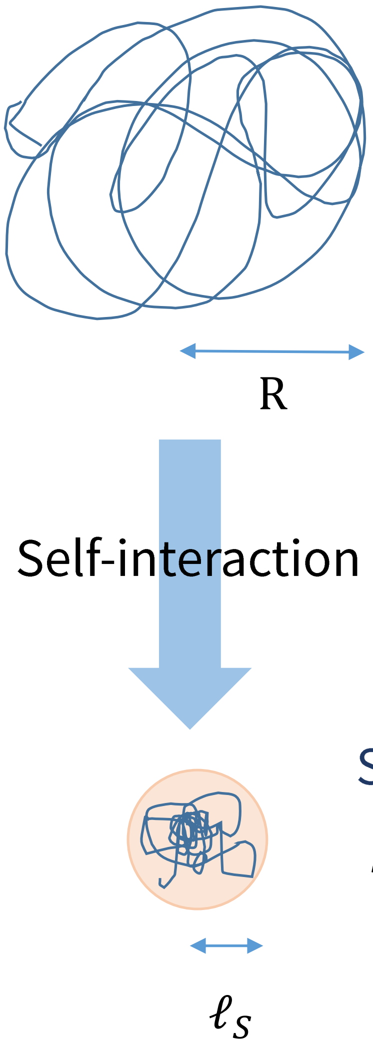
23 July 2015 at Karl Schwarzschild Meeting 2015, Frankfurt Institute for Advanced Studies

Size scaling of self-gravitating polymers and strings

Shoichi Kawamoto
(Chung Yuan Christian University, Taiwan)

arXiv:1506.01160 [hep-th] with Toshihiro Matsuo (NIT, Anan College, Japan)

Long Strings and Black Holes



Mean-size squared of a free string of level $\tilde{N}$

$$\langle R^2\rangle_{\tilde{N}} \sim \int d\sigma \langle :X^2(0,\sigma): \rangle_{\tilde{N}} \sim \sqrt{\tilde{N}}\alpha' \sim \mathrm{Mass}$$
$$\sim \mathrm{Length}$$

Free random walk of $N = \sqrt{\tilde{N}}$ steps [Mitchell-Turok, Mañes]

String/black hole correspondence

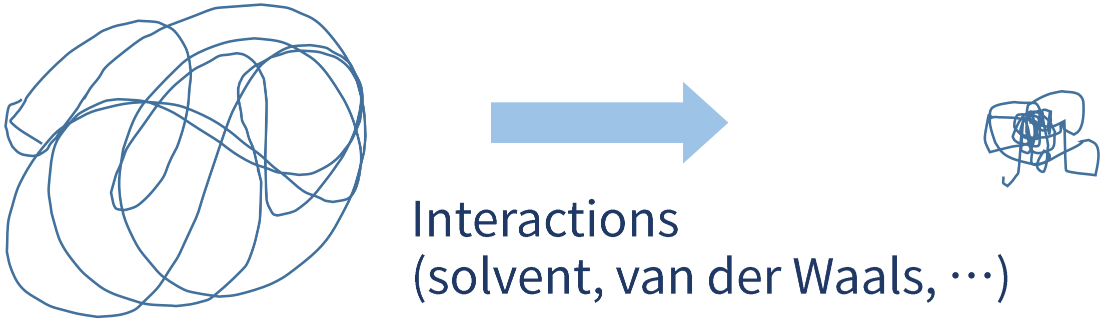
[Susskind, Horowitz-Polchinski]

Small black hole

$$S_{\mathrm{string}} \cong S_{\mathrm{BH}}$$

$$R \cong \frac{\ell_s}{g_s^2\sqrt{\tilde{N}}} \qquad (d = 3)$$

Random walks and Polymers



Interactions
(solvent, van der Waals, ⋯)

Coil$\sim N^{\frac{3}{5}}$

Θ-point

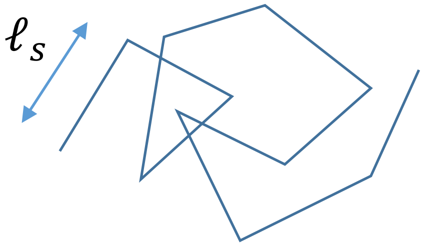
Globule$\sim N^{\frac{1}{3}}$

Many (self-)interacting 1D objects

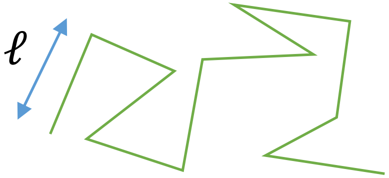
- Cosmic strings
- Vortex lines
- Polymer chains
- ⋯

We analyze the size scaling of string in terms of self-interacting polymers

Strings as self-avoiding walks



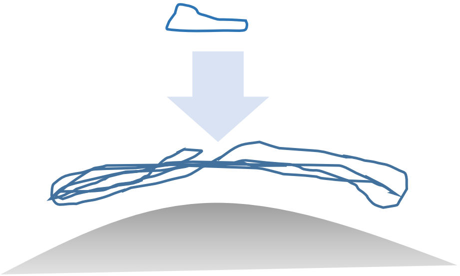
Fundamental strings are modelled by free (Gaussian) random walks with the bond length ℓ_s.



(Real) polymers are modelled by self-avoiding random walks with the (Kuhn) bond length ℓ.

→ $R \cong \ell N^{\frac{3}{d+2}}$ (Flory's exponent for real polymers)

Note: Repulsive property may emerge in high-density regime nonperturbatively, to explain an exponentially fast spreading of a string.



$L \propto e^t$

$R_T \propto t$ → $R_T \propto e^t$

[Susskind-Lindesay, Ropotenko]

Effective Hamiltonian

Edwards Hamiltonian

[Edwards-Muthukumar, Doi-Edwards]

$$\beta H = \frac{d}{2\ell^2}\int_0^N \left(\frac{\partial \boldsymbol{R}}{\partial \sigma}\right)^2 d\sigma \; + \int_0^N d\sigma_1 \int_0^N d\sigma_2 \, V\big(\boldsymbol{R}(\sigma_1), \boldsymbol{R}(\sigma_2)\big)$$

Interaction term

$$V = \underbrace{\frac{-g^2\,\ell^{d-2}}{|\boldsymbol{R}(\sigma_1) - \boldsymbol{R}(\sigma_2)|^{d-2}}}_{\text{Newton Interaction}} + \underbrace{u\,\ell^d \delta^{(d)}(\boldsymbol{R}(\sigma_1) - \boldsymbol{R}(\sigma_2))}_{\text{Repulsive force (Self-Avoiding effect)}}$$

Evaluate the size

$$\langle \boldsymbol{R}^2 \rangle_{V=0} = \ell^2 N \equiv R_0^2$$

Free random walk: $R_0 = \ell\sqrt{N}$

Consider the size scaling of self-gravitating self-avoiding walks

Phenomenological Free Energy:

$$\beta F \sim \underbrace{\underbrace{-\,(d-1)\ln R}_{\textbf{diffusion}} + \underbrace{\frac{R^2}{N\ell^2}}_{\textbf{elasticity}}}_{\textbf{Entropic force}} \underbrace{-\,\frac{g^2\ell^{d-2}N^2}{R^{d-2}}}_{\textbf{gravity}} + \underbrace{\frac{u\ell^d N^2}{R^d}}_{\substack{\textbf{repulsive}\\ \textbf{(excluded-volume effect)}}}$$

Free case: $g = u = 0$ $\Longrightarrow$ $R_0 = \ell\sqrt{N}$

Interaction becomes effective in free configurations:

$$\frac{g^2\ell^{d-2}N^2}{R_0^{d-2}} \sim O(1) \qquad \Longrightarrow \qquad g_o \sim N^{\frac{d-6}{4}}$$

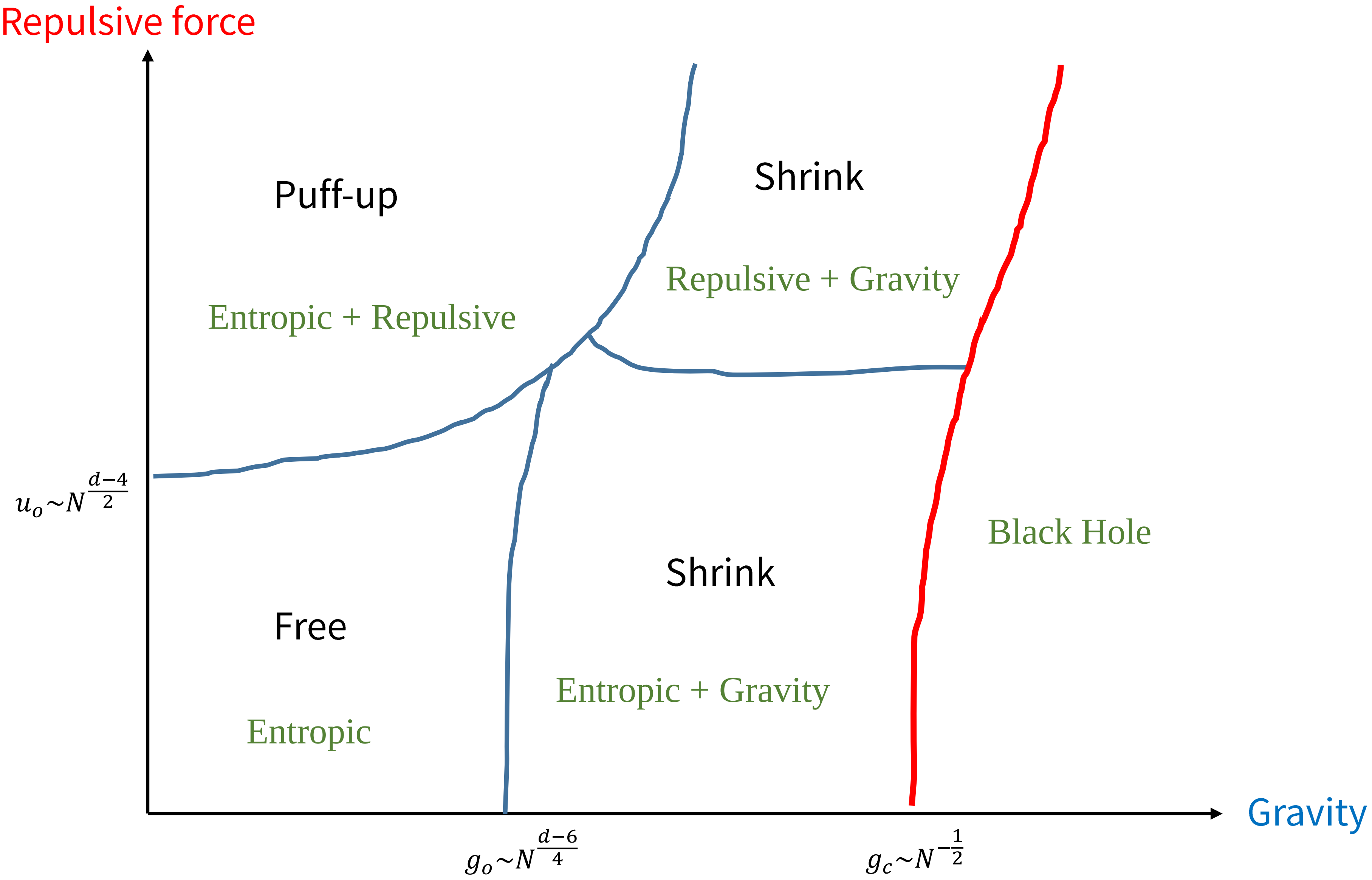
[Horowitz-Polchinski, Khuri]

$$\frac{u\ell^d N^2}{R_0^d} \sim O(1) \qquad \Longrightarrow \qquad u_o \sim N^{\frac{d-4}{2}}$$

($d = 4$ is critical dim.)

Gravity gets stronger: Schwarzschild Radius: $R_s \cong \ell(g^2N)^{\frac{1}{d-2}}$

Today's Goal


Repulsive force
Puff-up
Entropic + Repulsive
Shrink
Repulsive + Gravity
$u_o \sim N^{\frac{d-4}{2}}$
Free
Entropic
Shrink
Entropic + Gravity
Black Hole
$g_o \sim N^{\frac{d-6}{4}}$
$g_c \sim N^{-\frac{1}{2}}$
Gravity

PLAN

- Introduction
- Two Approximation Methods and Size Evaluation
- Summary of Size Scaling
- Conclusion

[Doi-Edwards]

Variational Method

Trial Hamiltonian: $$\beta H_0 = \frac{d}{2\ell^2}\int_0^N d\sigma\left(\frac{\partial \boldsymbol{R}}{\partial \sigma}\right)^2 + \frac{q^2 d}{2\ell^2}\int_0^N d\sigma\, \boldsymbol{R}(\sigma)^2$$

Convexity: $e^{-\beta F} = \int d\boldsymbol{R}\, e^{-\beta H} = \int d\boldsymbol{R}\, e^{-\beta H_0} e^{-\beta(H-H_0)} \geq \mathrm{e}^{\langle -\beta(H-H_0)\rangle_0} \mathrm{e}^{-\beta F_0}$

$$\beta F \leq \beta F_0(q) + \langle \beta(H - H_0)\rangle_0$$

Tune q to minimize RHS

$$\langle R^2\rangle_0 = \frac{\ell^2}{q_0}\tanh q_0 N \begin{cases} \ell^2 N\left(1 + O(q_0^2 N^2)\right) & (q_0 N \ll 1) \\ \dfrac{\ell^2}{q_0} & (q_0 N \geq O(1)) \end{cases}$$

Note: No expansion (see later)

Propagator:

$$G_0(\sigma,\sigma') = \left(\frac{qd}{2\pi\ell^2\ \sinh q|\sigma-\sigma'|}\right)^{\frac{\mathrm{d}}{2}} \exp\left(-\frac{qd\ [\boldsymbol{R}(\sigma)^2+\boldsymbol{R}(\sigma')^2]\cosh q|\sigma-\sigma'| - 2\boldsymbol{R}(\sigma)\cdot\boldsymbol{R}(\sigma')}{2\ell^2\sinh q|\sigma-\sigma'|}\right)$$

Evaluate $\beta F_0 = -\log Z_0$ and $\langle\beta(H-H_0)\rangle_0 = \left\langle \int_0^N d\sigma \int_0^N d\sigma'\, V - \frac{q^2 d}{2\ell^2}\int_0^N d\sigma\, \boldsymbol{R}(\sigma)^2 \right\rangle$

$$\beta F \le \frac{d}{2}\ln(\cosh qN) - \frac{qdN}{4}\tanh qN$$

$$-2\int_0^N d\sigma' \int_0^{\sigma'} d\sigma \left[\frac{g^2}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{qd}{2F_1(\sigma,\sigma';q)}\right)^{\frac{d-2}{2}} - u\left(\frac{qd}{2F_2(\sigma,\sigma';q)}\right)^{\frac{d}{2}}\right]$$

$$F_1(\sigma,\sigma';q) = \frac{\sinh q\sigma\cosh q(N-\sigma) + \sinh q\sigma'\cosh q(N-\sigma') - 2\sinh q\sigma\cosh q(N-\sigma')}{\cosh qN}$$

$$F_2(\sigma,\sigma';q) = \frac{\sinh q\sigma\sinh q(\sigma'-\sigma)}{\sinh q\sigma'} + \frac{\cosh q(N-\sigma')\sinh q\sigma'}{\cosh qN}\left(1-\frac{\sinh q\sigma}{\sinh q\sigma'}\right)^2$$

Simplification (an approximation): e^{-qN} $e^{-q(N-\sigma')}$ $e^{-q(N-\sigma)}$ $e^{-q\sigma}$ $e^{-q\sigma'}$ $e^{-q(\sigma'-\sigma)}$ $\ll 1$

Bound:

$$\beta F \leq qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$$

Stationary Cond.:

$$0 = 1 - N g^2 q_0{}^{\frac{d-4}{2}} + N u q_0{}^{\frac{d-2}{2}}$$

Omit (positive N-independent) numerical factors

$$\langle \boldsymbol{R}^2 \rangle_0 = \frac{\ell^2}{q_0} \tanh q_0 N \begin{cases} \ell^2 N & (q_0 N \ll 1) \\ \dfrac{\ell^2}{q_0} & (q_0 N \geq O(1)) \end{cases}$$

(Free walk size)

(shrink)

$$0 = 1 - N g^2 q_0^{\frac{d-4}{2}} + N u q_0^{\frac{d-2}{2}}$$

$(2 < d < 4)$

$$\langle \boldsymbol{R}^2 \rangle_0 = \frac{\ell^2}{q_0} \tanh q_0 N \begin{cases} \ell^2 N & (q_0 N \ll 1) \\ \dfrac{\ell^2}{q_0} & (q_0 N \geq O(1)) \end{cases}$$

1. No interaction $g = u = 0$:

Pure repulsive $g = 0,\ u > 0$:

$q_0 = 0$ → $R = R_0 = \ell\sqrt{N}$

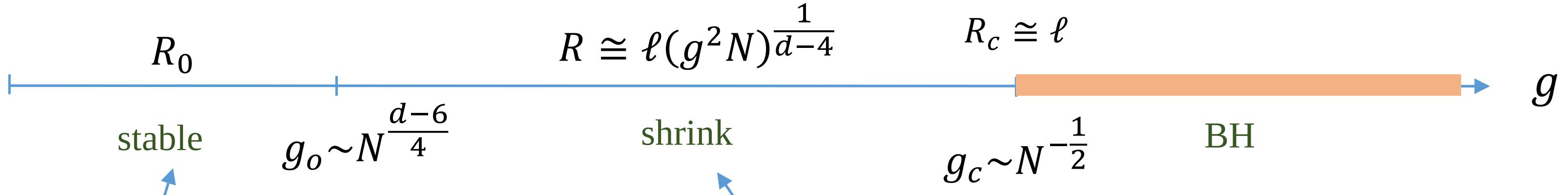
No expansion

2. Generic Case $g > 0, u > 0$: $u < N^{-1}$ (the repulsive force is too weak)

$q_0 = 0$

$q_0 \cong (g^2 N)^{\frac{2}{4-d}}$

Schwarzschild Radius: $R_s \cong \ell (g^2 N)^{\frac{1}{d-2}}$



R_0

$R \cong \ell (g^2 N)^{\frac{1}{d-4}}$

$R_c \cong \ell$

g

stable

$g_o \sim N^{\frac{d-6}{4}}$

shrink

$g_c \sim N^{-\frac{1}{2}}$

BH

$\beta F \leq qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$

$0 = 1 - N g^2 q_0^{\frac{d-4}{2}} + N u q_0^{\frac{d-2}{2}}$

The same as no repulsive force case $(u = 0)$

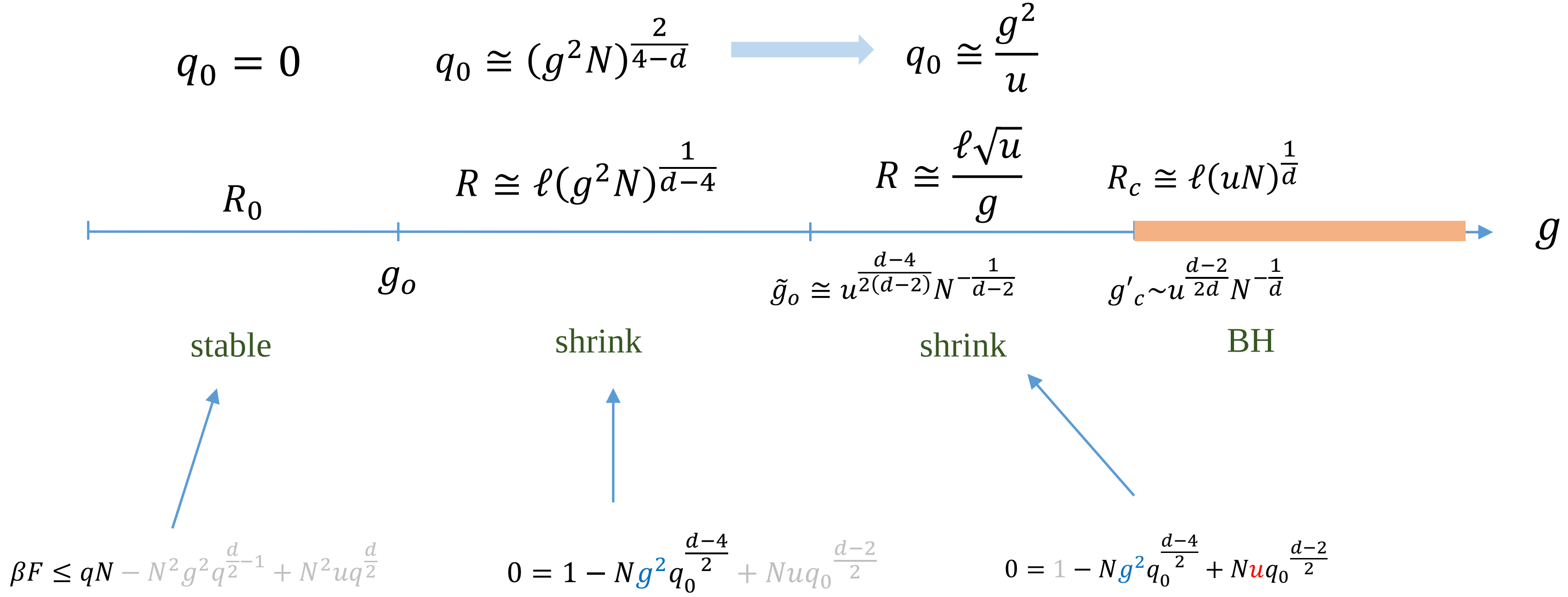
[Horowitz-Polchinski, Khuri]

$$0 = 1 - Ng^2 q_0^{\frac{d-4}{2}} + Nuq_0^{\frac{d-2}{2}}$$

$(2 < d < 4)$

$$\langle \boldsymbol{R}^2 \rangle_0 = \frac{\ell^2}{q_0} \tanh q_0 N \begin{cases} \ell^2 N & (q_0 N \ll 1) \\ \dfrac{\ell^2}{q_0} & (q_0 N \geq O(1)) \end{cases}$$

2. Generic Case $g > 0, u > 0:$ $N^{-1} < u < u_o$



$q_0 = 0$

$q_0 \cong (g^2 N)^{\frac{2}{4-d}}$ $\longrightarrow$ $q_0 \cong \dfrac{g^2}{u}$

R_0

$R \cong \ell (g^2 N)^{\frac{1}{d-4}}$

$R \cong \dfrac{\ell\sqrt{u}}{g}$

$R_c \cong \ell (uN)^{\frac{1}{d}}$

g

g_o

$\tilde{g}_o \cong u^{\frac{d-4}{2(d-2)}} N^{-\frac{1}{d-2}}$

$g'_c \sim u^{\frac{d-2}{2d}} N^{-\frac{1}{d}}$

stable

shrink

shrink

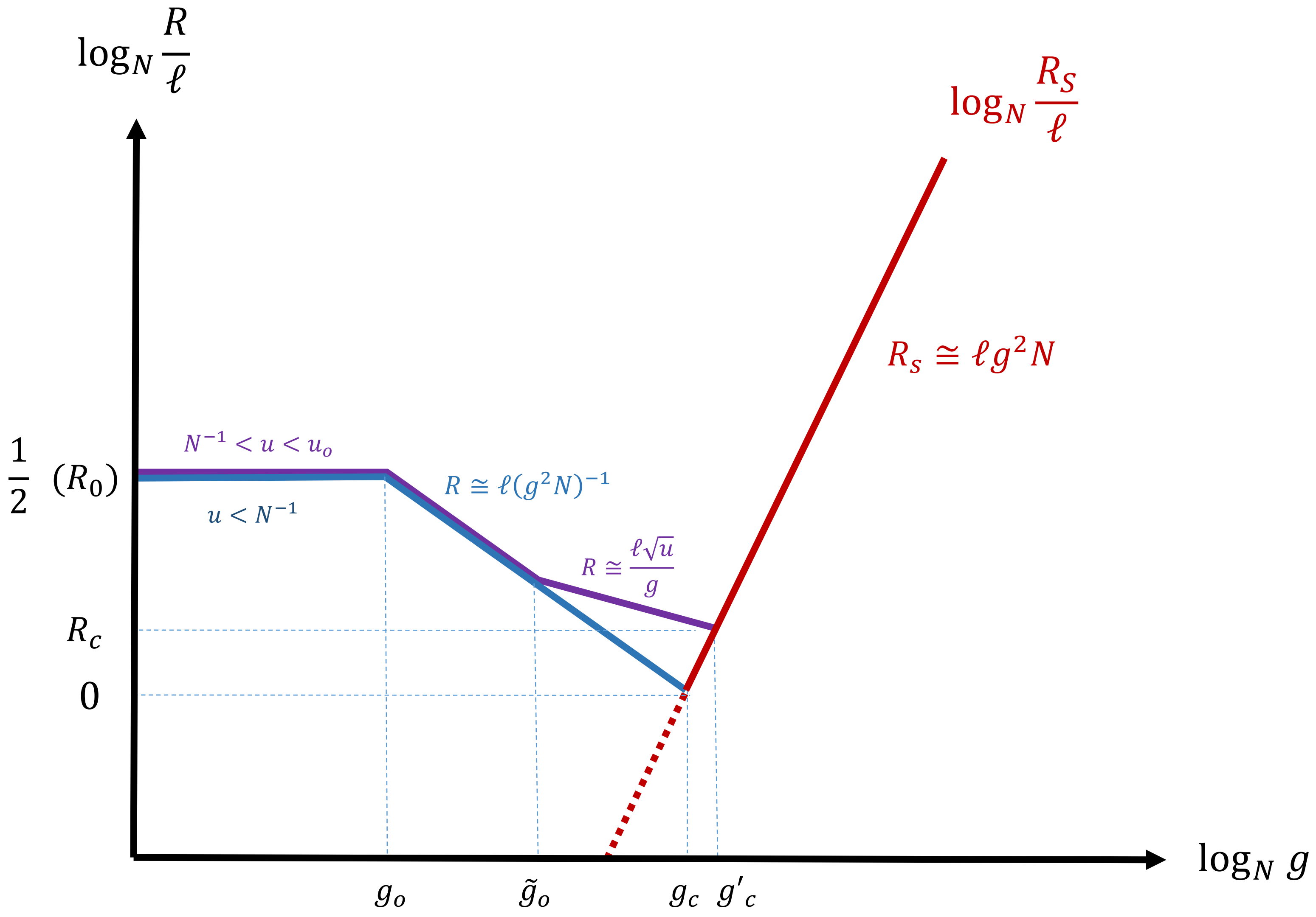
BH

$\beta F \leq qN - N^2 g^2 q^{\frac{d}{2}-1} + N^2 u q^{\frac{d}{2}}$

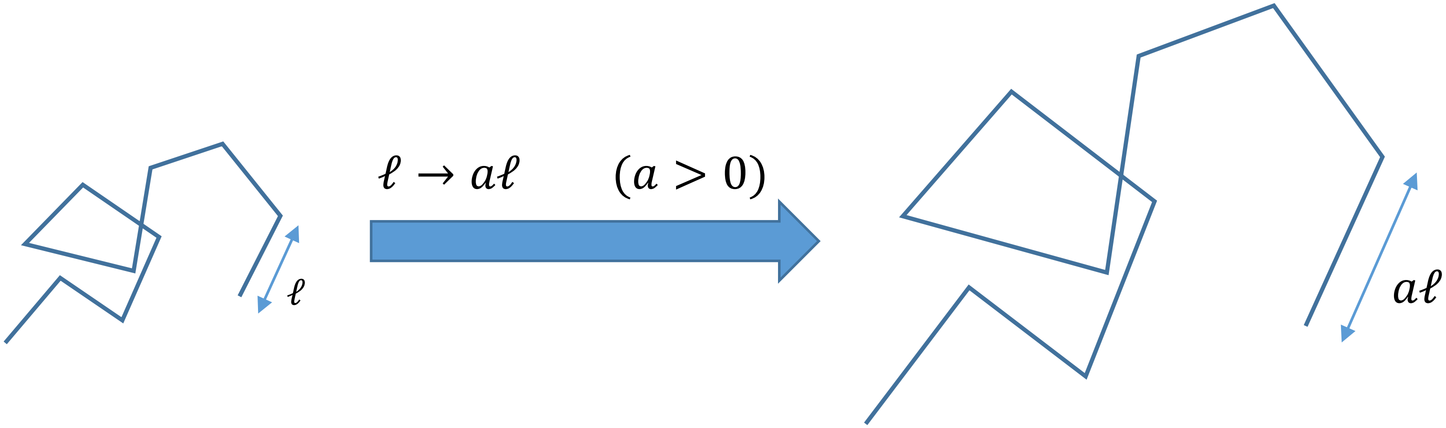
$0 = 1 - Ng^2 q_0^{\frac{d-4}{2}} + Nuq_0^{\frac{d-2}{2}}$

$0 = 1 - Ng^2 q_0^{\frac{d-4}{2}} + Nuq_0^{\frac{d-2}{2}}$

Size change ($d = 3$)



Uniform Expansion Model (UEM)



- Bond length is rescaled
- The configuration remains free walk one.

$$R = aR_0 = a\ell\sqrt{N}$$

Free (Gaussian) Hamiltonian with the bond length $a\ell$

$$\beta H' = \frac{d}{2a^2\ell^2}\int_0^N d\sigma \left(\frac{\partial \boldsymbol{R}}{\partial \sigma}\right)^2$$

Propagator: $G'(\sigma, \sigma') = \left(\frac{d}{2\pi a^2\ell^2\,|\sigma - \sigma'|}\right)^{\frac{\mathrm{d}}{2}} \exp\left(-\frac{d}{2a^2\ell^2|\sigma - \sigma'|}\left(\boldsymbol{R}(\sigma) - \boldsymbol{R}(\sigma')\right)^2\right)$

Calculate the mean-size-squared:

$$\langle A\rangle' \equiv \frac{1}{Z'}\int A e^{-\beta H'}$$

$$\langle \boldsymbol{R}^2\rangle = \frac{\int \left(\boldsymbol{R}(N)-\boldsymbol{R}(0)\right)^2 e^{-\beta H}}{\int e^{-\beta H}} = \frac{\left\langle e^{-\beta(H-H')}\left(\boldsymbol{R}(N)-\boldsymbol{R}(0)\right)^2\right\rangle'}{\left\langle e^{-\beta(H-H')}\right\rangle'}$$

$$\cong \left\langle\left(\boldsymbol{R}(N)-\boldsymbol{R}(0)\right)^2\right\rangle' \left(1+\langle\beta(H-H')\rangle'\right) - \left\langle\beta(H-H')\left(\boldsymbol{R}(N)-\boldsymbol{R}(0)\right)^2\right\rangle'$$

$$+O([\beta(H-H')]^2)$$

$$\beta(H-H') = \frac{d}{2\ell^2}\left(1-\frac{1}{a^2}\right)\int_0^N d\sigma\left(\frac{\partial\boldsymbol{R}}{\partial\sigma}\right)^2 + \int_0^N d\sigma_1\int_0^N d\sigma_2\, V\left(\boldsymbol{R}(\sigma_1),\boldsymbol{R}(\sigma_2)\right)$$

and Gaussian integrals by use of the propagator

$$\cong \underbrace{Na^2\ell^2}_{\text{Required Size}} + \underbrace{\left[a^d(1-a^2) + C_1 u N^{\frac{4-d}{2}} - C_2 g^2 N^{\frac{6-d}{2}} a^2\right]}_{=0} N\ell^2 a^{2-d}$$

C_1, C_2: Positive N independent constants

$$a^d - a^{d+2} + uN^{\frac{4-d}{2}} - g^2 N^{\frac{6-d}{2}} a^2 = 0$$

$$R = \ell\, a\sqrt{N}$$

0. No Interaction: $g^2 = u = 0$ $\quad a = 1 \quad$ $R = \ell\sqrt{N} = R_0$ (free walk)

1. Pure repulsive: $g^2 = 0, \;\; u > 0$

$a \cong 1$

$a^d - a^{d+2} + uN^{\frac{4-d}{2}} = 0$

$a = u^{\frac{1}{d+2}} N^{\frac{4-d}{2(d+2)}}$

$a^d - a^{d+2} + uN^{\frac{4-d}{2}} = 0$

$u \cong N^0 \longrightarrow R \cong \ell N^{\frac{3}{d+2}}$ Flory’s exponent for real polymers

Good description for expanded configurations

$$a^d - a^{d+2} + uN^{\frac{4-d}{2}} - g^2 N^{\frac{6-d}{2}} a^2 = 0$$

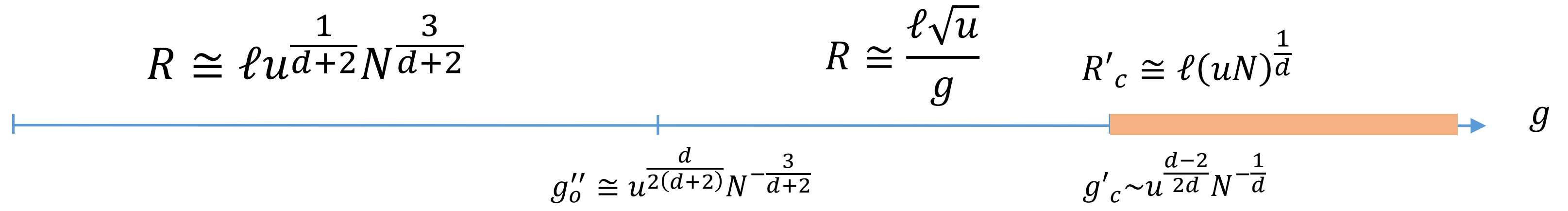
$$R = \ell\, a\sqrt{N}$$

2. Pure gravity: $g > 0, u = 0$

When gravity is effective ($g \geq g_o \sim N^{\frac{d-6}{4}}$), no solution of $a < 1$ is found.

→ Approximation is not good.

3. Generic Case: $g, u > 0$



$u > u_o$

$R \cong \ell u^{\frac{1}{d+2}} N^{\frac{3}{d+2}}$

$R \cong \frac{\ell\sqrt{u}}{g}$

$R'_c \cong \ell (uN)^{\frac{1}{d}}$

g

$g_o'' \cong u^{\frac{d}{2(d+2)}} N^{-\frac{3}{d+2}}$

$g'_c \sim u^{\frac{d-2}{2d}} N^{-\frac{1}{d}}$

Puff-up

shrink

BH

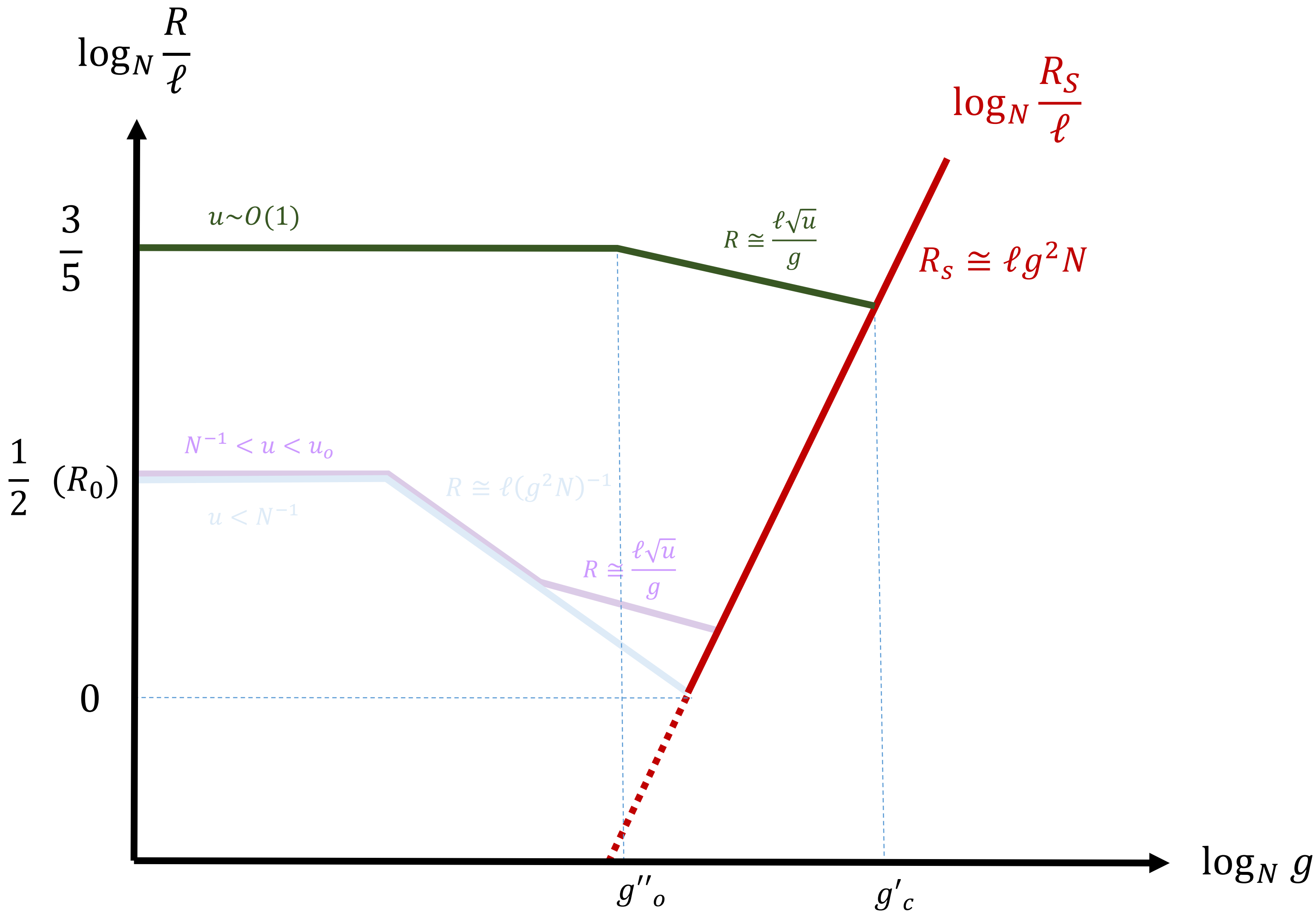
$$a = u^{\frac{1}{d+2}} N^{\frac{4-d}{2(d+2)}}$$

$$a \cong \sqrt{\frac{u}{g^2 N}}$$

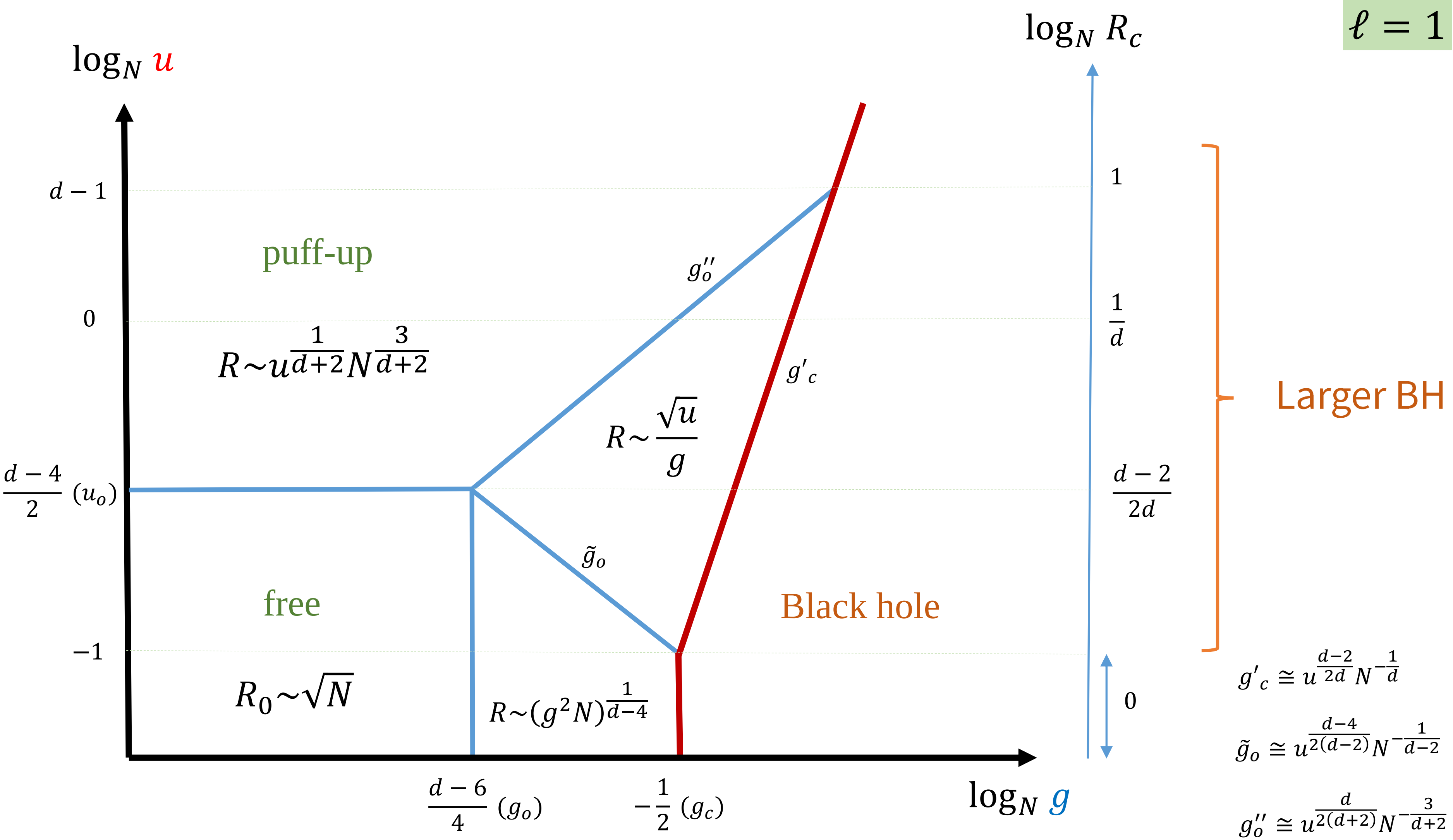
$$a^d - a^{d+2} + uN^{\frac{4-d}{2}} - g^2 N^{\frac{6-d}{2}} a^2 = 0$$

$$a^d - a^{d+2} + uN^{\frac{4-d}{2}} - g^2 N^{\frac{6-d}{2}} a^2 = 0$$

Size change ($d = 3$)



The Size Scaling ($2 < d < 4$)



$$g'_c \cong u^{\frac{d-2}{2d}} N^{-\frac{1}{d}}$$

$$\tilde{g}_o \cong u^{\frac{d-4}{2(d-2)}} N^{-\frac{1}{d-2}}$$

$$g''_o \cong u^{\frac{d}{2(d+2)}} N^{-\frac{3}{d+2}}$$

Summary

- Self-gravitating polymers (self-avoiding walks)

→ Collapse to a larger size black hole

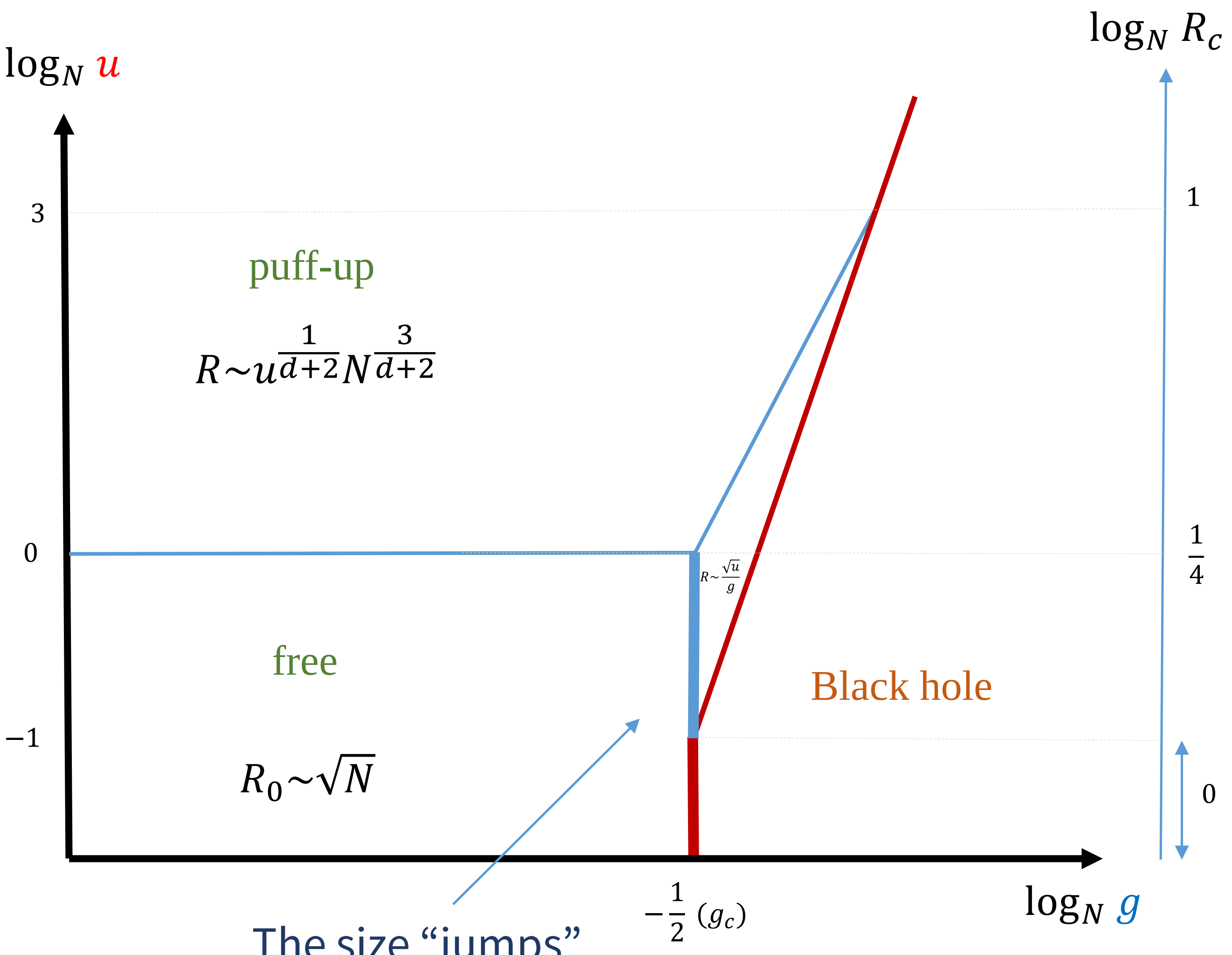
- Interesting size scaling behaviors are observed.

Next:

- Density distribution, elasticity (pressure), detailed gravitational collapse
  (Need GR ? -- Tolman-Oppenheimer-Volkoff eq.)
- Possible source of self-avoinding property
  Fermionic walk? Higher form field exchange?
- Beyond a mean field calculation. RG analysis
- Entropy? Corresponding point?

The Size Scaling ($d = 4$)

$\ell = 1$



The size “jumps”