Multisymplectic Geometry, Boundary Terms, and Black Hole Entropy

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Outline

I. Introduction to Multisymplectic Geometry
   a. Derivation of Multisymplectic Formalism
   b. Multisymplectic Phase Space

II. Covariant Approach
   a. Variational Principle and Boundary Terms in the Action
   b. Example: Rindler Path Integral

III. 3+1 Approach
   a. Derivation from Covariant Formalism
   b. Boundary Terms in the (Pre)symplectic Structure

IV. Conserved Currents and Black Hole Thermodynamics
   a. Noether Currents, Charges, and Their Boundary Terms
   b. First Law of Black Hole Thermodynamics

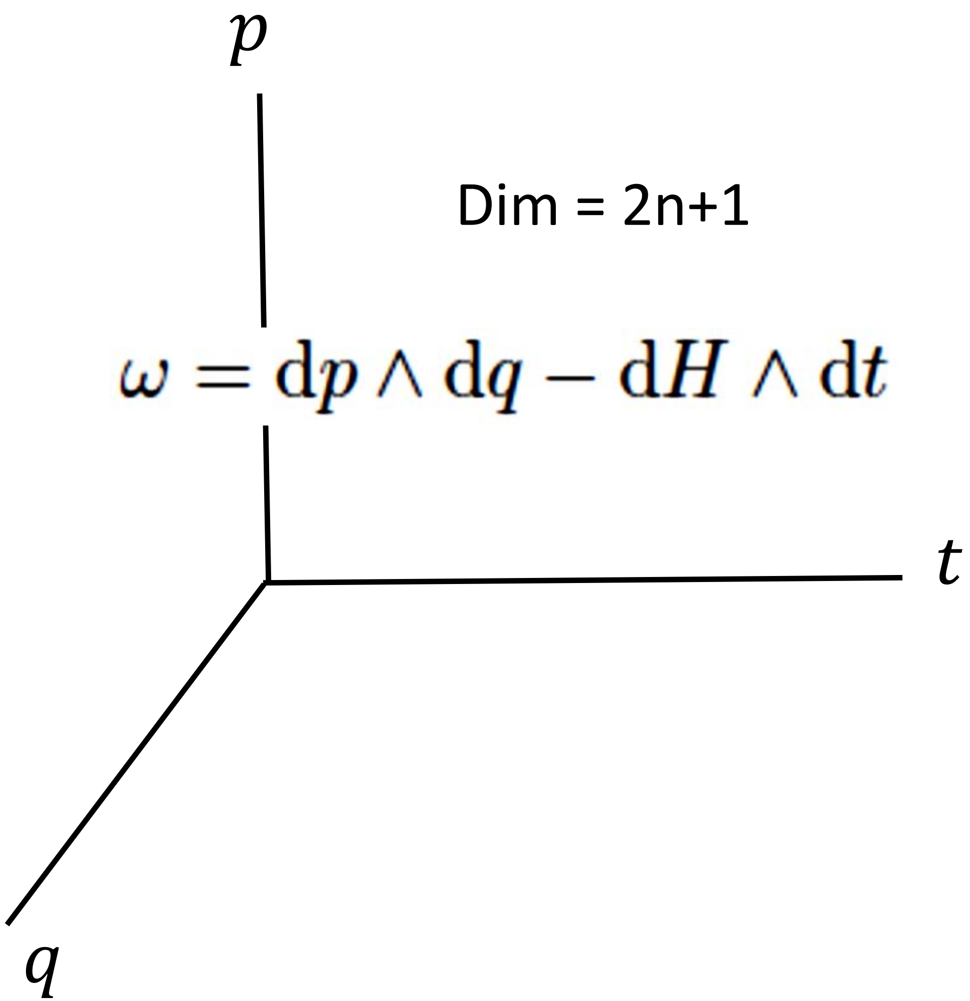
What is Multisymplectic Geometry?

- Covariant and Hamiltonian approach to field theory
- Generalization of symplectic geometry
- Yields finite dimensional “phase space” for field theories
- Separates spacetime geometry (contained in dynamics) from phase space geometry (contained in kinematics)
- Combines features of Lagrangian and Hamiltonian formulations of field theory
- Offers new perspectives on Noether currents, boundary terms, and standard field theoretic results

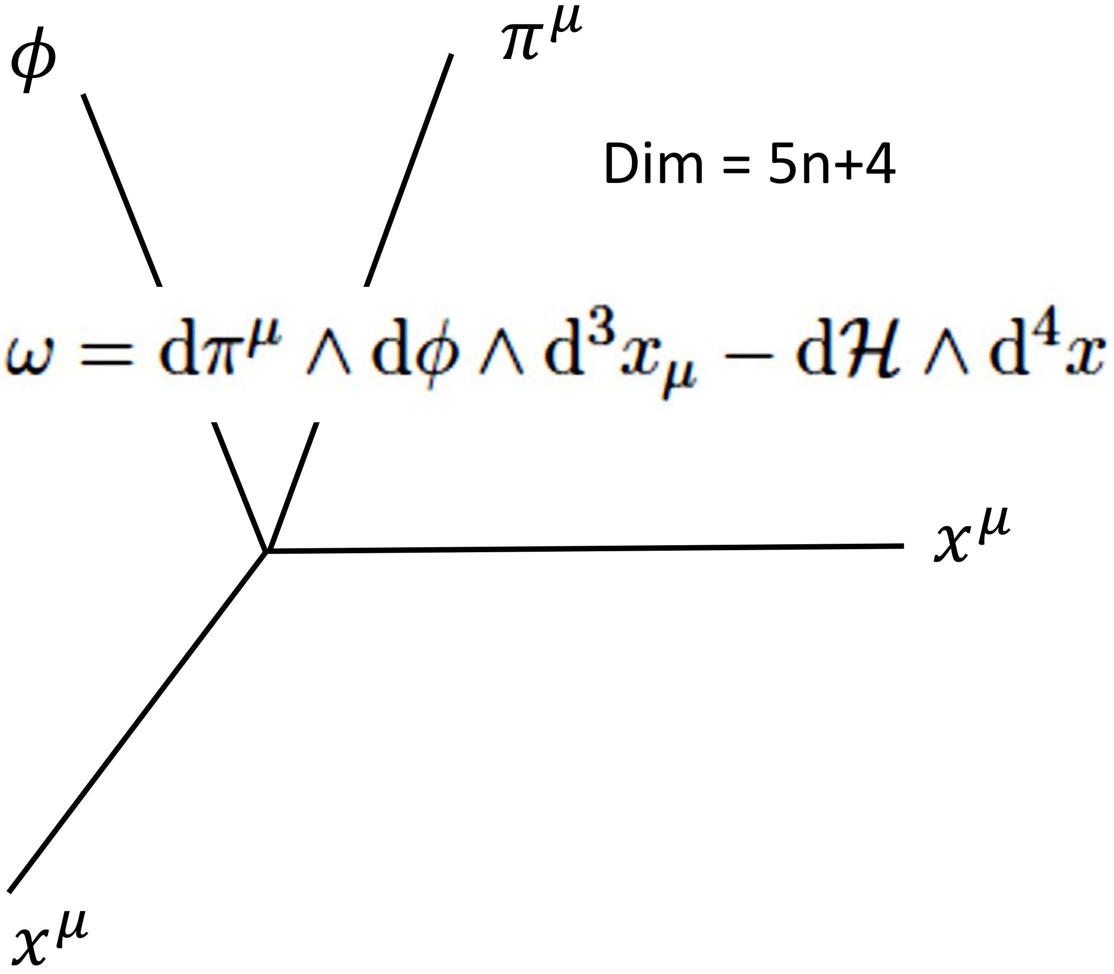
Derivation of Multisymplectic Formalism

	Hamiltonian Particle Mechanics	**Covariant Hamiltonian (Multisymplectic) Mechanics**
Legendre Transformation	$p = \frac{\partial L}{\partial \dot{q}}$ $H = p\dot{q} - L$	$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ $\mathcal{H}^{\text{De Donder-Weyl}} = \pi^\mu \partial_\mu \phi - \mathcal{L}$
Hamilton's Equations	$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$	$\partial_\mu \phi = \frac{\partial \mathcal{H}}{\partial \pi^\mu} \quad \partial_\mu \pi^\mu = -\frac{\partial \mathcal{H}}{\partial \phi}$
Geometric Structure	$\theta = p\mathrm{d}q - H\mathrm{d}t$ $\omega = \mathrm{d}\theta$	$\theta = \pi^\mu \mathrm{d}\phi \wedge \mathrm{d}^3 x_\mu - \mathcal{H}\mathrm{d}^4 x$ $\omega = \mathrm{d}\theta$

Multisymplectic Phase Space

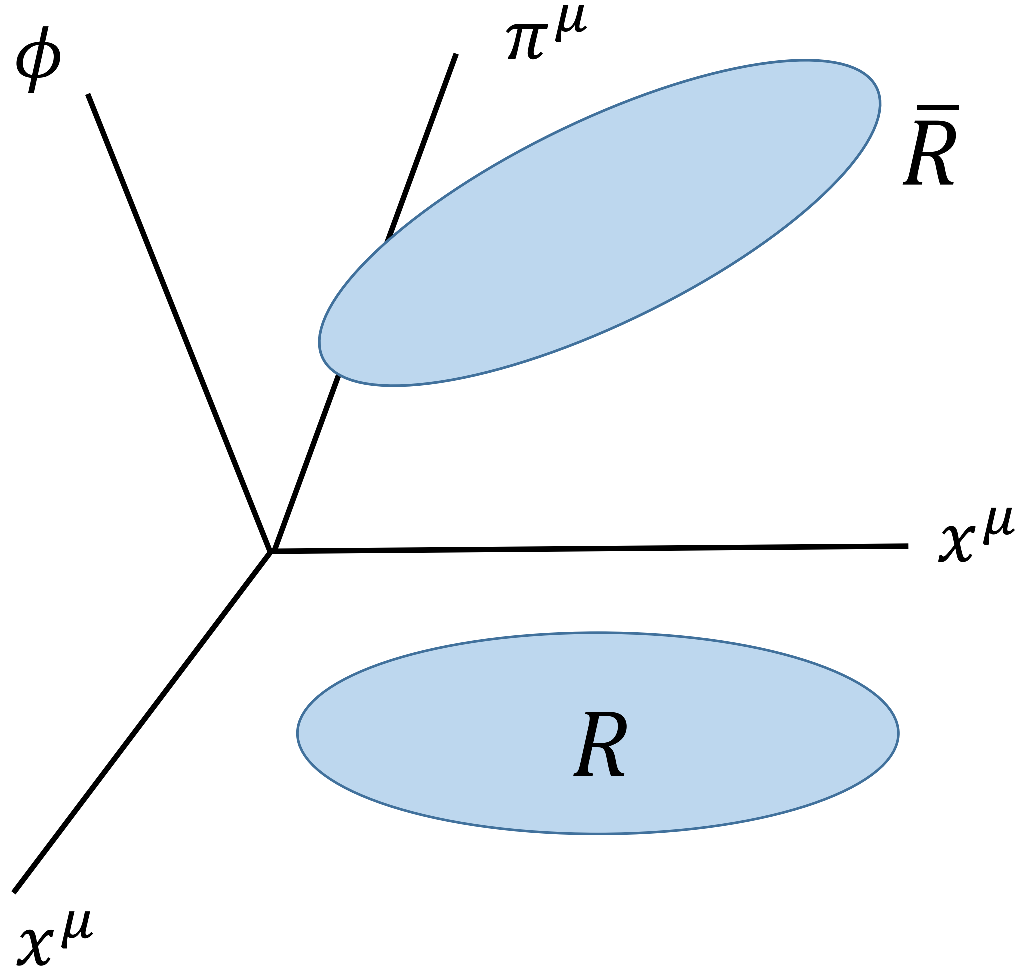


$$\omega = \mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t$$



$$\omega = \mathrm{d}\pi^{\mu} \wedge \mathrm{d}\phi \wedge \mathrm{d}^3 x_{\mu} - \mathrm{d}\mathcal{H} \wedge \mathrm{d}^4 x$$

Variational Principle



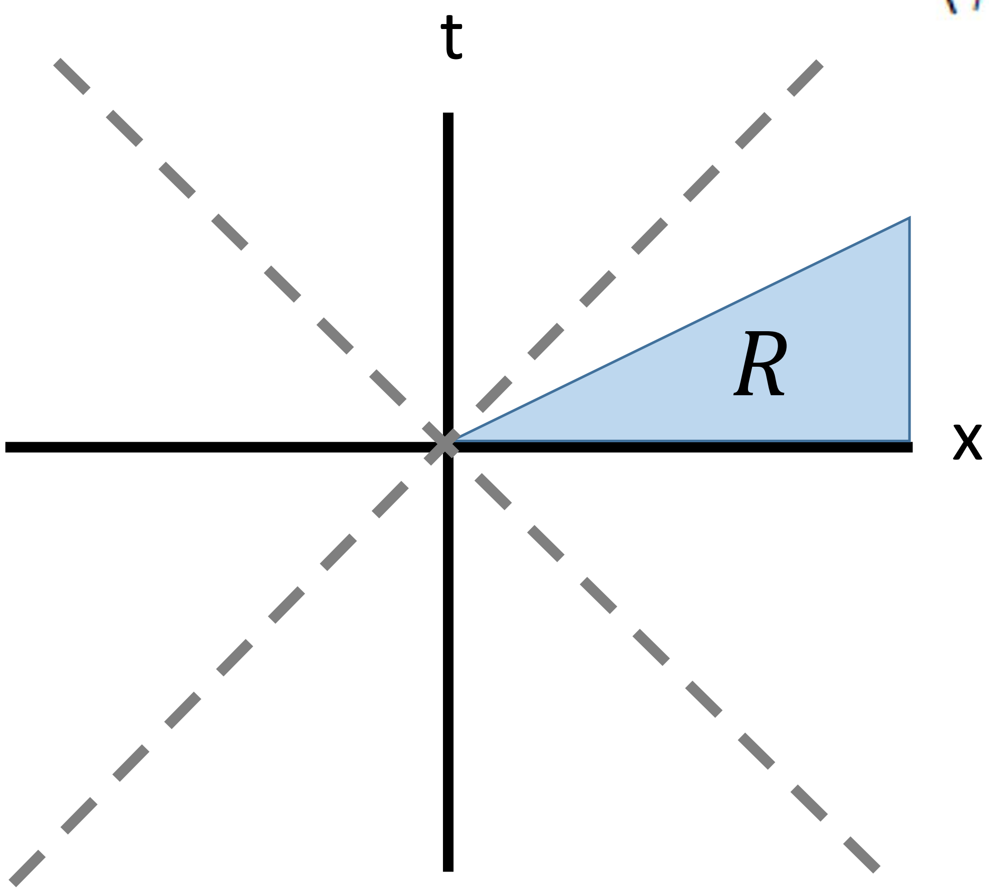
$\bar{R}$ is a solution iff $i_Y\omega\big|_{\bar{R}} = 0$ for all vectors Y

$$0 = \int_{\bar{R}} i_Y \mathrm{d}\theta = \int_{\bar{R}} \mathcal{L}_Y\theta - \mathrm{d}i_Y\theta$$

$$0 = \delta_Y S - \int_{\partial\bar{R}} i_Y\theta$$

$$S = \int_{\bar{R}} \theta$$

Example: Rindler Path Integral



$$\langle \phi_B | K \rangle = \int_{\phi|_{\partial R} = \phi_B} \mathcal{D}[\phi] e^{iS[\phi]/\hbar} \xrightarrow{\hbar \to 0} \delta S = 0$$

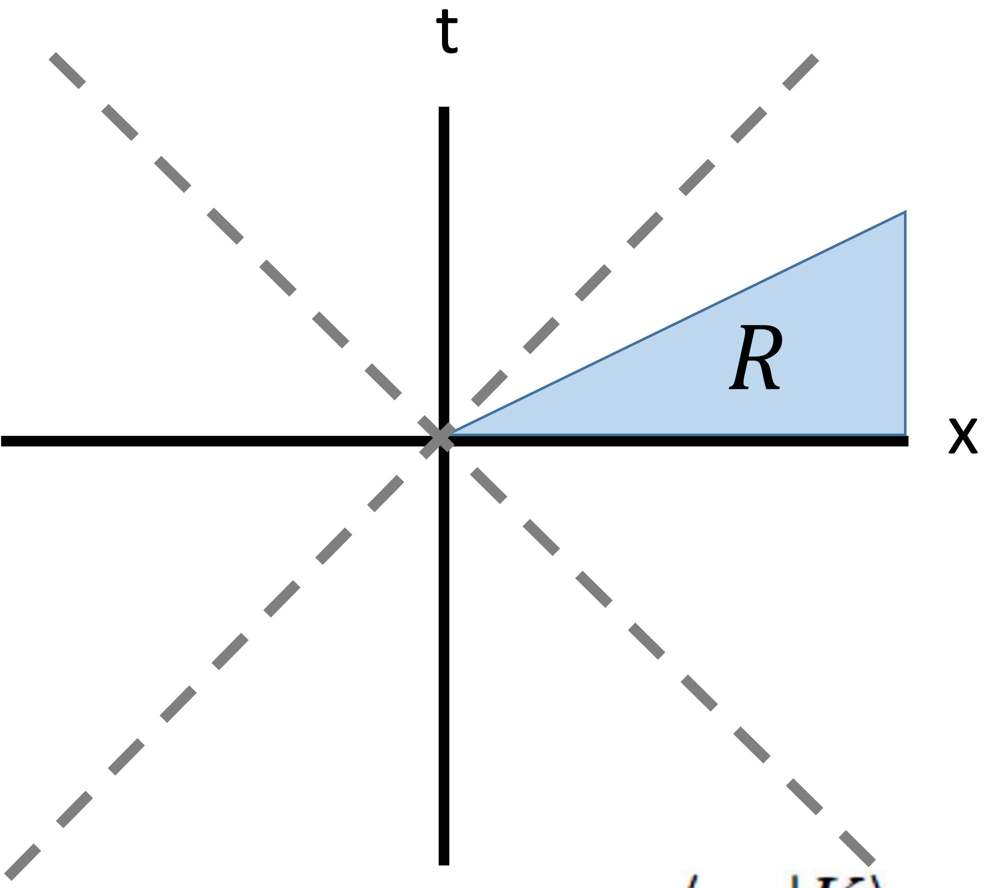
$$\int_{\partial R} i_Y \theta = \int_{\partial R} \pi^\mu \delta\phi \mathrm{d}x_\mu - \mathcal{H} \delta x^\mu \mathrm{d}x_\mu$$

Vanishes because field configuration fixed on boundary

Vanishes because boundary is fixed

$$\Longrightarrow S = \int_R \theta = \int_R \frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} \left(\frac{\partial \phi}{\partial \xi} \right)^2$$

Example: Rindler Path Integral



$$\langle \pi_B | K \rangle = \int_{\pi|_{\partial R} = \pi_B} \mathcal{D}[\phi] e^{iS[\phi]/\hbar} \xrightarrow{\hbar \to 0} \delta S = 0$$

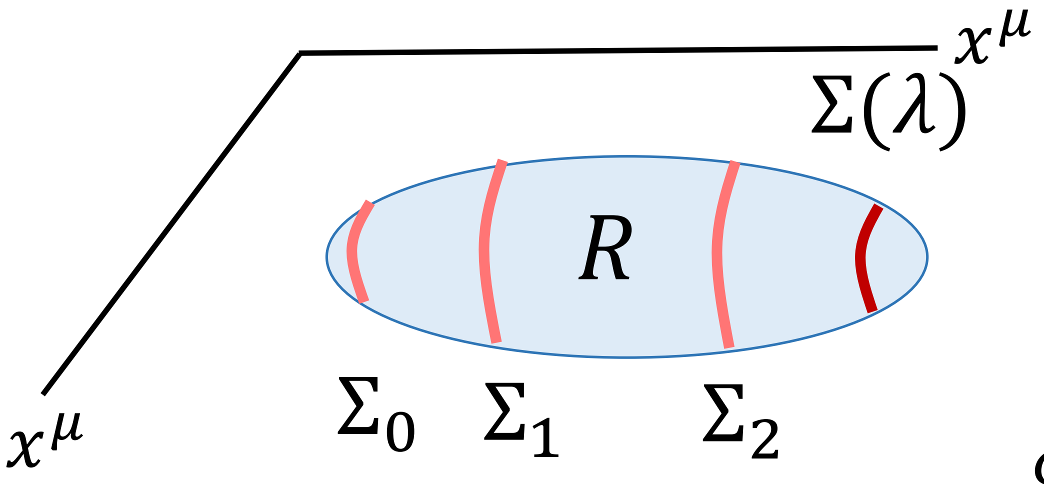
$$\int_{\partial R} i_Y \theta = \int_{\partial R} \pi^\mu \delta\phi \mathrm{d}x_\mu - \mathcal{H}\delta x^\mu \mathrm{d}x_\mu = \delta \int_{\partial R} \pi\,\phi$$

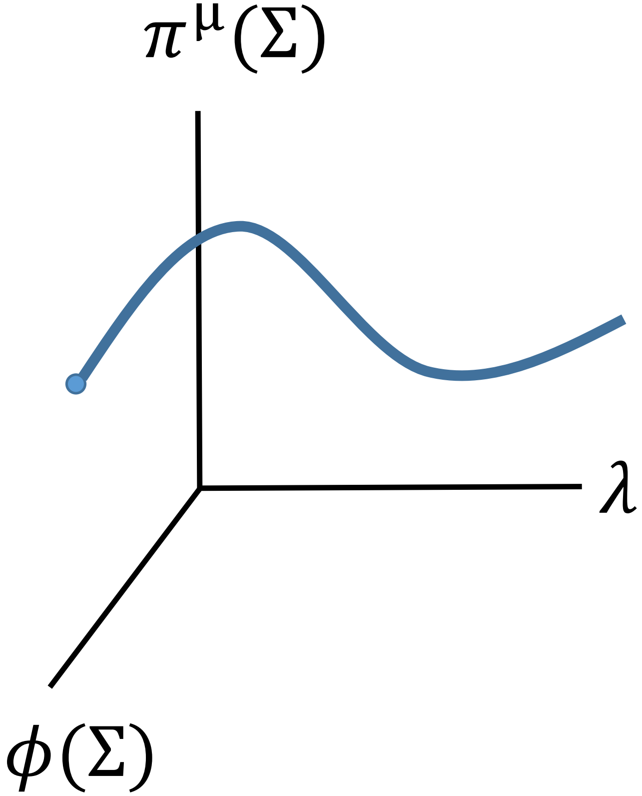
Vanishes because boundary is fixed

$$S = \int_R \theta + \int_{(\partial R)_f} \pi\,\phi - \int_{(\partial R)_i} \pi\,\phi$$

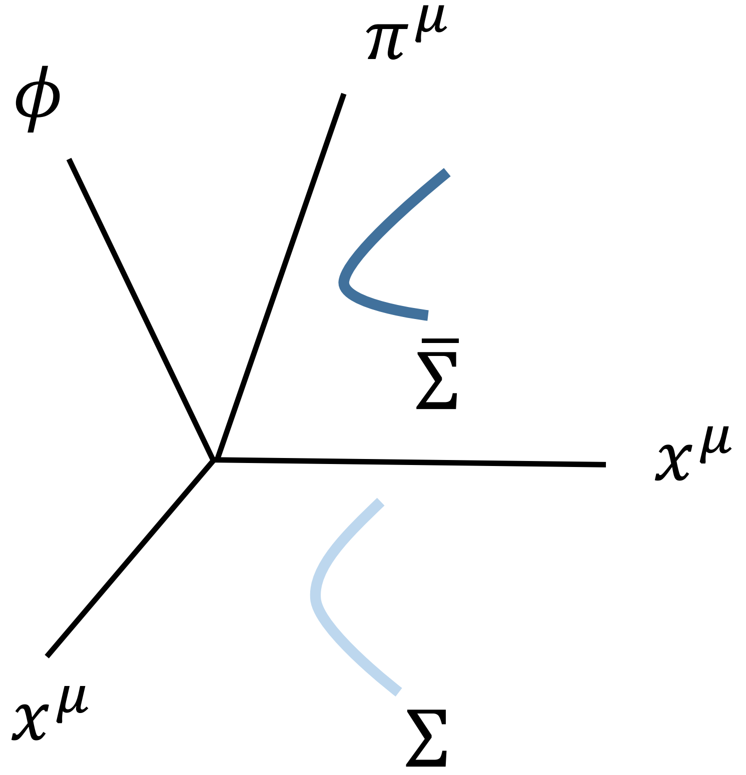
$$\langle \pi_B | K \rangle = \int_{\pi|_{\partial R} = \pi_B} \mathcal{D}[\phi_i]\mathcal{D}[\phi_f]\mathcal{D}[\phi] e^{\frac{i}{\hbar}\int_R \theta + \frac{i}{\hbar}\left(\int_{(\partial R)_f} \pi\,\phi_f - \int_{(\partial R)_i} \pi\,\phi_i \right)}$$

3+1 decomposition





$$\Omega(X,Y) = \int_{\bar{\Sigma}} \omega(X,Y,\cdot)$$



Presymplectic Structure

$$\Omega(X,Y) = \int_{\bar{\Sigma}} \omega(X,Y,\cdot)$$

$$\omega = \mathrm{d}\theta \implies \Omega(X,Y) = \mathrm{d}\Theta(X,Y) + \int_{\partial\bar{\Sigma}} \theta(X,Y,\cdot) \qquad \left[\Theta(X) = \int_{\bar{\Sigma}} \theta(X,\cdot)\right]$$

$\mathrm{d}\Omega \neq 0$ Unless boundary conditions imposed on fields such that

$$\int_{\partial\bar{\Sigma}} \theta(X,Y) = \mathrm{d}\Psi(X,Y)$$

Then $\Omega(X,Y) = \mathrm{d}\tilde{\Theta}(X,Y) \quad \left[\tilde{\Theta} = \Theta + \Psi\right]$

Conserved Currents

- For a symmetry ξ with generator X_ξ there is a corresponding conserved current (3-form) J_ξ satisfying $i_{X_\xi}\omega = -\mathrm{d}J_\xi$
- Typically $J_\xi = i_{X_\xi}\theta = (\pi^\mu \delta_\xi \phi - T^\mu{}_\nu \delta_\xi x^\nu)\,\mathrm{d}^3 x_\mu$
- After 3+1 split, expect conserved charge Q_ξ satisfying $i_{X_\xi}\Omega = -\mathrm{d}Q_\xi$
- Instead, we get $\Omega(X_\xi, Y) = -i_Y \mathrm{d} \int_{\bar\Sigma} J_\xi + \int_{\partial\bar\Sigma} i_Y J_\xi$
- Boundary conditions on fields typically guarantee $\int_{\partial\bar\Sigma} i_Y J_\xi = i_Y \mathrm{d} \int_{\partial\bar\Sigma} K_\xi$
- Hence a (modified) conserved charge $Q_\xi = \int_{\bar\Sigma} J_\xi - \int_{\partial\bar\Sigma} K_\xi = i_{X_\xi}\tilde\Theta$

Boundary Terms in Conserved Charges

$$Q_\xi = \int_{\bar{\Sigma}} J_\xi - \int_{\partial\bar{\Sigma}} K_\xi$$

- For gauge symmetries, J_ξ may contain derivatives of ξ
- Can use integration by parts to write Q_ξ free of such derivatives

$$Q_\xi = \underbrace{\int_{\bar{\Sigma}} \langle C, \xi \rangle}_{\substack{\text{Bulk term} \\ \text{vanishes on} \\ \text{solutions}}} + \underbrace{\int_{\partial\bar{\Sigma}} \langle B, \xi \rangle}_{\substack{\text{Conserved} \\ \text{charge is pure} \\ \text{boundary term}}}$$

Multisymplectic Formalism for GR

$$\theta = \frac{\sqrt{-g}}{16\pi G}\left[\left(g^{\nu\beta}\delta^{\mu}_{\alpha} - g^{\mu\beta}\delta^{\nu}_{\alpha}\right)\mathrm{d}\Gamma_{\nu}{}^{\alpha}{}_{\beta} \wedge \mathrm{d}^3 x_{\mu} + g^{\beta\nu}\left(\Gamma_{\alpha}{}^{\alpha}{}_{\gamma}\Gamma_{\nu}{}^{\gamma}{}_{\beta} - \Gamma_{\nu}{}^{\alpha}{}_{\gamma}\Gamma_{\alpha}{}^{\gamma}{}_{\beta}\right)\mathrm{d}^4 x\right]$$

- Gauge transformations are diffeomorphisms generated by vector fields ξ
- Associated conserved currents J_{ξ} and 3+1 charges Q_{ξ}
- On solutions, $Q_{\xi} = \int_{\Sigma} G^{\mu}{}_{\nu}\xi^{\nu}\,\mathrm{d}^3 x_{\mu} + \text{ boundary terms}$
- For asymptotically flat spacetimes with Σ extending out to spacelike infinity, and ξ asymptotically a translation, the outer boundary term is

$$(Q_{\xi})_{\infty} = \int_{I^0} (P_{\mathrm{ADM}})_{\mu}\xi^{\mu}$$

First Law of Black Hole Thermodynamics

- For an asymptotically flat black hole spacetime, if we terminate Σ on the event horizon, then Q_ξ will also have an inner boundary term
- May not be able to calculate term explicitly, but for asymptotically flat, stationary black holes, can use $\Omega(Y, X_\xi) = \delta_Y Q_\xi$
- Take ξ to be the Killing vector field of the black hole spacetime, X_ξ the corresponding Hamiltonian flow on phase space, so $\Omega(\cdot, X_\xi) = 0$
- Take Y to be a perturbation to a nearby stationary black hole solution with the same surface gravity κ

$$0 = -\frac{\kappa}{2\pi}\delta S + \delta E - \Omega\delta\mathcal{J} \qquad \left[S = \frac{2\pi}{\kappa}(Q_\xi)_{\text{Horizon}}\right]$$

Summary

- Multisymplectic geometry provides covariant Hamiltonian approach to field theory
- Covariant and canonical versions of multisymplectic formalism are closely connected
- Boundaries lead to modified action, presymplectic structure, and conserved charges
- Boundary terms of conserved charge in GR for stationary, asymptotically flat black holes yield first law of black hole thermodynamics