How to (path) integrate by differentiating

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The problem

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• Path integrals (which are also functional Fourier transforms)

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- If only integration could be expressed in terms of differentiation! **Or can it?**
Overview

• Main message?
  • New, convenient methods for integration and integral transforms such as Fourier and Laplace, using only derivatives.
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  • Often quicker, simpler.
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• Applications to QFT
  • Expresses functional integrations and functional transforms in terms of functional differentiation.
  • Offers new perturbative approaches.
New representations of integration:

\[ \int_a^b f(x) \, dx = \lim_{\epsilon \to 0} f(\partial_\epsilon) \frac{e^{\epsilon b} - e^{\epsilon a}}{\epsilon} \]

\[ \int_{-\infty}^{\infty} f(x) \, dx = \lim_{\epsilon \to 0^+} (f(\partial_\epsilon) + f(-\partial_\epsilon)) \frac{1}{\epsilon} \]

Compare with:

\[ f'(x) = \lim_{\epsilon \to 0} (f(x + \epsilon) - f(x)) \frac{1}{\epsilon} \]
And there are more methods:

Integration:
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Inverse Laplace: \[ \mathcal{L}^{-1}[f](x) = f(\partial_x) \delta(x) \]
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= \pi \lim_{x \to 0} \left( e^{\partial_x} - e^{-\partial_x} \right) \left( \Theta(x) + c \right)
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= \pi \lim_{x \to 0} \left( \Theta(x + 1) + c - \Theta(x - 1) - c \right)
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\[ = \pi \]
Examples for integration

Similarly, one quickly obtains, e.g.,

\[ \int_{-\infty}^{\infty} \frac{\sin^5(x)}{x} \, dx = \frac{3\pi}{8} \]

\[ \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} \, dx = \pi \]

\[ \int_{-\infty}^{\infty} \frac{(1 - \cos(tx))}{x^2} \, dx = \pi |t| \]

\[ \int_{-\infty}^{\infty} x^2 \cos(x) e^{-x^2} \, dx = \sqrt{\pi} e^{-1/4} / 4 \]

e tc ...
Examples for Fourier

Now how much harder is Fourier?
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Fourier transforming is even easier than integrating!
Examples for Fourier

Recall the new methods for integration and Fourier:

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How are they related?
The zero-frequency value of the Fourier transform is the integral (up to a prefactor of \( \sqrt{2\pi} \)).
Examples for Fourier

For example, for $f(x) = \sin(x)/x$, recall:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, dx = 2\pi \lim_{x \to 0} \frac{1}{2i} \left( e^{\partial_x} - e^{-\partial_x} \right) \frac{1}{-i\partial_x} \delta(x)$$

$$= \pi \lim_{x \to 0} \left( e^{\partial_x} - e^{-\partial_x} \right) (\Theta(x) + c')$$

$$= \pi \lim_{x \to 0} \left( \Theta(x + 1) - \Theta(x - 1) \right)$$

$$= \pi$$

By not taking the limit and by dividing by $\sqrt{2\pi}$, we obtain immediately:

$$\mathcal{F}[f](x) = \sqrt{\pi/2} \left( \Theta(x + 1) - \Theta(x - 1) \right)$$
Proof of the Fourier formula

Why does this work?
Proof of the Fourier formula

The claim is:

\[ \mathcal{F}[f](x) = \sqrt{2\pi} \ f(-i\partial_x) \ \delta(x) \]

Let us apply this to a plane wave: \( f(x) = e^{ixy} \).

We obtain the right answer:

\[
\mathcal{F}[f](x) = \sqrt{2\pi} \ e^{y\partial_x} \ \delta(x) \\
= \sqrt{2\pi} \ \delta(x + y)
\]

And the plane waves from a basis of the function space.
What is going on, intuitively?

\[
\int_{-\infty}^{\infty} f(x) \, dx = 2\pi \delta(i \partial_x) f(x)
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\[ \int_{-\infty}^{\infty} f(x) \, dx = 2\pi \delta(i\partial_x) \, f(x) \]

How does it work?

Regulate, e.g., this way:

\[
\delta(x) = \lim_{\sigma \to 0} \left( 2\pi\sigma \right)^{-1/2} e^{-x^2/2\sigma} \\
\int_{-\infty}^{\infty} f(x) \, dx = 2\pi \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma}} e^{\partial_x^2/2\sigma} f(x)
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What is going on, intuitively?

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Integration from asymptotics of heat flow!
Perturbative expansions

In QFT, we’d like to apply the new methods, e.g.:

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Inverse Laplace:
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\mathcal{L}^{-1}[f](x) = f(\partial_x) \delta(x)
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But what if in

\[ Z[J] = \int e^{iS[\phi] + i \int J \phi \, d^n x} D[\phi] \]

the action \( S[\phi] \) is not suitable to solve the integral or Fourier (or Laplace) transform with our new methods exactly?

And that’s the norm of course!
Perturbative expansions

• On the basic level, what if \( f(x) \) is too complicated, e.g., for:

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• Opportunity: Use any regularizations of \( \delta \) such as sinc or

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\delta_\sigma(x) = (2\pi \sigma)^{-1/2} e^{-x^2/2\sigma}
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• Also: applications to deblurring expansion of signals.
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• Obtain weak & strong coupling expansions and others...
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• What is the full size of the space of functions and distributions to which these methods apply?
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- Relation to Stoke’s theorem?

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- Relation to fermionic integration, a unifying formalism?

- A new perspective on integration measures and therefore anomalies in QFT?


**Bonus: Examples for Laplace**

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Examples for Laplace

If we apply the new Laplace transform method

\[ \mathcal{L}[f](x) = f(-\partial_x) \frac{1}{x} \]

to monomials \( f(x) = x^n \) we obtain:

\[ L[f](x) = (-\partial_x)^n \frac{1}{x} = \frac{n!}{x^{n+1}} \]

And the monomials form a basis in the function space.
Example for inverse Laplace

Consider a heat kernel trace:

\[ h(t) = \sum_{n} e^{-\lambda_n t} \]

Given \( h(t) \), the spectrum \( \{\lambda_n\} \) is known to be recoverable via inverse Laplace transform.

Why?
Example for inverse Laplace

Consider a heat kernel trace:

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Given $h(t)$, the spectrum $\{\lambda_n\}$ is known to be recoverable via inverse Laplace transform.

Why? Using the new inverse Laplace transform method, namely

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this is easy to see:
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\[ \mathcal{L}^{-1}[h](\lambda) = h(\partial_\lambda) \delta(\lambda) = \sum_{n} e^{-\lambda_n \partial_\lambda} \delta(\lambda) = \sum_{n} \delta(\lambda - \lambda_n) \]