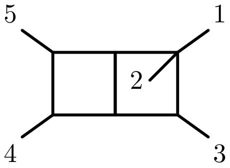




5 points, 2 loops:



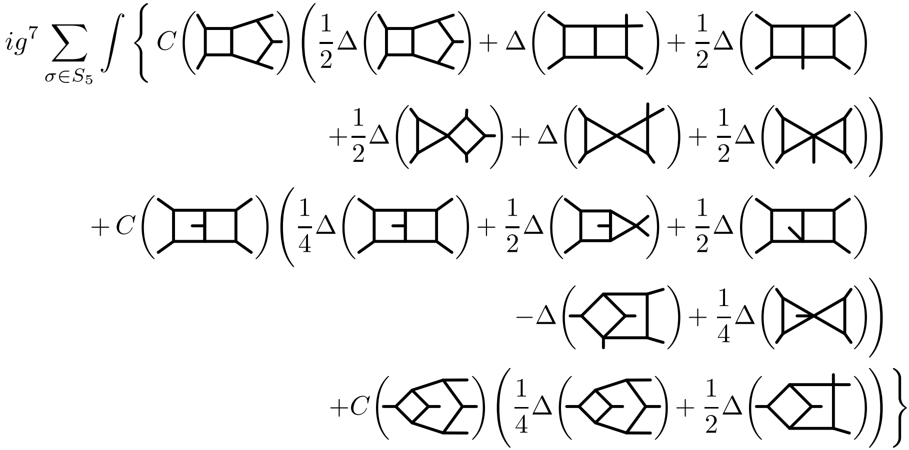




Invitation: Loop level application

Badger, Mogull, AO, O'Connell (2015) [4]

$$\mathcal{A}_5^{\text{2-loop}}(1^+,2^+,3^+,4^+,5^+) =$$



Invitation: Outline

$$\mathcal{A} = \sum_i C_i \, A_i$$

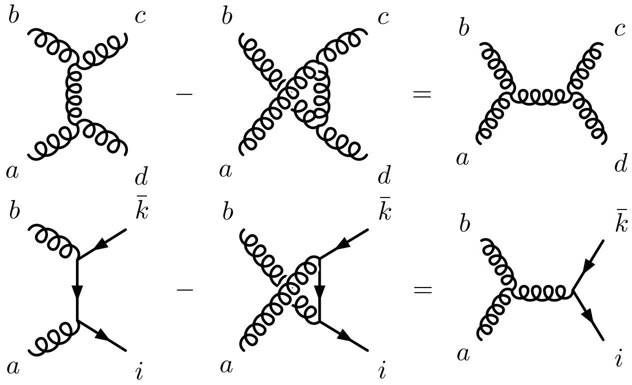
n particles:	color	kinematics
n gluons only	KK relations $\Rightarrow$ KK basis, $(n-2)!$ $\Rightarrow$ DDM decomposition	BCJ relations $\Rightarrow$ BCJ basis, $(n-3)!$
$(n-2k)$ gluons k quark pairs	KK relations $\Rightarrow$ Melia basis, $(n-2)!/k!$ $\Rightarrow$ new color decomposition?	new BCJ relations? reduced BCJ basis?

This talk: all question marks resolved and more!

QCD color structure

$$\tilde{f}^{dae}\tilde{f}^{ebc} - \tilde{f}^{dbe}\tilde{f}^{eac} = \tilde{f}^{abe}\tilde{f}^{dec}$$

$$T^a_{i\bar{\jmath}}T^b_{j\bar{k}} - T^b_{i\bar{\jmath}}T^a_{j\bar{k}} = \tilde{f}^{abe}T^e_{i\bar{k}}$$



Melia basis of primitive amplitudes

Melia (2013) [5, 6]

$$\left\{ A(\underline{1}, \overline{2}, \sigma) \;\middle|\; \sigma \in \mathrm{Dyck}_{k-1} \times \{\text{gluon insertions}\}_{n-2k} \right\}$$

$$\boxed{\text{Dyck word} = \text{well-formed brackets}}$$

Melia basis for $n = 4$, $k = 2$:

$$A(\underline{1}, \overline{2}, \underline{3}, \overline{4})$$

Melia basis for $n = 5$, $k = 2$:

$$A(\underline{1}, \overline{2}, 5, \underline{3}, \overline{4})\,, A(\underline{1}, \overline{2}, \underline{3}, 5, \overline{4})\,, A(\underline{1}, \overline{2}, \underline{3}, \overline{4}, 5)$$

Melia basis for $n = 6$, $k = 3$:

$$A(\underline{1}, \overline{2}, \underline{3}, \overline{4}, \underline{5}, \overline{6})\,, A(\underline{1}, \overline{2}, \underline{5}, \overline{6}, \underline{3}, \overline{4})\,, A(\underline{1}, \overline{2}, \underline{3}, \underline{5}, \overline{6}, \overline{4})\,, A(\underline{1}, \overline{2}, \underline{5}, \underline{3}, \overline{4}, \overline{6})$$

$$\begin{aligned} \text{XYXY} \;&\Rightarrow\; (\underline{3}, \overline{4}, \underline{5}, \overline{6}),\, (\underline{5}, \overline{6}, \underline{3}, \overline{4}) \;\Leftrightarrow\; \{3\,4\}\{5\,6\},\, \{5\,6\}\{3\,4\} \\ \text{XXYY} \;&\Rightarrow\; (\underline{3}, \underline{5}, \overline{6}, \overline{4}),\, (\underline{5}, \underline{3}, \overline{4}, \overline{6}) \;\Leftrightarrow\; \big\{3\{5\,6\}4\big\},\, \big\{5\{3\,4\}6\big\} \end{aligned}$$

New color decomposition

$$\mathcal{A}_{n,k}^{\text{tree}} = \sum_{\sigma \in \text{Melia basis}}^{\varkappa(n,k)} C(\underline{1}, \overline{2}, \sigma)\, A(\underline{1}, \overline{2}, \sigma)\,,$$

$$C(\underline{1}, \overline{2}, \sigma) = (-1)^{k-1} \{2|\sigma|1\}\Bigg|_{\substack{\underline{q} \;\to\; \{q|\, T^b \otimes \Xi_{l-1}^b \\ \overline{q} \;\to\; |q\} \\ g \;\to\; \Xi_l^{a_g}}}$$

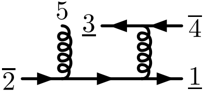
$$\Xi_l^a = \sum_{s=1}^{l} \underbrace{1 \otimes \cdots \otimes 1 \otimes \overbrace{T^a \otimes 1 \otimes \cdots \otimes 1 \otimes \overline{1}}^{s}}_{l}$$

checked analytically up to 8 points

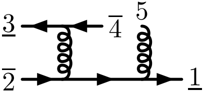
5-point color example

$$\mathcal{A}^{\text{tree}}_{5,2} = C_{\underline{1}\overline{2}5\underline{3}\overline{4}}\, A_{\underline{1}\overline{2}5\underline{3}\overline{4}} + C_{\underline{1}\overline{2}\underline{3}\overline{4}5}\, A_{\underline{1}\overline{2}\underline{3}\overline{4}5} + C_{\underline{1}\overline{2}\underline{3}5\overline{4}}\, A_{\underline{1}\overline{2}\underline{3}5\overline{4}}$$

$$C_{\underline{1}\overline{2}5\underline{3}\overline{4}} = -\{2|\Xi_1^{a_5}\{3|T^b \otimes \Xi_1^b|4\}|1\} = -(\overline{T}^{a_5}\overline{T}^b)_{\overline{\imath}_2 i_1} T^b_{i_3\overline{\imath}_4} =$$

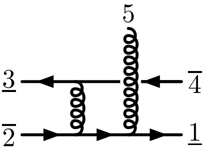


$$C_{\underline{1}\overline{2}\underline{3}\overline{4}5} = -\{2|\{3|T^b \otimes \Xi_1^b|4\}\Xi_1^{a_5}|1\} = -(\overline{T}^b\overline{T}^{a_5})_{\overline{\imath}_2 i_1} T^b_{i_3\overline{\imath}_4} =$$



$$C_{\underline{1}\overline{2}\underline{3}5\overline{4}} = -\{2|\{3|(T^b \otimes \Xi_1^b)\,\Xi_2^{a_5}|4\}|1\} = -(\overline{T}^b\overline{T}^{a_5})_{\overline{\imath}_2 i_1} T^b_{i_3\overline{\imath}_4} - \overline{T}^b_{\overline{\imath}_2 i_1}(T^b T^{a_5})_{i_3\overline{\imath}_4}$$

=



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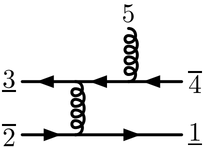
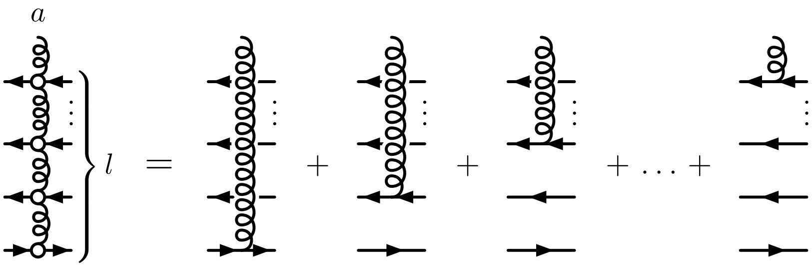


Reminder:

$$C(\underline{1},\overline{2},\sigma) = (-1)^{k-1}\,\{2|\sigma|1\}\Bigg|\begin{array}{lll} \underline{q} & \to & \{q|\,T^b \otimes \Xi^b_{l-1} \\ \overline{q} & \to & |q\} \\ g & \to & \Xi_l^{a_g} \end{array}$$

Tensor representation

$$\Xi_l^a = \sum_{s=1}^{l} \underbrace{1 \otimes \cdots \otimes 1 \otimes \overbrace{T^a \otimes 1 \otimes \cdots \otimes 1 \otimes \overline{1}}^{s}}_{l} \,.$$

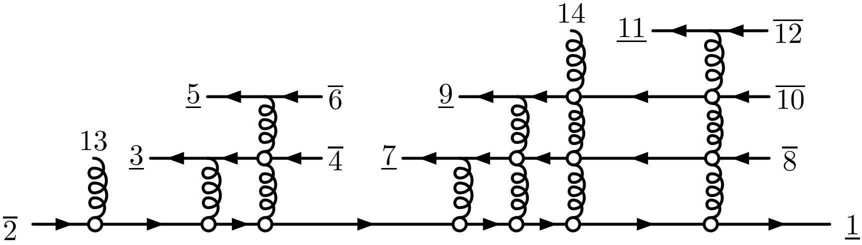


$$\left[\Xi_l^a, \, \Xi_l^b\right] = \tilde{f}^{abc}\, \Xi_l^c \,.$$

Higher-point color example

$$\mathcal{A}^{\text{tree}}_{14,6} = \sum_{\sigma \in \text{Melia basis}}^{665280} C(\underline{1}, \overline{2}, \sigma)\, A(\underline{1}, \overline{2}, \sigma)\,,$$

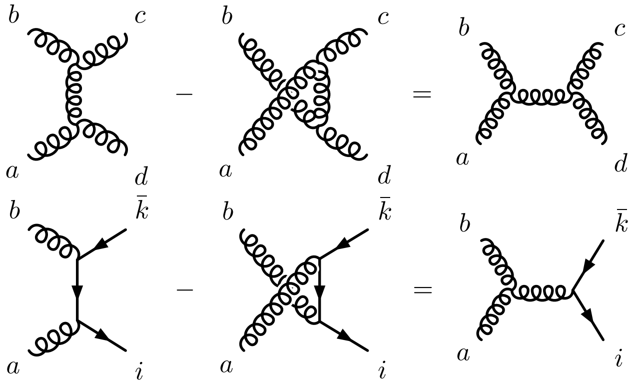
$$C(\underline{1}, \overline{2}, 13, \underline{3}, \underline{5}, \overline{6}, \overline{4}, \underline{7}, \underline{9}, 14, \underline{11}, \overline{12}, \overline{10}, \overline{8}) :$$



QCD color structure

$$\tilde{f}^{dae}\tilde{f}^{ebc} - \tilde{f}^{dbe}\tilde{f}^{eac} = \tilde{f}^{abe}\tilde{f}^{dec}$$

$$T^a_{i\bar{\jmath}}T^b_{j\bar{k}} - T^b_{i\bar{\jmath}}T^a_{j\bar{k}} = \tilde{f}^{abe}T^e_{i\bar{k}}$$

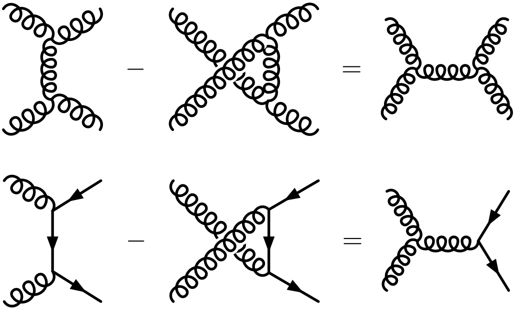


Review: BCJ color-kinematics duality

Bern, Carrasco, Johansson (2008,10) [7, 8]

Johansson, AO (2014) [9]

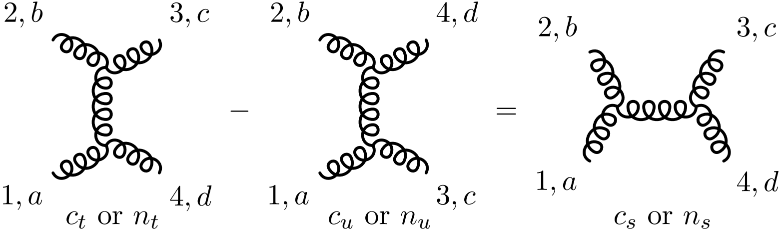
$$\mathcal{A}_4^{\text{tree}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$



works for color
check/impose on kinematics

Review: BCJ double copy

$$\mathcal{A}_4^{\text{tree}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$



$$\mathcal{M}_4^{\text{tree}} = i \left(\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u} \right)$$

Extends to any multiplicity, loop order and group rep.

Bern, Carrasco, Johansson (2008,10) [7, 8]

Johansson, AO (2014) [9]

Review: BCJ relations

Bern, Carrasco, Johansson (2008) [7]

$$\mathcal{A}^{\text{tree}}_{n,k} = \sum_{\text{cubic graphs } \Gamma_i} \frac{c_i n_i}{D_i}$$

$$c_i - c_j = c_k \quad \Leftrightarrow \quad n_i - n_j = n_k$$

$$\Rightarrow$$

$$\sum_{i=2}^{n-1} \Big(\sum_{j=2}^{i} s_{jn} \Big) A(1, 2, \ldots, i, n, i+1, \ldots, n-1) = 0$$

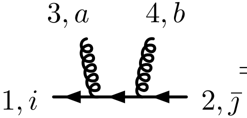
All other BCJ relations can be derived from relabelings thereof.

Feng, Huang, Jia (2010) [10]

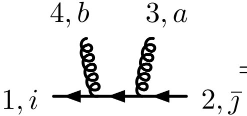
$$\Rightarrow$$

basis of $(n-3)!$ primitives

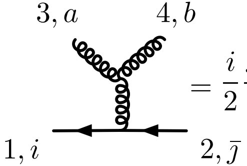
4-point kinematic example



$$= -\frac{i}{2}\,\frac{T^a_{i\bar{k}}T^b_{k\bar{\jmath}}}{s_{13}-m^2}\,(\bar{u}_1 \not{\varepsilon}_3(\not{k}_{1,3}+m)\not{\varepsilon}_4 v_2) = \frac{c_1 n_1}{D_1}$$



$$= -\frac{i}{2}\,\frac{T^b_{i\bar{k}}T^a_{k\bar{\jmath}}}{s_{14}-m^2}\,(\bar{u}_1 \not{\varepsilon}_4(\not{k}_{1,4}+m)\not{\varepsilon}_3 v_2) = \frac{c_2 n_2}{D_2}$$



$$= \frac{i}{2}\frac{\tilde{f}^{abc}T^c_{i\bar{\jmath}}}{s_{12}}\Big(2(k_4\cdot\varepsilon_3)(\bar{u}_1\not{\varepsilon}_4 v_2) - 2(k_3\cdot\varepsilon_4)(\bar{u}_1\not{\varepsilon}_3 v_2) + (\varepsilon_3\cdot\varepsilon_4)(\bar{u}_1(\not{k}_3 - \not{k}_4)v_2)\Big) = \frac{c_3 n_3}{D_3}$$

$c_1 - c_2 = c_3$ — commutation relation

$n_1 - n_2 - n_3 \propto \bar{u}_1\not{k}_1\not{\varepsilon}_3\not{\varepsilon}_4 v_2 + \bar{u}_1\not{\varepsilon}_3\not{\varepsilon}_4\not{k}_2 v_2 - (\varepsilon_3\cdot\varepsilon_4)(\bar{u}_1(\not{k}_1 + \not{k}_2)v_2) = 0$

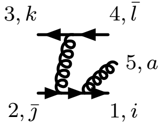
4-point amplitude relation

$$\mathcal{A}_{4,1}^{\text{tree}} = \frac{c_1 n_1}{D_1} + \frac{c_2 n_2}{D_2} + \frac{c_3 n_3}{D_3} = c_2 A_{\underline{1}\overline{2}34} + c_1 A_{\underline{1}\overline{2}43}$$

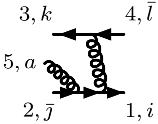
$$\begin{cases} A_{\underline{1}\overline{2}34} = \dfrac{n_2}{D_2} - \dfrac{n_3}{D_3} \\ A_{\underline{1}\overline{2}43} = \dfrac{n_1}{D_1} + \dfrac{n_3}{D_3} \\ n_1 - n_2 = n_3 \end{cases} \quad \Rightarrow \quad \boxed{(s_{14} - m^2) A_{\underline{1}\overline{2}34} = (s_{13} - m^2) A_{\underline{1}\overline{2}43}}$$

$$\mathcal{A}_{4,1}^{\text{tree}} = \left(T^{a_3}_{\bar{\imath}_2 j} T^{a_4}_{\bar{\jmath} i_1} + T^{a_4}_{\bar{\imath}_2 j} T^{a_3}_{\bar{\jmath} i_1} \frac{s_{14} - m^2}{s_{13} - m^2} \right) A_{\underline{1}\overline{2}34}$$

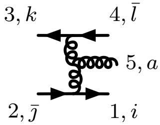
5-point kinematic example



$$= \frac{i}{2\sqrt{2}}\,\frac{1}{(s_{15}-m_1^2)s_{34}}\,T^a_{i\bar m}T^b_{m\bar j}T^b_{k\bar l}\,(\bar u_1\not\varepsilon_5(\not k_{1,5}+m_1)\gamma^\mu v_2)(\bar u_3\gamma_\mu v_4) = \frac{c_1 n_1}{D_1}$$



$$= \frac{-i}{2\sqrt{2}}\,\frac{1}{(s_{25}-m_2^2)s_{34}}\,T^b_{i\bar m}T^a_{m\bar j}T^b_{k\bar l}\,(\bar u_1\gamma^\mu(\not k_{2,5}-m_2)\not\varepsilon_5 v_2)(\bar u_3\gamma_\mu v_4) = \frac{c_2 n_2}{D_2}$$



$$= \frac{i}{\sqrt{2}}\,\frac{1}{s_{12}s_{34}}\,\tilde f^{abc}T^c_{i\bar j}T^b_{k\bar l}\Big((\bar u_1\not\varepsilon_5 v_2)(\bar u_3\not k_5 v_4) - (\bar u_1\not k_5 v_2)(\bar u_3\not\varepsilon_5 v_4) - (\bar u_1\gamma^\mu v_2)(\bar u_3\gamma_\mu v_4)(k_{12}\cdot\varepsilon_5)\Big) = \frac{c_5 n_5}{D_5}$$

$$c_1 - c_2 = c_5 \qquad c_3 - c_4 = -c_5$$

$$n_1 - n_2 = n_5 \qquad n_3 - n_4 = -n_5$$

$$\Rightarrow \quad (s_{35}-m_3^2)A_{\underline{1}\bar 2\underline{3}5\bar 4} + (s_{12}-s_{34})A_{\underline{1}\bar 2\underline{3}\bar 4 5} - (s_{25}-m_2^2)A_{\underline{1}5\bar 2\underline{3}\bar 4} = 0$$

$$\text{or} \quad (s_{25}-m_2^2)A_{\underline{1}\bar 2 5\underline{3}\bar 4} + (s_{14}-s_{23})A_{\underline{1}\bar 2\underline{3}5\bar 4} - (s_{15}-m_1^2)A_{\underline{1}\bar 2\underline{3}\bar 4 5} = 0$$

BCJ relations for QCD

purely gluonic: $\sum_{i=2}^{n-1}\left(\sum_{j=2}^{i} s_{jn}\right) A(1,2,\ldots i,n,i+1,\ldots,n-1) = 0$

becomes

$$\sum_{i=2}^{n-1}\left(\sum_{j=2}^{i} s_{jn} - m_j^2\right) A(1,2,\ldots i,n,i+1,\ldots,n-1) = 0$$

where n is a gluon

already proven by De la Cruz, Kniss, Weinzierl (2015) [11]

number of independent BCJ relations:

$k \setminus n$	3	4	5	6	7	8
0	0	1	4	18	96	600
1	0	1	4	18	96	600
2	-	0	1	6	36	240
3	-	-	-	0	4	40
4	-	-	-	-	-	0

Solution to BCJ relations for QCD

General BCJ relations:

$$A(\underline{1},\overline{2},\alpha,\underline{q},\beta)=\sum_{\sigma\in S(\alpha)\sqcup\beta}A(\underline{1},\overline{2},\underline{q},\sigma)\prod_{i=1}^{|\alpha|}\frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}-m_2^2}\,,$$

where α is purely gluonic

Melia basis of $(n-2)!/k!$ primitives

$$\left\{A(\underline{1},\overline{2},\sigma)\;\middle|\;\sigma\in\mathrm{Dyck}_{k-1}\times\{\text{gluon insertions}\}_{n-2k}\right\}$$

$\Rightarrow$ new BCJ basis of $(n-3)!(2k-2)/k!$ primitives

$$\left\{A(\underline{1},\overline{2},\underline{q},\sigma)\;\middle|\;\{\underline{q},\sigma\}\in\mathrm{Dyck}_{k-1}\times\{\text{gluon insertions in }\sigma\}_{n-2k}\right\}$$

New amplitude decomposition

Full color-dressed amplutude in terms of only
$(n-3)!(2k-2)/k!$ color-ordered primitive amplitudes:

$$\mathcal{A}^{\text{tree}}_{n,k\geq 2} = \sum_{(\underline{q},\sigma)\in \text{BCJ basis}} A(\underline{1},\overline{2},\underline{q},\sigma)$$

$$\times\left\{C(\underline{1},\overline{2},\underline{q},\sigma) + \sum_{\substack{\beta\subset\sigma \\ \sigma\backslash\beta \text{ gluonic}}} \sum_{\alpha\in S(\sigma\backslash\beta)} C(\underline{1},\overline{2},\alpha,\underline{q},\beta) \prod_{i=1}^{|\alpha|} \frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i} - m_2^2}\right\}$$

$$\mathcal{A}^{\text{tree}}_{n,k\leq 1} = \sum_{\alpha\in S_{n-3}(\{4,\ldots,n\})} A(1,2,3,\sigma)$$

$$\times\left\{C(1,2,3,\sigma) + \sum_{\beta\subset\sigma} \sum_{\alpha\in S(\sigma\backslash\beta)} C(1,2,\alpha,3,\beta) \prod_{i=1}^{|\alpha|} \frac{\mathcal{F}(3,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i} - m_2^2}\right\}$$

or $(n-3)!$ primitives for $k=0,1$

Conclusions

- New color decomposition for any quark-gluon tree amplitude in basis of $(n-2)!/k!$ primitives
- Can be used inside loops similarly to DDM decomposition
- Color-kinematics duality for massive quarks
- New BCJ relations for any quark-gluon tree amplitude
- Reduced basis of $(n-3)!(2k-2)!/k!$ primitives
  (or simply $(n-3)!$ for $k=1,0$)
- New amplitude decomposition after KK and BCJ relations

Thank you!

Backup slides

Pure-quark Melia basis

Melia (2013) [5]

$$\left\{ A(\underline{1}, \overline{2}, \sigma) \;\middle|\; \sigma \in \text{Dyck}_{k-1} \right\}$$

Dyck word = well-formed brackets

pure-quark $\mathcal{A}_{6,3}(\underline{1}, \overline{2}, \underline{3}, \overline{4}, \underline{5}, \overline{6})$:
fix $\underline{1}$, $\overline{2}$; rest must form Dyck words of length 4

$$\begin{aligned} \text{XYXY} \quad &\Rightarrow \quad (\underline{3}, \overline{4}, \underline{5}, \overline{6}),\ (\underline{5}, \overline{6}, \underline{3}, \overline{4}) \;\Leftrightarrow\; \{3\,4\}\{5\,6\},\ \{5\,6\}\{3\,4\} \\ \text{XXYY} \quad &\Rightarrow \quad (\underline{3}, \underline{5}, \overline{6}, \overline{4}),\ (\underline{5}, \underline{3}, \overline{4}, \overline{6}) \;\Leftrightarrow\; \big\{3\{5\,6\}4\big\},\ \big\{5\{3\,4\}6\big\} \end{aligned}$$

Melia basis for $n = 6$, $k = 3$:

$$A(\underline{1}, \overline{2}, \underline{3}, \overline{4}, \underline{5}, \overline{6})\,, A(\underline{1}, \overline{2}, \underline{5}, \overline{6}, \underline{3}, \overline{4})\,, A(\underline{1}, \overline{2}, \underline{3}, \underline{5}, \overline{6}, \overline{4})\,, A(\underline{1}, \overline{2}, \underline{5}, \underline{3}, \overline{4}, \overline{6})$$

Melia basis of primitive amplitudes

Melia (2013) [5, 6]

$$\left\{ A(\underline{1}, \overline{2}, \sigma) \;\middle|\; \sigma \in \mathrm{Dyck}_{k-1} \times \{\text{gluon insertions}\}_{n-2k} \right\}$$

Melia basis for $n = 5$, $k = 2$:

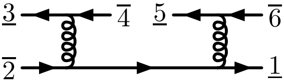
$$A(\underline{1}, \overline{2}, 5, \underline{3}, \overline{4}) \,, A(\underline{1}, \overline{2}, \underline{3}, 5, \overline{4}) \,, A(\underline{1}, \overline{2}, \underline{3}, \overline{4}, 5)$$

$$\varkappa(n,k) = \underbrace{\overbrace{\frac{(2k-2)!}{k!(k-1)!}}^{\text{empty brackets}} \times (k-1)!}_{\text{dressed quark brackets}} \times \underbrace{(2k-1)(2k)\dots(n-2)}_{\text{insertions of } (n-2k) \text{ gluons}} = \frac{(n-2)!}{k!}$$

6-point color example

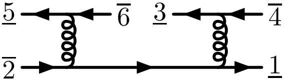
$$\mathcal{A}_{6,3}^{\text{tree}} = C_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}}\, A_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}} + C_{\underline{1}\overline{2}\underline{5}\overline{6}\underline{3}\overline{4}}\, A_{\underline{1}\overline{2}\underline{5}\overline{6}\underline{3}\overline{4}} + C_{\underline{1}\overline{2}\underline{35}\overline{64}}\, A_{\underline{1}\overline{2}\underline{35}\overline{64}} + C_{\underline{1}\overline{2}\underline{53}\overline{46}}\, A_{\underline{1}\overline{2}\underline{53}\overline{46}}$$

$$C_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}} =$$



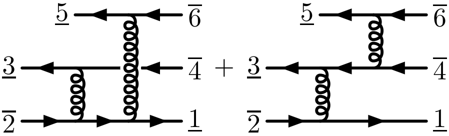
$$= \{2|\{3|T^a \otimes \Xi_1^a|4\}\{5|T^b \otimes \Xi_1^b|6\}|1\}$$

$$C_{\underline{1}\overline{2}\underline{5}\overline{6}\underline{3}\overline{4}} =$$



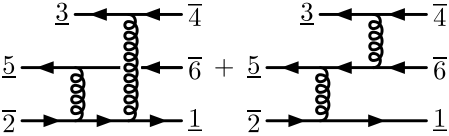
$$= \{2|\{5|T^a \otimes \Xi_1^a|6\}\{3|T^b \otimes \Xi_1^b|4\}|1\}$$

$$C_{\underline{1}\overline{2}\underline{35}\overline{64}} =$$



$$= \{2|\{3|T^a \otimes \Xi_1^a\{5|T^b \otimes \Xi_2^b|6\}|4\}|1\}$$

$$C_{\underline{1}\overline{2}\underline{53}\overline{46}} =$$



$$= \{2|\{5|T^a \otimes \Xi_1^a\{3|T^b \otimes \Xi_2^b|4\}|6\}|1\}$$

Review: Double copy for loops

Bern, Carrasco, Johansson (2010) [8]

$$\mathcal{A}_m^{L\text{-loop}} = i^L \sum_i \int \frac{d^{LD}\ell}{(2\pi)^{DL}} \frac{1}{S_i} \frac{n_i c_i}{D_i}$$

$$\begin{aligned} c_i - c_j = c_k \quad &\Leftrightarrow \quad n_i - n_j = n_k \\ c_i \to -c_i \quad &\Leftrightarrow \quad n_i \to -n_i \end{aligned}$$

$$\mathcal{M}_m^{L\text{-loop}} = i^{L+1} \sum_i \int \frac{d^{LD}\ell}{(2\pi)^{DL}} \frac{1}{S_i} \frac{n_i n_i'}{D_i}$$

Similarly for fundamental representation

Johansson, AO (2014) [9]

5-point amplitude relation

$$\mathcal{A}^{\text{tree}}_{5,2} = \frac{c_1 n_1}{D_1}+\frac{c_2 n_2}{D_2}+\frac{c_3 n_3}{D_3}+\frac{c_4 n_4}{D_4}+\frac{c_5 n_5}{D_5} = -c_2 A_{\underline{1}\bar{2}5\underline{3}\bar{4}} - c_1 A_{\underline{1}\bar{2}\underline{3}\bar{4}5} + (-c_1+c_4) A_{\underline{1}\bar{2}\underline{3}5\bar{4}}$$

$$\begin{aligned} A_{\underline{1}\bar{2}5\underline{3}\bar{4}} &= -\frac{n_2}{D_2} - \frac{n_3}{D_3} - \frac{n_5}{D_5} \\ A_{\underline{1}\bar{2}\underline{3}\bar{4}5} &= -\frac{n_1}{D_1} - \frac{n_4}{D_4} + \frac{n_5}{D_5} \\ A_{\underline{1}\bar{2}\underline{3}5\bar{4}} &= \frac{n_3}{D_3} + \frac{n_4}{D_4} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & (s_{35}-m_3^2) A_{\underline{1}\bar{2}\underline{3}5\bar{4}} + (s_{12}-s_{34}) A_{\underline{1}\bar{2}\underline{3}\bar{4}5} - (s_{25}-m_2^2) A_{\underline{1}5\bar{2}\underline{3}\bar{4}} = 0 \\ \text{or} \quad & (s_{25}-m_2^2) A_{\underline{1}\bar{2}5\underline{3}\bar{4}} + (s_{14}-s_{23}) A_{\underline{1}\bar{2}\underline{3}5\bar{4}} - (s_{15}-m_1^2) A_{\underline{1}\bar{2}\underline{3}\bar{4}5} = 0 \end{aligned}$$

$$\begin{aligned} \mathcal{A}^{\text{tree}}_{5,2} = & \left(T^b_{i_1 \bar{\imath}_2} T^{a_5}_{i_3 \bar{\jmath}} T^b_{j\,\bar{\imath}_4} + T^b_{i_1 \bar{\jmath}} T^{a_5}_{j\,\bar{\imath}_2} T^b_{i_3 \bar{\imath}_4} \frac{s_{35}-m_3^2}{s_{25}-m_2^2} \right) A_{\underline{1}\bar{2}\underline{3}5\bar{4}} \\ & - \left(T^{a_5}_{i_1 \bar{\jmath}} T^b_{j\,\bar{\imath}_2} T^b_{i_3 \bar{\imath}_4} + T^b_{i_1 \bar{\jmath}} T^{a_5}_{j\,\bar{\imath}_2} T^b_{i_3 \bar{\imath}_4} \frac{s_{15}-m_1^2}{s_{25}-m_2^2} \right) A_{\underline{1}\bar{2}\underline{3}\bar{4}5} \end{aligned}$$

New formula for gravitational scattering amplitudes

$$\mathcal{A}_{n,k}^{\text{tree}} = \sum_{\sigma \in \text{Melia basis}} C(1,2,\sigma) A(1,2,\sigma)\,,$$

$C(1,2,\sigma)$ is constructed out of c_i

$$\mathcal{M}_{n,k}^{\text{tree}} = \sum_{\sigma \in \text{Melia basis}} K(1,2,\sigma) A(1,2,\sigma)\,,$$

$K(1,2,\sigma)$ is constructed out of n_i

Gravity coupled to massive scalars, fermions, vectors

References

R. Kleiss and H. Kuijf *Nucl.Phys.* **B312** (1989) 616.

V. Del Duca, A. Frizzo, and F. Maltoni *Nucl.Phys.* **B568** (2000) 211–262.

V. Del Duca, L. J. Dixon, and F. Maltoni *Nucl.Phys.* **B571** (2000) 51–70.

S. Badger, G. Mogull, A. Ochirov, and D. O'Connell `arXiv:1507.08797`.

T. Melia *Phys.Rev.* **D88** (2013), no. 1 014020.

T. Melia *Phys.Rev.* **D89** (2014), no. 7 074012.

Z. Bern, J. Carrasco, and H. Johansson *Phys.Rev.* **D78** (2008) 085011.

Z. Bern, J. J. M. Carrasco, and H. Johansson *Phys.Rev.Lett.* **105** (2010) 061602.

H. Johansson and A. Ochirov `arXiv:1407.4772`.

B. Feng, R. Huang, and Y. Jia *Phys.Lett.* **B695** (2011) 350–353.

L. De la Cruz, A. Kniss, and S. Weinzierl `arXiv:1508.01432`.