General Relativity from a Canonical Transformation Formalism

J. Struckmeier[1,2], D. Vasak[3], J. Kirsch[3], H. Stöcker[1,2,3]

[1]GSI Helmholtzzentrum für Schwerionenforschung GmbH, Darmstadt, Germany
j.struckmeier@gsi.de, web-docs.gsi.de/~struck

[2]Fachbereich Physik, Goethe Universität, Frankfurt am Main, Germany

[3]Fankfurt Institute for Advanced Studies (FIAS), Frankfurt am Main, Germany

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Outline

1. Classical field theory with variable space-time
2. Extended Lagrangians in field theory
   - Example: Einstein-Hilbert Lagrangian
3. Extended covariant Hamiltonians in field theory
4. Extended canonical transformations
5. General Relativity as an extended canonical gauge theory
6. Conclusions and Outlook

The talk is based on a paper published in Phys. Rev. D **91**, 085030 (2015)

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Rationale

General Relativity should obey the following principles:

1. Action Principle: The fundamental laws of nature should follow from action principles.
2. General Principle of Relativity: The form of the action principle — and hence the resulting field equations — should be the same in any frame of reference.
3. ⇝ The change of reference frame must constitute an extended canonical transformation, which by construction maintains the form of the action principle.
4. For a system of tensor fields, the affine connection coefficients $\Gamma^{\alpha}{}_{\mu\nu}$ turn out to be the relevant gauge quantities.
5. This confirms Einstein's conclusion: "…the essential achievement of general relativity is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the 'displacement field' $\Gamma^{\alpha}{}_{\mu\nu}$ …."

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Extended action principle, extended Lagrangian

Generalized action functional for dynamical space-time: treat $\partial x^\nu / \partial y^\mu$ as dynamical variable in the Lagrangian $\mathcal{L}$

Extended action principle

$$S = \int_{R'} \mathcal{L}\left(a_\mu, \frac{\partial a_\mu}{\partial x^\nu}\right) \det\Lambda \, \mathrm{d}^4 y, \quad \delta S \stackrel{!}{=} 0, \quad \delta a_\mu\big|_{\partial R'} = \delta x^\mu\big|_{\partial R'} \stackrel{!}{=} 0$$

with y^μ the new set of independent variables and $x^\nu = x^\nu(y)$

$$\Lambda = \begin{pmatrix} \frac{\partial x^0}{\partial y^0} & \cdots & \frac{\partial x^0}{\partial y^3} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^3}{\partial y^0} & \cdots & \frac{\partial x^3}{\partial y^3} \end{pmatrix}, \qquad \det\Lambda = \frac{\partial(x^0, \ldots, x^3)}{\partial(y^0, \ldots, y^3)} \neq 0.$$

The integrand defines the extended Lagrangian $\mathcal{L}_\mathrm{e} = \mathcal{L} \det\Lambda$

$$\mathcal{L}_\mathrm{e}\left(a_\mu(y), \frac{\partial a_\mu(y)}{\partial y^\nu}, \frac{\partial x^\mu(y)}{\partial y^\nu}\right) = \mathcal{L}\left(a_\mu(y), \frac{\partial y^\alpha}{\partial x^\nu}\frac{\partial a_\mu(y)}{\partial y^\alpha}\right) \det\Lambda$$

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Example: Einstein-Hilbert Lagrangian

The Einstein equations follow from the extended Lagrangian

$$\mathcal{L}_{\mathrm{e,EH}} = (\mathcal{L}_R + \mathcal{L}_{\mathrm{M}}) \det \Lambda, \qquad \mathcal{L}_R = \frac{R}{2\kappa} = \frac{1}{2\kappa} g^{\mu\nu} R_{\mu\nu},$$

wherein $R = g^{\mu\nu} R_{\mu\nu}$ denotes the Riemann curvature scalar, κ [Lenght]^2 a coupling constant, and $\mathcal{L}_{\mathrm{M}}$ the conventional Lagrangian of a given system.

The Ricci tensor $R_{\mu\nu} = R^{\eta}{}_{\mu\eta\nu}$ is the contraction $\eta = \beta$ of the

Riemann-Christoffel curvature tensor

$$R^{\eta}{}_{\mu\beta\nu} = \frac{\partial \Gamma^{\eta}{}_{\mu\nu}}{\partial y^{\beta}} - \frac{\partial \Gamma^{\eta}{}_{\mu\beta}}{\partial y^{\nu}} + \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\eta}{}_{\lambda\beta} - \Gamma^{\lambda}{}_{\mu\beta}\Gamma^{\eta}{}_{\lambda\nu}.$$

In the Palatini approach, the metric and the connection coefficients are *a priori* independent quantities, hence the Euler-Lagrange equations are here

$$\frac{\partial \mathcal{L}_{\mathrm{e,EH}}}{\partial \left(\frac{\partial x_{\nu}}{\partial y^{\mu}}\right)} = 0, \qquad \frac{\partial}{\partial y^{\beta}} \frac{\partial \mathcal{L}_{\mathrm{e,EH}}}{\partial \left(\frac{\partial \Gamma^{\eta}{}_{\alpha\xi}}{\partial y^{\beta}}\right)} - \frac{\partial \mathcal{L}_{\mathrm{e,EH}}}{\partial \Gamma^{\eta}{}_{\alpha\xi}} = 0.$$

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Extended covariant Hamiltonian

We define the $p^{\mu\nu}$ and the tensor densities $\tilde{p}^{\mu\nu} = p^{\mu\nu} \det \Lambda$ as the dual quantities of the derivatives of the fields according to

$$p^{\mu\nu}(x) = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial a_\mu(x)}{\partial x^\nu} \right)}, \qquad \tilde{p}^{\mu\nu}(y) = \frac{\partial \mathcal{L}_\mathrm{e}}{\partial \left(\frac{\partial a_\mu(y)}{\partial y^\mu} \right)}.$$

Similarly, the two-point tensor $\tilde{t}_\nu{}^\mu$ defines the dual quantity to $\partial x^\nu / \partial y^\mu$

$$\tilde{t}_\nu{}^\mu = -\frac{\partial \mathcal{L}_\mathrm{e}}{\partial \left(\frac{\partial x^\nu}{\partial y^\mu} \right)}$$

An extended Lagrangian $\mathcal{L}_\mathrm{e} = \mathcal{L} \det \Lambda$ is thus Legendre-transformed to the

Extended Hamiltonian $\mathcal{H}_\mathrm{e}$

$$\mathcal{H} = p^{\beta\alpha} \frac{\partial a_\beta}{\partial x^\alpha} - \mathcal{L}, \qquad \mathcal{H}_\mathrm{e} = \mathcal{H} \det \Lambda - \tilde{t}_\alpha{}^\beta \frac{\partial x^\alpha}{\partial y^\beta}.$$

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Form-invariance for the extended action principle

The extended action principle must be maintained for extended canonical transformations that map $a_\mu \mapsto A_\mu$, $\tilde{p}^{\mu\nu} \mapsto \tilde{P}^{\mu\nu}$, $x^\mu \mapsto X^\mu$, $\tilde{t}_\nu{}^\mu \mapsto \tilde{T}_\nu{}^\mu$

Condition for extended canonical transformations

$$\delta \int_{R'} \left[\tilde{p}^{\beta\alpha} \frac{\partial a_\beta}{\partial y^\alpha} - \tilde{t}_\beta{}^\alpha \frac{\partial x^\beta}{\partial y^\alpha} - \mathcal{H}_\mathrm{e} \right] \mathrm{d}^4 y$$
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This condition implies that the integrands may differ by the divergence of a vector field $\mathcal{F}_1^\mu$ with $\delta \mathcal{F}_1^\mu |_{\partial R'} = 0$

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$\mathcal{F}_1^\alpha$ may be defined to depend on a_β, A_β, x^ν, and X^ν only.
$\rightsquigarrow$ This defines the extended generating function of type $\mathcal{F}_1^\alpha$.

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Transformation rules for a generating function $\mathcal{F}_1^\mu$

The divergence of a vector function $\mathcal{F}_1^\alpha(a_\beta, A_\beta, x^\nu, X^\nu)$ is

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$$\tilde{p}^{\beta\mu} = \frac{\partial \mathcal{F}_1^\mu}{\partial a_\beta}, \quad \tilde{P}^{\beta\mu} = -\frac{\partial \mathcal{F}_1^\mu}{\partial A_\beta}, \quad \tilde{t}_\nu{}^\mu = -\frac{\partial \mathcal{F}_1^\mu}{\partial x^\nu}, \quad \tilde{T}_\nu{}^\mu = \frac{\partial \mathcal{F}_1^\mu}{\partial X^\nu}, \quad \mathcal{H}_\mathrm{e}' = \mathcal{H}_\mathrm{e}.$$

The transformation rule for the extended Hamiltonian translates into the following rule for the given covariant Hamiltonian $\mathcal{H}$

$$\mathcal{H}' \det\Lambda' - \tilde{T}_\alpha{}^\beta \frac{\partial X^\alpha}{\partial y^\beta} = \mathcal{H}\det\Lambda - \tilde{t}_\alpha{}^\beta \frac{\partial x^\alpha}{\partial y^\beta}.$$

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Extended generating function of type $\mathcal{F}_2^{\mu}$

By means of a Legendre transformation

$$\mathcal{F}_2^{\alpha}(a_\beta, \tilde{P}^{\beta\nu}, x^\nu, \tilde{T}_\nu{}^{\mu}) = \mathcal{F}_1^{\alpha}(a_\beta, A_\beta, x^\nu, X^\nu) + A_\beta \tilde{P}^{\beta\alpha} - X^\beta \tilde{T}_\beta{}^{\alpha},$$

an equivalent set of transformation rules is encountered, hence the

Rules for an extended generating function $\mathcal{F}_2^{\mu}$

$$\tilde{p}^{\beta\mu} = \frac{\partial \mathcal{F}_2^{\mu}}{\partial a_\beta}, \quad A_\beta \delta_\nu^\mu = \frac{\partial \mathcal{F}_2^{\mu}}{\partial \tilde{P}^{\beta\nu}}, \quad \tilde{t}_\nu{}^{\mu} = -\frac{\partial \mathcal{F}_2^{\mu}}{\partial x^\nu}, \quad X^\alpha \delta_\nu^\mu = -\frac{\partial \mathcal{F}_2^{\mu}}{\partial \tilde{T}_\alpha{}^{\nu}}, \quad \mathcal{H}_e' = \mathcal{H}_e$$

There are 6 symmetry relations for $\mathcal{F}_2^{\mu}$ of the type

$$\frac{\partial \tilde{p}^{\beta\mu}}{\partial \tilde{P}^{\alpha\nu}} = \frac{\partial^2 \mathcal{F}_2^{\mu}}{\partial a_\beta \partial \tilde{P}^{\alpha\nu}} = \delta_\nu^\mu \frac{\partial A_\alpha}{\partial a_\beta}, \qquad \frac{\partial \tilde{t}_\beta{}^{\mu}}{\partial \tilde{T}_\alpha{}^{\nu}} = -\frac{\partial^2 \mathcal{F}_2^{\mu}}{\partial x^\beta \partial \tilde{T}_\alpha{}^{\nu}} = \delta_\nu^\mu \frac{\partial X^\alpha}{\partial x^\beta}.$$

$\rightsquigarrow$ Note that all quantities in the derivations must refer to the same space-time event in order to be well-defined.

Extended generating function of type $\mathcal{F}_2^{\mu}$

By means of a Legendre transformation

$$\mathcal{F}_2^{\alpha}(a_\beta, \tilde{P}^{\beta\nu}, x^\nu, \tilde{T}_\nu{}^\mu) = \mathcal{F}_1^{\alpha}(a_\beta, A_\beta, x^\nu, X^\nu) + A_\beta \tilde{P}^{\beta\alpha} - X^\beta \tilde{T}_\beta{}^\alpha,$$

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$$\tilde{p}^{\beta\mu} = \frac{\partial \mathcal{F}_2^{\mu}}{\partial a_\beta}, \quad A_\beta \delta_\nu^\mu = \frac{\partial \mathcal{F}_2^{\mu}}{\partial \tilde{P}^{\beta\nu}}, \quad \tilde{t}_\nu{}^\mu = -\frac{\partial \mathcal{F}_2^{\mu}}{\partial x^\nu}, \quad X^\alpha \delta_\nu^\mu = -\frac{\partial \mathcal{F}_2^{\mu}}{\partial \tilde{T}_\alpha{}^\nu}, \; \mathcal{H}_\mathrm{e}' = \mathcal{H}_\mathrm{e}$$

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Coordinate transformation of a Proca field

Under a coordinate transformation $x^\nu \mapsto X^\nu$, a vector field a_μ transforms as

$$A_\mu(X) = a_\xi(x)\,\frac{\partial x^\xi}{\partial X^\mu}.$$

Regarded as a canonical transformation, the mapping of the vector field is generated by

$$\mathcal{F}_2^\mu(y) = -\tilde{T}_\alpha{}^\mu h^\alpha(x) + \tilde{P}^{\alpha\beta}(X)\,a_\xi(x)\frac{\partial x^\xi}{\partial X^\alpha}\frac{\partial y^\mu}{\partial X^\beta}.$$

We thus get the additional canonical transformation rules

$$\tilde{p}^{\mu\nu}(x) = \tilde{P}^{\alpha\beta}(X)\frac{\partial x^\mu}{\partial X^\alpha}\frac{\partial x^\nu}{\partial X^\beta}$$

$$X^\alpha = h^\alpha(x)$$

$$\tilde{t}_\nu{}^\mu = -\tilde{P}^{\alpha\beta}(X)\,a_\xi(x)\frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\lambda}\frac{\partial X^\lambda}{\partial x^\nu}\frac{\partial y^\mu}{\partial X^\beta} + \tilde{T}_\alpha{}^\mu\frac{\partial h^\alpha}{\partial x^\nu}.$$

The last rule yields the transformation of the Hamiltonian.

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Transformation rule for the Hamiltonians

According to the general prescription, the Hamiltonians transform as

$$\mathcal{H}' \det\Lambda' = \mathcal{H} \det\Lambda + \tilde{P}^{\alpha\beta}(X) A_\lambda(X) \frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\beta} \frac{\partial X^\lambda}{\partial x^\xi}.$$

$\rightsquigarrow$ The Hamiltonians do not maintain their form if $\partial^2 x^\xi / \partial X^\alpha \partial X^\beta \neq 0$.

In order to find the desired form-invariant Hamiltonian, we must formally introduce "gauge Hamiltonians" $\mathcal{H}_\mathrm{g}$ as

$$\mathcal{H}'_\mathrm{g} \det\Lambda' = \tilde{P}^{\alpha\beta}(X)\, A_\lambda(X)\, \Gamma^\lambda{}_{\alpha\beta}(X), \qquad \mathcal{H}_\mathrm{g} \det\Lambda = \tilde{p}^{\alpha\beta}(x)\, a_\lambda(x)\, \gamma^\lambda{}_{\alpha\beta}(x)$$

The amended Hamiltonian $(\mathcal{H} + \mathcal{H}_\mathrm{g}) \det\Lambda$ is then form-invariant, provided that the formally introduced gauge coefficients transform as

$$\Gamma^\lambda{}_{\alpha\beta}(X) = \gamma^k{}_{ij}(x) \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\beta} \frac{\partial X^\lambda}{\partial x^k} + \frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\beta} \frac{\partial X^\lambda}{\partial x^\xi}.$$

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Treating the gauge coefficients $\Gamma^{\lambda}{}_{\alpha\beta}$ as internal “fields”

We include the description of the dynamics of the gauge coefficients by incorporating their transformation rule into the CT’s generating function

$$\bar{\mathcal{F}}_2^{\mu} = \mathcal{F}_2^{\mu} + g_1\, \tilde{Q}_{\eta}{}^{\alpha\xi\lambda} \frac{\partial y^{\mu}}{\partial X^{\lambda}} \left(\gamma^{k}{}_{ij} \frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial x^{i}}{\partial X^{\alpha}} \frac{\partial x^{j}}{\partial X^{\xi}} + \frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial^2 x^{k}}{\partial X^{\alpha} \partial X^{\xi}} \right),$$

with $\tilde{Q}_{\eta}{}^{\alpha\xi\mu}$ the canonical conjugates of the gauge fields $\Gamma^{\eta}{}_{\alpha\xi}$ and g_1 a dimensionless coupling constant.

- The amended generating function $\bar{\mathcal{F}}_2^{\mu}$ now defines the transformation rule for the vector fields a_{μ} and for the gauge coefficients $\gamma^{\eta}{}_{\alpha\xi}$.
- As a feature of the canonical formalism, the generating function simultaneously defines the rules for the respective conjugates, $\tilde{p}^{\mu\nu}$ and $\tilde{q}_{\eta}{}^{\alpha\xi\lambda}$, and for the Hamiltonian.
- This additional structure ensures the action principle to be maintained.

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Transformation rules for $\gamma^{\eta}{}_{\alpha\xi}(x) \mapsto \Gamma^{\eta}{}_{\alpha\xi}(X)$

The additional transformation rules are:

$$
\begin{aligned}
\Gamma^{\eta}{}_{\alpha\xi} &= \gamma^{k}{}_{ij}\,\frac{\partial X^{\eta}}{\partial x^{k}}\frac{\partial x^{i}}{\partial X^{\alpha}}\frac{\partial x^{j}}{\partial X^{\xi}} + \frac{\partial X^{\eta}}{\partial x^{k}}\frac{\partial^2 x^{k}}{\partial X^{\alpha}\partial X^{\xi}} \\
\tilde{q}_{k}{}^{ij\mu} &= \tilde{Q}_{\eta}{}^{\alpha\xi\lambda}\frac{\partial X^{\eta}}{\partial x^{k}}\frac{\partial x^{i}}{\partial X^{\alpha}}\frac{\partial x^{j}}{\partial X^{\xi}}\frac{\partial x^{\mu}}{\partial X^{\lambda}} \\
\tilde{t}_{\nu}{}^{\mu} &= -\tilde{P}^{\alpha\beta}(X)\,a_{\xi}(x)\frac{\partial^2 x^{\xi}}{\partial X^{\alpha}\partial X^{\lambda}}\frac{\partial X^{\lambda}}{\partial x^{\nu}}\frac{\partial y^{\mu}}{\partial X^{\beta}} + \tilde{T}_{\alpha}{}^{\mu}\frac{\partial h^{\alpha}}{\partial x^{\nu}} \\
&\quad - \tilde{Q}_{\eta}{}^{\alpha\xi\lambda}\frac{\partial y^{\mu}}{\partial X^{\lambda}}\left[\gamma^{k}{}_{ij}\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial X^{\eta}}{\partial x^{k}}\frac{\partial x^{i}}{\partial X^{\alpha}}\frac{\partial x^{j}}{\partial X^{\xi}}\right) + \frac{\partial}{\partial x^{\nu}}\left(\frac{\partial X^{\eta}}{\partial x^{k}}\frac{\partial^2 x^{k}}{\partial X^{\alpha}\partial X^{\xi}}\right)\right]
\end{aligned}
$$

We observe:

- The required transformation rule for the connection coefficients $\gamma^{\eta}{}_{\alpha\xi}$ is reproduced.
- Their conjugates $\tilde{q}_{\eta}{}^{\alpha\xi\mu}$ transform as a tensor!
- The last rule defines the transformation of the amended Hamiltonians.

Transformation rules for $\gamma^{\eta}{}_{\alpha\xi}(x) \mapsto \Gamma^{\eta}{}_{\alpha\xi}(X)$

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$$- \tilde{Q}_{\eta}{}^{\alpha\xi\lambda} \frac{\partial y^{\mu}}{\partial X^{\lambda}} \left[\gamma^{k}{}_{ij} \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial x^{i}}{\partial X^{\alpha}} \frac{\partial x^{j}}{\partial X^{\xi}} \right) + \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial X^{\eta}}{\partial x^{k}} \frac{\partial^2 x^{k}}{\partial X^{\alpha} \partial X^{\xi}} \right) \right]$$

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Transformation rule for the Hamiltonian

Recipe to derive the physical Hamiltonian

The task is now to express all derivatives of the X^{μ} and x^{μ} in terms of the gauge coefficients ${\gamma^{\eta}}_{\alpha\xi}$ and ${\Gamma^{\eta}}_{\alpha\xi}$, and their conjugates, ${\tilde{q}_{\eta}}^{\alpha\xi\mu}$ and ${\tilde{Q}_{\eta}}^{\alpha\xi\mu}$, according to the canonical transformation rules.

Remarkably, this works well: all terms match up perfectly. The result is:

$$\begin{aligned}\mathcal{H}'\det\Lambda' - \mathcal{H}\det\Lambda = {} & \tfrac{1}{2}{\tilde{Q}_{\eta}}^{\alpha\xi\mu}\left(\frac{\partial{\Gamma^{\eta}}_{\alpha\xi}}{\partial X^{\mu}} + \frac{\partial{\Gamma^{\eta}}_{\alpha\mu}}{\partial X^{\xi}} - {\Gamma^{i}}_{\alpha\xi}{\Gamma^{\eta}}_{i\mu} + {\Gamma^{i}}_{\alpha\mu}{\Gamma^{\eta}}_{i\xi}\right)\\ & - \tfrac{1}{2}{\tilde{q}_{\eta}}^{\alpha\xi\mu}\left(\frac{\partial{\gamma^{\eta}}_{\alpha\xi}}{\partial x^{\mu}} + \frac{\partial{\gamma^{\eta}}_{\alpha\mu}}{\partial x^{\xi}} - {\gamma^{i}}_{\alpha\xi}{\gamma^{\eta}}_{i\mu} + {\gamma^{i}}_{\alpha\mu}{\gamma^{\eta}}_{i\xi}\right)\\ & + \tilde{P}^{\alpha\beta}A_{\lambda}{\Gamma^{\lambda}}_{\alpha\beta} - \tilde{p}^{\alpha\beta}a_{\lambda}{\gamma^{\lambda}}_{\alpha\beta}\end{aligned}$$

- The terms emerge in a symmetric form with opposite sign in the original and the transformed dynamical variables.
- No new gauge quantities are required, hence, the dynamical system is now closed.

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$$\begin{aligned}\mathcal{H}'\det\Lambda' - \mathcal{H}\det\Lambda = {} & \tfrac{1}{2}\tilde{Q}_{\eta}{}^{\alpha\xi\mu}\left(\frac{\partial\Gamma^{\eta}{}_{\alpha\xi}}{\partial X^{\mu}} + \frac{\partial\Gamma^{\eta}{}_{\alpha\mu}}{\partial X^{\xi}} - \Gamma^{i}{}_{\alpha\xi}\Gamma^{\eta}{}_{i\mu} + \Gamma^{i}{}_{\alpha\mu}\Gamma^{\eta}{}_{i\xi}\right) \\ & - \tfrac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\mu}\left(\frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\mu}} + \frac{\partial\gamma^{\eta}{}_{\alpha\mu}}{\partial x^{\xi}} - \gamma^{i}{}_{\alpha\xi}\gamma^{\eta}{}_{i\mu} + \gamma^{i}{}_{\alpha\mu}\gamma^{\eta}{}_{i\xi}\right) \\ & + \tilde{P}^{\alpha\beta}A_{\lambda}\Gamma^{\lambda}{}_{\alpha\beta} - \tilde{p}^{\alpha\beta}a_{\lambda}\gamma^{\lambda}{}_{\alpha\beta}\end{aligned}$$

- The terms emerge in a symmetric form with opposite sign in the original and the transformed dynamical variables.
- No new gauge quantities are required, hence, the dynamical system is now closed.

Transformation rule for the Hamiltonian

Recipe to derive the physical Hamiltonian

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Final form-invariant Hamiltonian

Similar to conventional gauge theories, the final form-invariant Hamiltonian must contain in addition a dynamics term $\mathcal{H}_{\mathrm{e,dyn}}$ to allow for a non-static space-time. Furthermore

$$\mathcal{H}'_{\mathrm{e,dyn}}(\tilde{Q}) = \mathcal{H}_{\mathrm{e,dyn}}(\tilde{q})$$

must hold in order for the final extended Hamiltonians to satisfy the required transformation rule $\mathcal{H}'_{\mathrm{e}} = \mathcal{H}_{\mathrm{e}}$.

The final form-invariant extended Hamiltonian is now given by

$$\mathcal{H}_{\mathrm{e,GR}}\left(a, \tilde{p}, \tilde{r}, \gamma, \tilde{q}, \tilde{t}\,\right) = \mathcal{H}_{\mathrm{e}}\left(a, \tilde{p}, \tilde{t}\,\right) - \tfrac{1}{2} g_1 \mathcal{H}_{\mathrm{e,dyn}}(\tilde{q}) + \tilde{p}^{\alpha\beta}\, a_\lambda\, {\gamma^\lambda}_{\alpha\beta}$$
$$+ \tfrac{1}{2} g_1 {\tilde{q}_\eta}^{\alpha\xi\mu} \left(\frac{\partial {\gamma^\eta}_{\alpha\mu}}{\partial y^\xi} + \frac{\partial {\gamma^\eta}_{\alpha\xi}}{\partial y^\mu} + {\gamma^k}_{\alpha\mu} {\gamma^\eta}_{k\xi} - {\gamma^k}_{\alpha\xi} {\gamma^\eta}_{k\mu} \right)$$

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Canonical equation for $a_\mu, p^{\mu\nu}$

Due to the coupling term $\tilde{p}^{\alpha\beta}\, a_\eta\, \gamma^{\eta}{}_{\alpha\beta}$ in $\mathcal{H}_{\mathrm{e,GR}}$, the field equations for a_μ and $p^{\mu\nu}$ acquire an additional term

$$\frac{\partial a_\nu}{\partial y^\mu} = \frac{\partial \mathcal{H}_{\mathrm{e,GR}}}{\partial \tilde{p}^{\nu\mu}} = \frac{\partial \mathcal{H}}{\partial p^{\nu\mu}} + a_\eta\, \gamma^{\eta}{}_{\nu\mu}$$

$$\frac{\partial \tilde{p}^{\nu\beta}}{\partial y^\beta} = -\frac{\partial \mathcal{H}_{\mathrm{e,GR}}}{\partial a_\nu} = -\frac{\partial \mathcal{H}}{\partial a_\nu} \det \Lambda - \tilde{p}^{\alpha\beta} \gamma^{\nu}{}_{\alpha\beta}\,.$$

If we now interpret the $\gamma^{\nu}{}_{\alpha\beta}$ as affine connections, then the partial derivatives of the fields and the terms proportional to γ can be combined to yield covariant derivatives, which yields the tensor equations

$$a_{\nu;\mu} = \frac{\partial \mathcal{H}}{\partial p^{\nu\mu}}, \qquad p^{\nu\beta}{}_{;\beta} = -\frac{\partial \mathcal{H}}{\partial a_\nu}.$$

The coupling term in $\mathcal{H}_{\mathrm{e,GR}}$ thus converts the non-tensor equations for a_μ and $p^{\mu\nu}$ emerging from $\mathcal{H}$ into tensor equations, provided that the gauge coefficients $\gamma^{\nu}{}_{\alpha\beta}$ are interpreted as affine connections.

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Canonical equation for $\gamma^{\eta}{}_{\alpha\xi}$

The canonical equation for the gauge coefficients follows as

$$\frac{\partial \gamma^{\eta}{}_{\alpha\xi}}{\partial y^{\mu}} = \frac{\partial \mathcal{H}_{\mathrm{e,GR}}}{\partial \left(g_1 \tilde{q}_{\eta}{}^{\alpha\xi\mu} \right)}$$

$$= -\frac{1}{2} \frac{\partial \mathcal{H}_{\mathrm{e,dyn}}}{\partial \tilde{q}_{\eta}{}^{\alpha\xi\mu}} + \frac{1}{2} \left(\frac{\partial \gamma^{\eta}{}_{\alpha\mu}}{\partial y^{\xi}} + \frac{\partial \gamma^{\eta}{}_{\alpha\xi}}{\partial y^{\mu}} + \gamma^{k}{}_{\alpha\mu} \gamma^{\eta}{}_{k\xi} - \gamma^{k}{}_{\alpha\xi} \gamma^{\eta}{}_{k\mu} \right)$$

Solved for $\mathcal{H}_{\mathrm{e,dyn}}$, one finds exactly the representation of the

Riemann curvature tensor $r^{\eta}{}_{\alpha\xi\beta}$

$$\frac{\partial \mathcal{H}_{\mathrm{e,dyn}}}{\partial \tilde{q}_{\eta}{}^{\alpha\xi\mu}} = \frac{\partial \gamma^{\eta}{}_{\alpha\mu}}{\partial y^{\xi}} - \frac{\partial \gamma^{\eta}{}_{\alpha\xi}}{\partial y^{\mu}} + \gamma^{k}{}_{\alpha\mu} \gamma^{\eta}{}_{k\xi} - \gamma^{k}{}_{\alpha\xi} \gamma^{\eta}{}_{k\mu} = r^{\eta}{}_{\alpha\xi\beta}(\gamma, \partial\gamma)$$

We observe:

- The CT requirement for $q^{\eta}{}_{\alpha\xi\mu}$ to constitute a tensor can be satisfied.
- $\mathcal{H}_{\mathrm{e,dyn}}$ linear in $\tilde{q}_{\eta}{}^{\alpha\xi\mu}$ only allows for constant (or zero) curvature.

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Options for choosing $\mathcal{H}_{\mathrm{e,dyn}}$

For the particular choice of a "free-field" Hamiltonian that is a quadratic function of q

$$\mathcal{H}_{\mathrm{e,dyn}} = \tfrac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta} q^{\eta}{}_{\alpha\xi\beta},$$

we find

$$q^{\eta}{}_{\alpha\xi\beta} = \frac{\partial \mathcal{H}_{\mathrm{e,dyn}}}{\partial \tilde{q}_{\eta}{}^{\alpha\xi\beta}} = r^{\eta}{}_{\alpha\xi\beta}(\gamma, \partial\gamma).$$

⇝ With this choice of $\mathcal{H}_{\mathrm{e,dyn}}$, the quantity q — introduced formally in the generating function — emerges as the Riemann curvature tensor.

We could as well define $\mathcal{H}_{\mathrm{e,dyn}}$ as

$$\mathcal{H}_{\mathrm{e,dyn}} = -\frac{1}{\ell^2}\tilde{q}_{\eta}{}^{\alpha\xi\beta}\left(\delta^{\eta}_{\xi} g_{\alpha\beta} - \delta^{\eta}_{\beta} g_{\alpha\xi}\right).$$

The above field equation then yields

$$r_{\eta\alpha\xi\beta}(\gamma, \partial\gamma) = -\frac{1}{\ell^2}\left(g_{\xi\eta} g_{\alpha\beta} - g_{\beta\eta} g_{\alpha\xi}\right).$$

⇝ The dynamical quantities γ then describe a 4-dimensional Anti-de Sitter space (AdS_4) with radius ℓ, hence a solution of the Einstein-Hilbert action.

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Canonical equation for $\tilde{q}_{\kappa}{}^{\tau\sigma\alpha}$

The derivative of $\mathcal{H}_{\mathrm{e,GR}}$ with respect to $\gamma^{\kappa}{}_{\tau\sigma}$ follows as

$$\frac{\partial \tilde{q}_{\kappa}{}^{\tau\sigma\alpha}}{\partial y^{\alpha}} = -\frac{\partial \mathcal{H}_{\mathrm{e,GR}}}{\partial\left(g_1 \gamma^{\kappa}{}_{\tau\sigma}\right)} = \gamma^{\beta}{}_{\kappa\alpha}\, \tilde{q}_{\beta}{}^{\tau\sigma\alpha} - \gamma^{\tau}{}_{\alpha\beta}\, \tilde{q}_{\kappa}{}^{\beta\sigma\alpha} + \frac{1}{g_1} \tilde{p}^{\tau\sigma}\, a_{\kappa}.$$

This equation is actually a tensor equation

$$g_1 \left(\tilde{q}_{\kappa}{}^{\tau\sigma\alpha}\right)_{;\alpha} = \tilde{p}^{\tau\sigma}\, a_{\kappa}$$

⇝ For a torsion-free space-time, the canonical equation states that the fields a_{κ} and their conjugates, $\tilde{p}^{\tau\sigma}$ act as the source of $\tilde{q}_{\kappa}{}^{\tau\sigma\alpha}$.

For the particular case of $\mathcal{H}_{\mathrm{e,dyn}}$ quadratic in $q_{\kappa}{}^{\tau\sigma\alpha}$, we have

$$q_{\kappa}{}^{\tau\sigma\alpha} = r_{\kappa}{}^{\tau\sigma\alpha}.$$

In contracted form this yields a Poisson-like equation for the Ricci tensor $r^{\tau\alpha}$

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Conclusions

- The canonical transformation formalism ensures the action principle to be maintained.
- The gauge principle was applied to theories that are form-invariant under Lorentz transformations as the system's global symmetry group.
- The theories can be rendered form-invariant under the corresponding local group (i.e., local Lorentz transformations) with the affine connection coefficients ${\gamma^{\eta}}_{\alpha\xi}$ acting as the respective gauge quantities. The resulting theory then satisfies the general principle of relativity.
- Up to a "free-field" Hamiltonian $\mathcal{H}_{\mathrm{e,dyn}}$, the canonical formalism yields unambiguously a Hamiltonian that describes the dynamics of the connection coefficients ("displacement fields") ${\gamma^{\eta}}_{\alpha\xi}$.
- For $\mathcal{H}_{\mathrm{e,dyn}}$ quadratic in the Riemann tensor, a "Poisson-type" equation for the Ricci tensor emerges.
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- The theories can be rendered form-invariant under the corresponding local group (i.e., local Lorentz transformations) with the affine connection coefficients ${\gamma^{\eta}}_{\alpha\xi}$ acting as the respective gauge quantities. The resulting theory then satisfies the general principle of relativity.
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- The canonical formalism does not yield a unique GR theory, but restricts the freedom to merely choosing the appropriate $\mathcal{H}_{\mathrm{e,dyn}}$.
- A Lagrangian that is quadratic in the curvature tensor was already proposed by A. Einstein in a personal letter to H. Weyl, reasoning analogies with other classical field theories.
- The formalism can easily be generalized by introducing the metric $g_{\mu\nu}$ as an additional canonical variable. The theory then allows for non-zero torsion and non-metricity tensors.
- The formalism can be further generalized by introducing tetrads instead of the metric. As we can then distinguish the internal and external (space-time) degrees of freedom of spinors, this allows to describe the interaction of fermions with the space-time dynamics.

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