Five-point two-loop master integrals in QCD

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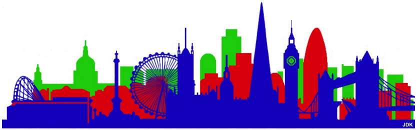

work in collaboration with Thomas Gehrmann and Johannes Henn

Introduction

With LHC's RUN II higher precision of theoretical predictions is expected.

1-loop NLO established in the last decade as the new standard for high-multiplicity processes. BlackHat, Gosam, OpenLoops, NJet ...

2-loop NNLO is the current frontier
(although: N^3LO for inclusive Higgs production done [Anastasiou et al.]).

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- $2 \to 2$ processes calculated recently ($\gamma\gamma$, ZZ, $Z\gamma$, $W\gamma$, WW, $t\bar{t}$, Hj, Wj, jj)
  [Catani, Cieri, de Florian, Ferrera, Grazzini, Gehrmann, G.-De Ridder, Glover, Boughezal, Focke, Liu,Petriello, Czakon, Fiedler, Mitov, Kallweit, Maierhöfer, Rathlev, Chen, Jaquier, Giele, Melnikov, Caola, Schulze, Tejeda-Yeomans, Huss, Morgan . . .]

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- **$2 \to 3$ processes still open**

Introduction

Among NNLO bottle-necks:
two-loop scattering amplitudes $\longrightarrow$ **purely virtual contribution**.

At one-loop Feynman diagrams can be decomposed
into a small set of master integrals (MIs), all of which are known.

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Taking derivatives of the integr-als/-ands delivers a very powerful tool
- to reduce the amplitude to MIs (Laporta algorithm),
- **to evaluate the integrals** (Differential Equation method).

Review of Differential-Equations method

Integration by part identities

$$\int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \left(\frac{\partial}{\partial k_j^\mu} v^\mu \frac{1}{D_1^{a_1} \dots D_n^{a_n}} \right) = 0$$

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Derivaties w.r.t external kinematic invariants

$$\frac{\partial}{\partial p^2} \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{D_1^{a_1} \dots D_n^{a_n}} = \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{2p^2} \left(p^\mu \frac{\partial}{\partial p^\mu} \right) \frac{1}{D_1^{a_1} \dots D_n^{a_n}}$$

yield differential equations for MIs. [Gehrmann, Remiddi]

Codes used: Fire [Smirnov], Reduze [von Manteuffel],

Review of Differential-Equations method

MIs basis is not unique. Suitable choice considerably simplifies diff. eqs.:

$\partial_x \vec{f} = A(x,\varepsilon)\vec{f} \longrightarrow \partial_x \vec{f} = \varepsilon A(x)\vec{f}$ can be integrated order by order in ε.

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Further simplification:

$$\partial_x \vec{f} = \varepsilon \sum_k \frac{A_k}{x - x_k}\vec{f} \quad \longrightarrow \quad d\vec{f}(\vec{x},\varepsilon) = \varepsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right]\vec{f}(\vec{x},\varepsilon)$$

where the list of functions $\{\alpha_1, \ldots \alpha_n\}$ is called **alphabet**.

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Solution (symbolic): $\vec{f}(\vec{x},\varepsilon) = P\exp\left[\varepsilon \int_\gamma dA\right]\vec{f}(\vec{x}_0,\varepsilon)$

Review of Differential-Equations method

Solutions expressed in terms of multiple polylogarithms

[Remiddi,Vermaseren; Gehrmann, Remiddi; Goncharov]

$$G(a_1,a_2,\ldots,a_n\,;x) = \int_0^x \frac{dt}{t-a_1} G(a_2,\ldots,a_n\,;t)\,,$$

$$\text{with } G(x) = 1\,, \quad G(0) = 0 \quad \text{and} \quad G(\vec{0}_n\,;x) = \frac{1}{n!}\log^n x\,.$$

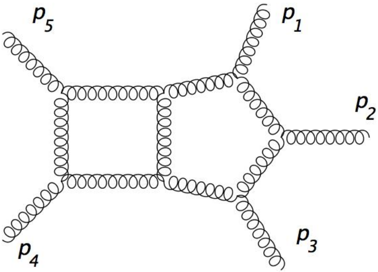
Simple example: $G(\vec{a}_n\,;x) = \frac{1}{n!}\log^n\left(1-\frac{x}{a}\right)$ with $\vec{a}_n = \{a,\ldots,a\}$

If $a_i \in \{1,-1,0\}$ $\longrightarrow$ Harmonic Polylogarithms.

2-loop five-point planar integrals

$$G_{\{a_1,\ldots,a_{11}\}} = \int \frac{d^D k_1\, d^D k_2}{(i\pi^{D/2})^2} \frac{D_9^{-a_9}\, D_{10}^{-a_{10}}\, D_{11}^{-a_{11}}}{D_1^{a_1}\, D_2^{a_2}\, D_3^{a_3}\, D_4^{a_4}\, D_5^{a_5}\, D_6^{a_6}\, D_7^{a_7}\, D_8^{a_8}}$$

$$\begin{aligned}
D_1 &= -k_1^2\,, \\
D_2 &= -(k_1+p_1)^2\,, \\
D_3 &= -(k_1+p_1+p_2)^2\,, \\
D_4 &= -(k_1+p_1+p_2+p_3)^2\,, \\
D_5 &= -k_2^2\,, \\
D_6 &= -(k_2+p_1+p_2+p_3)^2\,, \\
D_7 &= -(k_2+p_1+p_2+p_3+p_4)^2\,, \\
D_8 &= -(k_1-k_2)^2\,, \\
D_9 &= -(k_1+p_1+p_2+p_3+p_4)^2\,, \\
D_{10} &= -(k_2+p_1)^2\,, \\
D_{11} &= -(k_2+p_1+p_2)^2
\end{aligned}$$

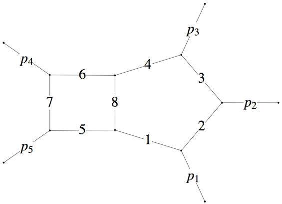


$$s_{ij} = (p_i + p_j)^2$$

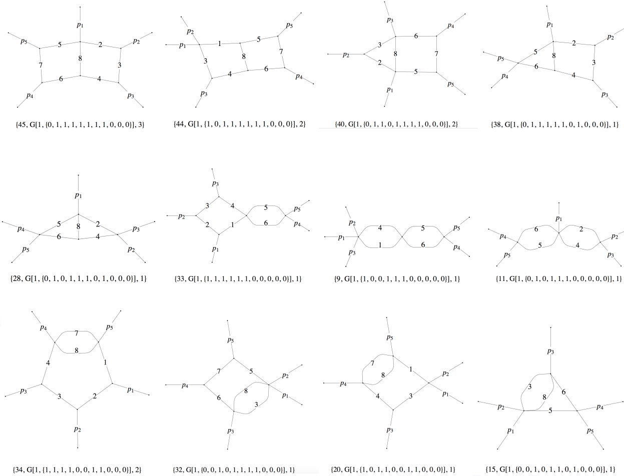
$$\vec{x} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\}$$

2-loop five-point planar integrals

61 MIs

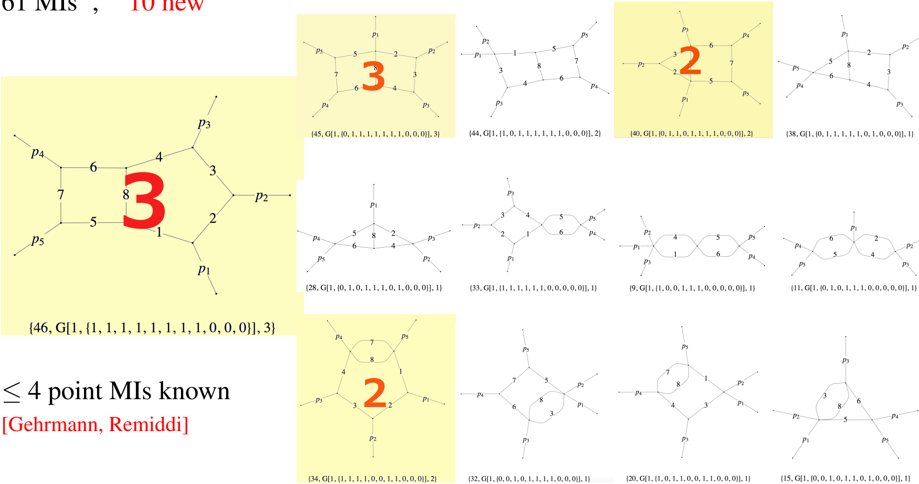


{46, G[1, {1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0}], 3}



2-loop five-point planar integrals

61 MIs , 10 new



≤ 4 point MIs known

[Gehrmann, Remiddi]

2-loop five-point planar integrals

Alphabet of 24 letter

$$\{ \; s_{12} \;,\; s_{12}-s_{34} \;,\; s_{12}+s_{23} \;,\; s_{12}-s_{34}-s_{45} \;,\; \ldots \;,$$
$$(s_{23}-s_{51})\sqrt{\Delta} + s_{12}\,s_{23}^2 - s_{34}\,s_{23}^2 + s_{34}\,s_{45}\,s_{23} - 2\,s_{12}\,s_{51}\,s_{23}$$
$$+ s_{34}\,s_{51}\,s_{23} + s_{45}\,s_{45}\,s_{23} + s_{12}\,s_{51}^2 - s_{45}\,s_{51}^2 + s_{34}\,s_{45}\,s_{23} \;\}$$

Δ is the Gram determinant.
With a suitably chosen parametrization, $\Delta \to$ perfect square

$$\begin{aligned} s_{12} &= z_1 \,, \\ s_{23} &= z_1\,z_2\,z_4 \,, \\ s_{34} &= (z_1/z_2)\left[z_3\,(z_4-1) + z_2\,z_4 + z_2\,z_3\,(z_4-z_5)\right] , \\ s_{45} &= z_1\,z_2\,(z_4-z_5) \,, \\ s_{51} &= z_1\,z_3\,(1-z_5) \end{aligned}$$

obtained by using Momentum Twistor variables [Hodges 0905.1473]

Boundary conditions

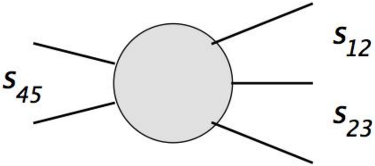
Boundary values can be obtained from physical conditions,
in kinematic limits with singular diff. eq. but regular integrals.

No singularities in the Euclidean region $s_{i,i+1} < 0$.

Un-physical singularities appear in the limit

$$s_{45} \to s_{12} + s_{23}$$

and they need to cancel.



$\longrightarrow$ no need to compute any additional integrals.

Boundary conditions

$\Delta = 0$ defines hypersurface where divergencies need to cancel.

The symmetric point $\vec{x}_{sym} = \{-1,-1,-1,-1,-1\}$

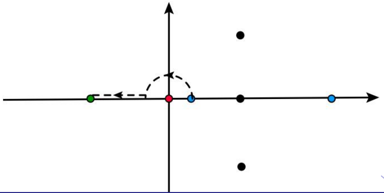
is connected to the $\Delta = 0$ surface by

$$\vec{f}(\vec{x},\varepsilon) = P\exp\left[\varepsilon \int_\gamma dA\right] \vec{f}(\vec{x}_0,\varepsilon)$$

path $\gamma = \{-\frac{y}{(1-y)^2},-1,-1,-1,-1\} \longrightarrow$ reduced alphabet .

$$\text{Sym. pt} \rightarrow y = \frac{3 \pm \sqrt{5}}{2}$$

$$\Delta = 0 \rightarrow y = -1$$



Applications

- We have applied our integrals to the **all-plus amplitude**.
In particular we have checked the pole structure

$$A_5^{(2)}(1^+,2^+,3^+,4^+,5^+) = -\left(\frac{1}{\varepsilon^2}\sum_{i=1}^{5}\left(\frac{\mu_R^2}{-s_{i,i+1}}\right)^{\varepsilon} + \frac{11}{3\varepsilon}\right) A_5^{(1)}(1^+,2^+,3^+,4^+,5^+) + O(\varepsilon^0)$$

Finite part: ongoing numerical comparisons against [Badger, Frellesvig, Zhang].

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- Check done: **infrared** structure and **finite** part of $\mathcal{N} = 4$ sYM ampl.

$$\log M_5 = \sum_{L\geq 1} a^L \left[-\frac{\gamma^{(L)}}{8\,(L\varepsilon)^2} - \frac{\mathcal{G}_0^{(L)}}{4\,L\varepsilon} + f^L\right] \sum_{i=1}^{5}\left(\frac{\mu^2}{s_i, i+1}\right)^{L\varepsilon} + \frac{\gamma(a)}{4} F_n^{(1)}(a) + C(a) + O(\varepsilon)$$

Formula conjectured by [Bern, Dixon, Smornov (BDS)].
Follows from dual-conformal symmetry [Drummond,Henn,Korchemsky,Sokatchev].
Proven numerically [Cachazo,Spradlin,Volovich; Bern,Czakon,Kosower,Roiban,Smirnov]

Summary and Outlook

- I have presented the computation of five-point two-loop MIs (planar).
- Results obtained using the Differential-Equation method, with MIs basis that makes the diff. eq. system canonical.
- Boundary conditions obtained by requiring the cancellation of spurious singularities in diff. eqs. $\rightarrow$ No further integration required.

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- Results obtained using the Differential-Equation method, with MIs basis that makes the diff. eq. system canonical.
- Boundary conditions obtained by requiring the cancellation of spurious singularities in diff. eqs. $\rightarrow$ No further integration required.
- Analytic continuation outside Euclidean region ($\rightarrow$ physical region).
- Non-planar integrals: in progress.