Particle acceleration by Black Holes in a model of $f(R)$ gravity

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Particle collisions are considered within the context of $f(R)$ gravity described by $f(R) = R + 2\alpha\sqrt{R}$, where R stands for the Ricci scalar and α is a non-zero constant. The center of mass (CM) energy of colliding particles near the horizon grows unbounded. Addition of a cosmological constant does not change the outcome. When the collision occurs near a non- black hole, i.e. a naked singularity (for $\alpha > 0$), the particles are absorbed with zero total CM energy. Collisions of a massless outgoing Hawking photon with an infalling particle and collision of two photons following null-geodesics are also taken into account.

I. INTRODUCTION

Bañados, Silk and West (BSW) [1] showed first that collision of geodesic particles in the vicinity of black hole horizon yields a total unbounded center of mass (CM) energy. This amounts to a natural collision similar to the artificially tested process in the high energy laboratory at CERN . The difference is that the latter is under strict human control albeit a too expensive process whereas the former one is free of charge, occurring in cosmos frequently as an ordinary event. Not only the black holes but also naked singularities as well as the throat regions of wormholes may create a similar BSW effect [2]. Rotating black holes and wormholes act more efficiently in comparison with the static ones to yield a high CM energy [3–17]Another aspect of the BSW effect is that it occurs irrespective of the dimensionality of spacetime or the nature of the underlying theory. That is, even in lower/higher dimensions of 3+1- spacetime we can also have an accelerator effect[6, 18–35]. From this token we have shown recently that in 1+1- dimensional Hořava-Lifshitz theory of gravity[36] we did have the BSW effect[37]. Collision must take place near the horizon of the formed black hole so that the particles get boost from the unlimited attraction of the black hole[1, 38–40].

In this paper we investigate the possibility of BSW effect in the modified Einstein theory known as the $f(R)$ gravity[41]. In this theory the Einstein-Hilbert action characterized by the Ricci scalar (R) is extended to cover an arbitrary function of R. Theoretically such a theory has infinite number of possibilities which are to be severely restricted by experimental tests. Naturally any higher power of R hosts higher order derivatives of the metric and expectedly obtaining exact solutions is not an easy task at all. The solution for $f(R)$ gravity that we shall consider in this study is $f(R) = R + 2\alpha\sqrt{R - 4\Lambda} - 2\Lambda$, in which the constant $\alpha \neq 0$, so that our model of $f(R)$ has no vacuum Einstein limit. By comparison with the Scwarzchild - de Sitter line element the second integration constant Λ can be interpreted as a cosmological constant. In the first part we shall choose $\alpha < 0$ with $\Lambda = 0$, so that our solution will represent a black hole. Near its horizon we shall show that collision of particles exhibits BSW effect. Next, we asssume that $\Lambda \neq 0$ and repeat the similar calculation which yields again an unbounded CM energy. In the following stage of our study we choose $\alpha > 0$ (with $\Lambda = 0$) and consider the collision process in the vicinity of $r = 0$, which stands for a naked singularity. It turns out that E_{cm}(= the CM energy) $\to 0$, which amounts to the fact that particles come to rest by radiating out all their kinetic energy. In contrast to mostly finite energy cases emergence of a zero total energy is not encountered in other collision processes. Another piece of work that we take into consideration is the analog of a Compton process in which an infalling particle collides with an outgoing photon. The source of the photon can be attributed to the Hawking process where both massless and massive emissions from the black hole are admissible. It is observed once more that the BSW effect occurs even with such photons. Finally, head-on collision of two photons near a black-hole also is shown to yield an unbounded energy.

II. BLACK HOLES IN $f(R)$ THEORY

The action of general, sourceless $f(R)$ gravity theories in four dimension is

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\, f(R) \tag{1}$$

in which $f(R)$ is the function of the scalar curvature R ,and g stands for the determinant of the metric tensor.

By using the variation of $f(R)$action, the corrosponding field equation is obtained as

$$FR_\mu^\nu - \frac{1}{2} f\delta_\mu^\nu - \nabla^\nu\nabla_\mu F + \delta_\mu^\nu \Box F = 0 \tag{2}$$

where the covariant Laplacian is $\Box = \nabla_\gamma\nabla^\gamma$ and $F = \frac{df}{dR}$. For the spherically symmetric line element,
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$$ds^2 = -Adt^2 + \frac{1}{B}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (3)$$

in which A and B are functions depending only on r. The non-zero Ricci tensors for this line element are

$$R_t^t = -\frac{1}{4A}(A'B' + 2BA'' - \frac{B}{A}A'^2) - \frac{BA'}{rA}, \quad (4)$$

$$R_r^r = -\frac{1}{4A}(A'B' + 2BA'' - \frac{B}{A}A'^2) - \frac{B'}{r}, \quad (5)$$

and

$$R_\theta^\theta = R_\phi^\phi = -\frac{AB'}{2rA} - \frac{B'}{2r} + \frac{1-B}{r^2} \quad (6)$$

with the Ricci scalar

$$R = -B'' - \frac{4rB' + 2(B-1)}{r^2} + \frac{2r^3BF'''(rF' + 2F)}{(rF' + 2F)^2 r^2} + \frac{3r^2F''\left[r^2B'F' + 2F(rB' + 2B)\right]}{(rF' + 2F)^2 r^2} \quad (7)$$

in which a prime implies derivative with respect to r. The field equations amount to ;

$$FR_t^t = \frac{f}{2} - BF'' - \frac{B'F'}{2} - \frac{2BF'}{r}, \quad (8)$$

$$FR_r^r = \frac{f}{2} - \frac{BF'A'}{2A} - \frac{2BF'}{r}, \quad (9)$$

and

$$FR_\theta^\theta = \frac{f}{2} - \frac{BF'}{r} - \frac{BA'F'}{2A} - B\left(F'' + \frac{B'F'}{2B}\right). \quad (10)$$

In this paper, we search for a particular solution $A = B$, which is given in summary as follows:[41]

The $f(R)$ firstly is found as

$$f(R) = R + 2\alpha\sqrt{R} \quad (11)$$

in which α is a constant ($-\infty < \alpha < 0$) and the metric function $A(= B)$takes the form

$$A = B = \frac{1}{2} + \frac{1}{3\alpha r} \quad (12)$$

with the Ricci scalar

$$R = \frac{1}{r^2}. \quad (13)$$

Secondly, we shall choose $f(R)$ with an integration constant that can be interpreted to be the cosmological constant as[42]

$$f(R) = R + 2\alpha\sqrt{R - 4\Lambda} - 2\Lambda. \quad (14)$$

The metric functions in this case are given by

$$A = B = \frac{1}{2} + \frac{1}{3\alpha r} - \frac{\Lambda}{3}r^2 \quad (15)$$

with the Ricci scalar

$$R = \frac{1}{r^2} + 4\Lambda. \quad (16)$$

In the sequel for both cases, $\Lambda = 0$ and $\Lambda \neq 0$ we shall investigate the possibility of BSW effect.

Lastly, for the case of $\alpha > 0$, ($\Lambda = 0$) which corresponds to a naked singular solution at $r = 0$ we shall search for the collider effect.

Obviously in the two parametric solution employed, Λ is a dispensable parameter whereas α not. That is, our choice of $f(R)$ gravity lacks the Einstein's general relativity limit. With deliberation we have made such a choice to see the significance of the BSW effect in a $f(R)$ model that is not connected with the general relativity. This is precisely the case with $\alpha \neq 0$.

III. PARTICLE COLLISION NEAR THE $f(R)$ BLACK HOLES

For different cases we investigate the CM energy for the collision, 4-d velocity components of the colliding particles in the background of the 4-d $f(R)$ black holes by taking the radial motion on equatorial plane ($\theta = \frac{\pi}{2}$).

Our Lagrangian is chosen by

$$L = \frac{1}{2}\left(-A\dot{t}^2 + \frac{1}{A}\dot{r}^2 + r^2\dot{\varphi}^2\right)$$

in which a dot implies derivative with respect to proper distance. The velocities follow as

$$u^t = \dot{t} = \frac{E}{A} \quad (17)$$

and

$$u^\varphi = \dot{\varphi} = \frac{L}{r^2} \quad (18)$$

where E and L are energy and angular momentum constants, respectively.After that, by using the normalization condition ($u.u = -1$), it is found that the radial part is

$$u^r = \dot{r} = \pm\sqrt{E^2 - A\left(1 + \frac{L^2}{r^2}\right)} \quad (19)$$

and clearly we are interested in time-like geodesics. We proceed now to present the CM energy of two particles with four-velocity u_1^μ and u_2^μ. We will assume that both have rest mass $m_0 = 1$. The CM energy is given by,

$$E_{cm} = \sqrt{2}\sqrt{(1 - g_{\mu\nu}u_1^\mu u_2^\nu)} \tag{20}$$

so that it can be expressed as

$$\frac{E_{cm}^2}{2} = 1 + \frac{E_1 E_2}{A} - \kappa\frac{|L_1||L_2|}{r^2} - \kappa\frac{1}{A}\sqrt{E_1^2 - A\left(1 + \frac{L_1^2}{r^2}\right)}\sqrt{E_2^2 - A\left(1 + \frac{L_2^2}{r^2}\right)} \tag{21}$$

where $\kappa = \pm 1$ correspond to particles moving in the same direction ($\kappa = +1$) or opposite direction ($\kappa = -1$), respectively. Furthermore E_1 and E_2/ L_1 and L_2 are defined as the energy/ angular momentum constants corresponding to each particle. Upon taking the lowest order terms in the vicinity of the horizon, since $A \approx 0$, we can make the expansion

$$\sqrt{\left[E^2 - A\left(1 + \frac{L^2}{r^2}\right)\right]} \cong E\left[1 - \frac{A}{2E^2}\left(1 + \frac{L^2}{r^2}\right) + ...\right], \tag{22}$$

so that the CM energy of two particles is obtained as

$$\frac{E_{cm}^2}{2} \cong 1 + (1-\kappa)\frac{E_1 E_2}{A} - \kappa\frac{|L_1||L_2|}{r^2} - \kappa\left[\frac{E_2}{2E_1}\left(1 + \frac{L_1^2}{r^2}\right) + \frac{1}{2E_2^2}\left(1 + \frac{L_2^2}{r^2}\right)\right]. \tag{23}$$

By using different cases, we will investigate the BSW effect whether occurs or not for A(r)$\rightarrow 0$ whenever there is a horizon. In the case of the collision of ingoing and outgoing particles (i.e. motion in opposite directions) $\kappa = -1$, it reduces to

$$\frac{E_{c.m.}^2}{2m^2} \cong \frac{2E_1 E_2}{A} \tag{24}$$

since the other terms are all finite near the horizon.

For the first case for the f(R) black holes without cosmological constant where $\alpha < 0$, horizon is at $r_h = \frac{2}{3|\alpha|}$, and A goes to zero at horizon of $f(R)$ black holes, so that it diverges ($E_{c.m.}^2 \rightarrow \infty$) . So there is a BSW effect. On the other case, when $\kappa = +1$,i.e when the particles move in the same direction, the center of mass energy $E_{c.m.}^2$gives finite value.

Other possibility is when choosing $\kappa = -1$ where there are opposite particles with their angular momenta close to zero ($L_1 = L_2 = 0$). In this case it yields

$$\frac{E_{cm}^2}{2} = 1 + \frac{E_1 E_2}{A} + \kappa\frac{1}{A}\left[\sqrt{E_1^2 - A}\sqrt{E_2^2 - A}\right]. \tag{25}$$

Upon expanding the square roots and taking the lowest order terms we obtain

$$\frac{E_{cm}^2}{2} \cong 1 + \frac{E_1 E_2 - \kappa E_1 E_2}{A} + \frac{(E_1 E_2)}{2} \tag{26}$$

or equivalently for $\kappa = -1$

$$\frac{E_{cm}^2}{2} \cong 1 + \frac{2E_1 E_2}{A} + \frac{(E_1 E_2)}{2} \tag{27}$$

so that for the case $A \longrightarrow 0$,

$$E_{c.m.}^2|_{r_h} \rightarrow \infty. \tag{28}$$

Hence the CM energy of collision between the opposite particles are always infinite for the case of $\alpha < 0$.

A. Particle Collision near the $f(R)$ Black Holes with a Cosmological Constant

The second case of interest is for the $f(R)$ black holes with a cosmological constant in which the metric function A is

$$A = \frac{-\Lambda r^2}{3} + \frac{1}{2} + \frac{1}{3\alpha r} \tag{29}$$

where the horizon is located at

$$r_h = \frac{\Xi}{2\alpha\Lambda} + \frac{\alpha}{\Xi} \tag{30}$$

for

$$\Xi = \left(4 + 2\alpha^2\Lambda^2\sqrt{\frac{-2\alpha^2 - 4\Lambda}{\Lambda}}\right)^{\frac{1}{3}}. \tag{31}$$

It is observed that for real r_h we must have $\frac{-2\alpha^2}{\Lambda} - 4 > 0$, which restricts the cosmological constant to the case of $\Lambda < 0$.

The center of mass energy for the case of colliding ingoing and outgoing particles is obtained with reference to (25) as

$$\frac{E_{c.m.}^2}{2m^2} \cong \frac{2E_1 E_2}{A}. \tag{32}$$

It is clear that as A goes to zero at horizon of $f(R)$ black holes , it diverges ($E_{c.m.}^2|_{r_h} \rightarrow \infty$) . So there is a BSW effect. On the other hand, the case $\kappa = +1$ demands that the center of mass energy $E_{c.m.}^2$is finite.

B. Particle Collision near the Naked Singularity

There is a naked singularity for our $f(R)$ model at the location of $r = 0$, with $\alpha > 0$,where the metric function is given by

$$A = \frac{1}{2} + \frac{1}{3\alpha r}. \tag{33}$$

As it is calculated above the collision of two particles in opposite directions ($\kappa=-1$) is determined as

$$\frac{E_{c.m.}^2}{2m^2} \cong \frac{2E_1E_2}{A} \cong \frac{12E_1E_2\alpha r}{3\alpha r+2}. \quad (34)$$

When r goes to zero, the center of mass energy also goes to zero.

$$\frac{E_{c.m.}^2}{2m^2}|_{r=0} \to 0 \quad (35)$$

This suggests that the particles are absorbed by the naked singularity and strictly there is no BSW effect near such a singularitiy.

IV. COLLISIONAL PROCESSES WITH PHOTONS

The massless photon coming from the Hawking radiation can naturally scatter an infalling particle or vice versa. This phenomenou is analogous to a Compton scattering process which was originally introduced for a photon and an electron. The null-geodesics for a photon satisfies

$$\frac{dt}{d\lambda} = \dot{t} = \frac{E_\gamma}{A} \quad (36)$$

and

$$\frac{d\varphi}{d\lambda} = \dot{\varphi} = \frac{L_\gamma}{r^2} \quad (37)$$

$$\dot{r} = \pm\sqrt{\left[E_\gamma^2 - \frac{AL^2}{r^2}\right]} \quad (38)$$

where λ and $E_\gamma$are an affine parameter and the photon energy, respectively. Defining $E_\gamma = \hbar\omega_0$, where ω_0 is the frequency (with the choice $\hbar = 1$) we can parametrize the energy of the photon by ω_0 alone. The center-of-mass energy of a Hawking photon and the infalling particle can be taken now as

$$E_{cm}^2 = -(p^\mu + k^\mu)^2 \quad (39)$$

in which p^μ and k^μ refer to the particle and photon, 4-momenta, respectively. This amounts to

$$E_{cm}^2 = m^2 - 2mg_{\mu\nu}u^\mu k^\nu. \quad (40)$$

Since we have for the particle

$$p^\mu = m\left(\frac{E_2}{A}, \sqrt{E_2^2 - A\left(1+\frac{L_2^2}{r^2}\right)}, 0, \frac{L_2}{r^2}\right) \quad (41)$$

and for the photon

$$k^\mu = \left(\frac{E_\gamma}{A}, \sqrt{E_\gamma^2 - \frac{AL_\gamma^2}{r^2}}, 0, \frac{L_\gamma}{r^2}\right) \quad (42)$$

one obtains

$$E_{cm}^2 = m^2 + \frac{2mE_\gamma E_2}{A} - \frac{2mL_\gamma L_2}{r^2} - \frac{1}{A}\sqrt{E_\gamma^2 - \frac{AL_\gamma^2}{r^2}}\sqrt{E_2^2 - A\left(1+\frac{L_2^2}{r^2}\right)}. \quad (43)$$

In the near horizon limit this reduces to

$$E_{cm}^2 \cong m^2 + \frac{2mE_\gamma E_2}{A} - \frac{2mL_\gamma L_2}{r^2} - \frac{2m}{A}\left(E_2^2 - \frac{A}{2} - \frac{L_2^2}{2r^2}\right)\left(E_\gamma^2 - \frac{L_\gamma^2}{2Ar^2}\right). \quad (44)$$

so that when $A \longrightarrow 0$, we have

$$E_{cm}^2|_{r_h} \longrightarrow \infty. \quad (45)$$

Thus, there is a BSW effect for a Compton- type scattering near the $f(R)$ black holes considered here.

A. Compton Scattering Process with $\Lambda \neq 0$

For the $f(R)$ black holes. with cosmological constant the center of mass energy near horizon limit is given for the metric function (29) as

$$E_{cm}^2 = m^2 + \frac{2mE_\gamma E_2}{A} - \frac{2mL_\gamma L_2}{r^2} - \frac{2m}{A}\left(E_2^2 - \frac{A}{2} - \frac{L_2^2}{2r^2}\right)\left(E_\gamma^2 - \frac{L_\gamma^2}{2Ar^2}\right). \quad (46)$$

so when $A \longrightarrow 0$, $(orr \to\ r_h$,from(30)) , it diverges

$$E_{cm}^2|_{r_h} \longrightarrow \infty. \quad (47)$$

Thus, there is also a BSW effect for a Compton- like scattering near the $f(R)$ black holes with cosmological constant.

B. Compton Scattering For Infalling Particles Near Naked Singularity

Last case is when the $f(R)$ has naked singularity. The center of mass energy in the Compton- type scattering process can be written as

$$E_{cm}^2 = m^2 + \frac{2mE_\gamma E_2}{A} - \frac{2mL_\gamma L_2}{r^2} - \frac{2m}{A}\left(E_2^2 - \frac{A}{2} - \frac{L_2^2}{2r^2}\right)\left(E_\gamma^2 - \frac{L_\gamma^2}{2Ar^2}\right). \quad (48)$$

so in the vicinity of the naked singularity when r$\longrightarrow 0$, also E_{cm}^2 diverges

$$E_{cm}^2|_{r_h} \longrightarrow \infty. \tag{49}$$

Hence, there is a BSW effect for Compton scattering with naked singularity. We should add, however, that physically this cases is of little interest since neither the photon nor the particle has the chance to escape from a naked singularity.

C. Photon - Photon Collision

As a final example we consider the problem of collision between two photons in the vicinity of our $f(R)$ black hole. The photons follow null geodesics in opposite directions and make head-on collision. In quantum electrodynamics colliding energetic photons can transmute into particles. Since our analysis here is entirely classical we shall refer only to the center of mass energy of the yield without further specification. The center of mass energy of the product satisfies

$$E_{cm}^2 = -(k_1^\mu + k_2^\mu)^2 = -2g_{\mu\nu}k_1^\mu k_2^\nu \tag{50}$$

where k_1 and k_2 correspond to the 4- momenta of respective photons. From the null- geodesic analysis in the $\theta = \frac{\pi}{2}$ plane, we have

$$k_1^\mu = \{\frac{E_1}{A}, \sqrt{E_1^2 - \frac{AL_1^2}{r^2}}, 0, \frac{L_1}{r^2}\} \tag{51}$$

$$k_2^\mu = \{\frac{E_2}{A}, \sqrt{E_2^2 - \frac{AL_2^2}{r^2}}, 0, \frac{L_2}{r^2}\} \tag{52}$$

where E_1 and E_2 are the corresponding energies of different photons. Upon substitution into 50 we obtain

$$\frac{1}{2}E_{cm}^2 = \frac{E_1E_2}{A} - \frac{\kappa}{A}\sqrt{E_1^2 - \frac{AL_1^2}{r^2}}\sqrt{E_2^2 - \frac{AL_2^2}{r^2}} - \frac{L_1L_2}{r^2} \tag{53}$$

in which we inserted $\kappa = \pm 1$ to specify the parallel/anti-parallel propagation of the photons. Expansion near the horizon for $A \to 0$, gives

$$\frac{1}{2}E_{cm}^2 \cong (1-\kappa)\frac{E_1E_2}{A} + O(A) \tag{54}$$

where $O(A) \to 0$ as $A \to 0$. For $\kappa = 1$, which implies two parallel photons moving at the speed of light naturally don't scatter, so we observe no noticable effect. For $\kappa = -1$, however, the photons are moving in opposite directions and inevitably they collide. Their corresponding center of mass energy diverges, consequently for two colliding photons we obtain a BSW effect.

Let us comment that this is a collision of test photons on a given geometry without backreaction effect. On the other hand, exact collision of electromagnetic shock plane waves in Einstein's gravity, as a highly non- linear process [43] [44] is entirely different. As a result of mutual focusing the latter develops null- singularities after the collision process.

V. CONCLUSION

Collision of particles near black hole horizons in Einstein's general relativity, i.e. the BSW effeect, has been considered in detals during recent years . Besides static black holes charged and rotating black holes are investigated as well . In particular, rotational effects was shown from original Penrose process long ago [45], that has significant role in the extraction of energy from the black holes. We extended the idea of BSW to the modified theory known as $f(R)$ gravity. In particular we concentrated on $f(R) = R+2\alpha\sqrt{R-4\Lambda}-2\Lambda$, which arises as an exact, source- free spherically symmetric solution. That is, the external energy- momentum tensor vanishes but yet the curvature makes its own source. We can easily set $\Lambda = 0$, but $\alpha \neq 0$ is an essential parameter of the model so that our model does not have the general relativity limit of $f(R) = R$. For $\alpha < 0$ we have black hole while for $\alpha > 0$ we obtain a naked singularity at $r = 0$. In case of black hole we show the existence of BSW effect near the horizon. A similar result can also be obtained for a Compton-like process between a photon and a particle. Collision of two photons near the horizon also does not change the picture, i.e. we have still the BSW effect. Finally, since our model of $f(R)$ admits such a singularity (for $\alpha > 0$) we consider the collision process near the naked singularity. In conclusion, it remains to be seen whether these ultra- high energetic collisions taking place near black holes has implications as far as dark matter, cosmic rays, energetic jets etc. are concerned.

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