How to (path) integrate by differentiating

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Achim Kempf

Departments of Applied Mathematics and Physics
and Institute for Quantum Computing, University of Waterloo
and Perimeter Institute

Joint work with D.M. Jackson (UW) and A.H. Morales (UCLA)

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- If only integration could be expressed in terms of differentiation! **Or can it?**

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- Applications to QFT
  - Expresses functional integrations and functional transforms in terms of functional differentiation.
  - Offers new perturbative approaches.

New representations of integration:

$$\int_a^b f(x)\, dx \quad = \quad \lim_{\epsilon \to 0} \; f(\partial_\epsilon)\; \frac{e^{\epsilon b} - e^{\epsilon a}}{\epsilon}$$

$$\int_{-\infty}^{\infty} f(x)\, dx \quad = \quad \lim_{\epsilon \to 0^+} \left(f(\partial_\epsilon) + f(-\partial_\epsilon) \right)\; \frac{1}{\epsilon}$$

Compare with:

$$f'(x) \;=\; \lim_{\epsilon \to 0} (f(x+\epsilon) - f(x))\; \frac{1}{\epsilon}$$

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&= \pi
\end{aligned}$$

Examples for integration

Similarly, one quickly obtains, e.g.,

$$\int_{-\infty}^{\infty} \frac{\sin^5(x)}{x}\, dx = 3\pi/8$$

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, dx = \pi$$

$$\int_{-\infty}^{\infty} \frac{(1-\cos(tx))}{x^2}\, dx = \pi|t|$$

$$\int_{-\infty}^{\infty} x^2 \cos(x)\, e^{-x^2}\, dx = \sqrt{\pi} e^{-1/4}/4$$

etc ...

Examples for Fourier

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Fourier transforming is even easier than integrating!

Examples for Fourier

Recall the new methods for integration and Fourier:

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How are they related?

The zero-frequency value of the Fourier transform is the integral (up to a prefactor of $\sqrt{2\pi}$).

Examples for Fourier

For example, for $f(x) = \sin(x)/x$, recall:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx &= 2\pi \lim_{x\to 0} \frac{1}{2i}\left(e^{\partial_x} - e^{-\partial_x}\right) \frac{1}{-i\partial_x}\, \delta(x) \\
&= \pi \lim_{x\to 0} \left(e^{\partial_x} - e^{-\partial_x}\right) \left(\Theta(x) + c'\right) \\
&= \pi \lim_{x\to 0} \left(\Theta(x+1) - \Theta(x-1)\right) \\
&= \pi
\end{aligned}
$$

By not taking the limit and by dividing by $\sqrt{2\pi}$, we obtain immediately:

$$
\mathcal{F}[f](x) \;=\; \sqrt{\pi/2}\;\; \left(\Theta(x+1) - \Theta(x-1)\right)
$$

Proof of the Fourier formula

Why does this work?

Proof of the Fourier formula

The claim is:

$$\mathcal{F}[f](x) = \sqrt{2\pi}\; f(-i\partial_x)\; \delta(x)$$

Let us apply this to a plane wave: $f(x) = e^{ixy}$.

We obtain the right answer:

$$\begin{aligned} \mathcal{F}[f](x) &= \sqrt{2\pi}\; e^{y\partial_x}\; \delta(x) \\ &= \sqrt{2\pi}\; \delta(x+y) \end{aligned}$$

And the plane waves from a basis of the function space.

What is going on, intuitively?

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How does it work?

Regulate, e.g., this way: $\delta(x) = \lim_{\sigma\to 0}(2\pi\sigma)^{-1/2} e^{-x^2/2\sigma}$

$$\int_{-\infty}^{\infty} f(x)\, dx \;=\; 2\pi\ \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma}}\ e^{\partial_x^2/2\sigma}\ f(x)$$

Integration from asymptotics of heat flow !

Perturbative expansions

In QFT, we'd like to apply the new methods, e.g.:

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Perturbative expansions

But what if in

$$Z[J] = \int e^{iS[\phi]+i\int J\phi \ d^nx} D[\phi]$$

the action $S[\phi]$ is not suitable to solve the integral or Fourier (or Laplace) transform with our new methods exactly?

And that's the norm of course!

Perturbative expansions

- On the basic level, what if $f(x)$ is too complicated, e.g., for:

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- Obtain weak & strong coupling expansions and others...
- Also: applications to deblurring expansion of signals.

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- Relation to fermionic integration, a unifying formalism?

- A new perspective on integration measures and therefore anomalies in QFT?

Bonus: Examples for Laplace

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Examples for Laplace

If we apply the new Laplace transform method

$$\mathcal{L}[f](x) \;=\; f(-\partial_x)\,\frac{1}{x}$$

to monomials $f(x) = x^n$ we obtain:

$$L[f](x) = (-\partial_x)^n\,\frac{1}{x} \;=\; \frac{n!}{x^{n+1}}$$

And the monomials form a basis in the function space.

Example for inverse Laplace

Consider a heat kernel trace:

$$h(t) = \sum_n e^{-\lambda_n t}$$

Given $h(t)$, the spectrum $\{\lambda_n\}$ is known to be recoverable via inverse Laplace transform.

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this is easy to see:

$$\begin{aligned}\mathcal{L}^{-1}[h](\lambda) &= h(\partial_\lambda)\,\delta(\lambda)\\ &= \sum_n e^{-\lambda_n \partial_\lambda}\,\delta(\lambda) = \sum_n \delta(\lambda - \lambda_n)\end{aligned}$$