3D gravity with a conformally coupled scalar field: Chern-Simons-like formulation and black hole thermodynamics

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Outline

Motivation

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- It has been proved that the scalar field can exhibit a slow fall-off at infinity in such a way that it contributes to the mass of the black hole.
- It is constructed a Chern-Simons-like description for gravity in 3D with $\Lambda < 0$ and a conformally coupled scalar field.

Metric formulation

The action for gravity in 3D with $\Lambda < 0$ and a conformally coupled scalar field is given by

$$I^{(2)}\left[g_{\mu\nu},\phi\right] = \int d^3x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{16}R\phi^2 - \lambda\phi^6\right),$$

The matter part is conformally invariant.
The stress tensor is traceless so that the scalar curvature is constant $R = -6\ell^{-2}$.

This action can be seen in a different fashion, inspired in what was firstly proposed by *S. Deser, "Scale invariance and gravitational coupling" Annals Phys. (1970)*. It is consider that a conformally invariant action

$$I\left[g_{\mu\nu},\phi\right]=\int dx^{3}\sqrt{-g}\left(-\frac{1}{16}R\phi^{2}-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi-\lambda\phi^{6}\right)\,,$$

is equivalent (up to boundary terms) to

$$I\left[\bar{g}_{\mu\nu}\right]=\frac{1}{2\bar{\kappa}}\int dx^{3}\sqrt{-\bar{g}}\left(\bar{R}-2\bar{\Lambda}\right)\,,$$

after performing

$$\bar{g}_{\mu\nu}(x)=\phi(x)^{4}g_{\mu\nu}(x)\,,$$

and considering

$$\bar{\Lambda}=\bar{\kappa}\lambda\qquad,\qquad\bar{\kappa}=-8\,.$$

Then, we can writte

$$I^{(2)}\left[g_{\mu\nu},\phi\right]=\int d^3x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa}-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{16}R\phi^2-\lambda\phi^6\right),$$

as

$$I^{(2)}\left[g_{\mu\nu},\phi\right]=\frac{1}{2\kappa}\int dx^3\sqrt{-g}\left(R-2\Lambda\right)+\frac{1}{2\bar{\kappa}}\int dx^3\sqrt{-\bar{g}}\left(\bar{R}-2\bar{\Lambda}\right),$$

due to the conformal mapping

$$\bar{g}_{\mu\nu}(x)=\phi(x)^4 g_{\mu\nu}(x)\,,$$

and a siutable choice of the constants $\bar{\Lambda}$ and $\bar{\kappa}$.

First order formulation

It is proposed the following action in terms of 1-form fields and a scalar field ϕ,

$$I^{(1)}\left[e,\omega,\phi\right]=\frac{1}{2\kappa}\int\left(2R^{a}e_{a}-\frac{\Lambda}{3}\epsilon_{abc}e^{a}e^{b}e^{c}\right)+\frac{1}{2\bar{\kappa}}\int\left(2\bar{R}^{a}\bar{e}_{a}-\frac{\bar{\Lambda}}{3}\epsilon_{abc}\bar{e}^{a}\bar{e}^{b}\bar{e}^{c}\right).$$

Here, $e^{a}=e_{\mu}^{a}dx^{\mu}$ is the dreibein, $\omega^{a}=\omega_{\mu}^{a}dx^{\mu}$ is the spin connection and and $R^{a}=d\omega^{a}+\frac{1}{2}\epsilon^{abc}\omega_{b}\omega_{c}$ is the curvature two-form.

The field $\bar{e}^a = \bar{e}^a_\mu dx^\mu$ is a function of the scalar field ϕ and e^a through the relation

$$\boxed{\bar{e}^a_\mu = \phi^2 e^a_\mu \,.}$$

$\bar{\omega}^a$ is a 1-form and $\bar{R}^a$ is a 2-form such that

$$\bar{R}^a = d\bar{\omega}^a + \frac{1}{2}\epsilon^{abc}\bar{\omega}_b\bar{\omega}_c.$$

Equations of motion

The equations of motion of this action are given by

$$\begin{aligned}
\delta e^{a} &: \quad 2R_{a}-\Lambda\epsilon_{abc}e^{b}e^{c}=-\frac{\kappa}{\bar{\kappa}}\phi^{2}\left(2\bar{R}_{a}-\bar{\Lambda}\epsilon_{abc}\bar{e}^{b}\bar{e}^{c}\right),\\
\delta\phi &: \quad \phi e^{a}\left(2\bar{R}_{a}-\bar{\Lambda}\epsilon_{abc}\bar{e}^{b}\bar{e}^{c}\right)=0,\\
\delta\omega^{a} &: \quad de_{a}+\epsilon_{abc}\omega^{b}e^{c}=0,\\
\delta\bar{\omega}^{a} &: \quad d\bar{e}_{a}+\epsilon_{abc}\bar{\omega}^{b}\bar{e}^{c}=0.
\end{aligned}$$

The equivalence among them and second order fomulation ones occurs when

$$g_{\mu\nu}=\eta_{ab}e_{\mu}^{a}e_{\nu}^{b} \quad , \quad R^{ab}{}_{\mu\nu}\left(\omega\right)=e_{\lambda}^{a}e_{\rho}^{b}R^{\lambda\rho}{}_{\mu\nu}\left(\Gamma\right) .$$

At the level of the action the equivalence is also obtained provided a vanishing torsion.

Action

It is possible to give a description of gravity in the presence of a conformally coupled scalar field by means of a Chern-Simons-like picture,

$$I^{(1)}\left[A\left[e,\omega\right],\bar{A}\left[e,\phi,\bar{\omega}\right]\right]=I_{CS}\left[A\right]+I_{CS}\left[\bar{A}\right]\ ,$$

$$I_{CS}\left[A\right]=\frac{k}{4\pi}\int\limits_{\Sigma}\left\langle AdA+\frac{2}{3}A^{3}\right\rangle,\quad I_{CS}\left[\bar{A}\right]=\frac{\bar{k}}{4\pi}\int\limits_{\Sigma}\left\langle \bar{A}d\bar{A}+\frac{2}{3}\bar{A}^{3}\right\rangle.$$

where $k=2\pi/\kappa$ and $\bar{k}=2\pi/\bar{\kappa}$.

The gauge connection A is valued in the AdS algebra,

$$A = \omega^a J_a + e^a P_a \,,$$

because $\Lambda < 0$. The gauge connection $\bar{A}$ is written as

$$\bar{A}(e, \phi, \bar{\omega}) = \bar{\omega}^a \bar{J}_a + \bar{e}^a \bar{P}_a \,,$$

and is defined according to the value of the self-interacting parameter $\bar{\Lambda} = -8\lambda$. It is considered

$$\boxed{\bar{e}^a_\mu = \phi^2 e^a_\mu \,,}$$

In general, the generators $\bar{J}_a$ and $\bar{P}_a$ for any value of $\bar{\Lambda} = \pm 1/\bar{\ell}^2$ or $\bar{\Lambda} = 0$ (and also J_a , P_a) share the algebra,

$$\left[\bar{J}_a, \bar{J}_b\right] = \epsilon_{abc}\bar{J}^c \quad , \quad \left[\bar{J}_a, \bar{P}_b\right] = \epsilon_{abc}\bar{P}^c \quad , \quad \left[\bar{P}_a, \bar{P}_b\right] = -\bar{\Lambda}\epsilon_{abc}\bar{J}^c \, ,$$

and the invariant bilinear form is

$$\left\langle \bar{J}_a, \bar{J}_a \right\rangle = \left\langle \bar{P}_a, \bar{P}_a \right\rangle = 0, \qquad \left\langle \bar{J}_a, \bar{P}_a \right\rangle = \eta_{ab}.$$

The variation of the Chern-Simons-like action with respect to the dynamical fields $(\phi, e, \omega, \bar{\omega})$ leads to the same equations of motion found in the previous fomulation.

Black hole thermodynamics

The thermodynamical properties will be discussed by means of the Euclidean approach, where $\tau = -it$ and

$$0 < \tau < \beta \quad , \quad \beta = T^{-1}.$$

The variation of the Hamiltonian Euclidean action is given by

$$\delta I_E = \delta B(\infty) - \delta B(r_+)$$

once the constraints are fulfilled and considering stationary configurations. Here B is a boundary term added in order to ensure well-defined functional derivatives.

In general, the variation of Hamiltonian Euclidean action on-shell reads as

$$\begin{aligned} \delta I_E &= \delta B\left(\infty\right) - \delta B\left(r_+\right) && (1) \\ &= \beta\delta M + \beta\Omega\delta J - \delta S && (2) \end{aligned}$$

where

$$\delta B = -\frac{k}{2\pi}\int\limits_{\partial\Sigma}\left\langle A_\tau \delta A_\theta\right\rangle d\theta d\tau - \frac{k}{2\pi}\int\limits_{\partial\Sigma}\left\langle \bar{A}_\tau \delta \bar{A}_\theta\right\rangle d\theta d\tau\,.$$

Despite of the fact that this is not in a Chern-Simons theory, we have a Chern-Simons action with the “usual” boundary terms.

Hairy Black hole

The theory admits hairy black hole solutions.
The rotating version of the solution presented by *M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, "Black holes and asymptotics of 2+1 gravity coupled to a scalar field," Phys. Rev. D (2002)*. Its line element reads as

$$ds^2 = -G(r)^2 dt^2 + F(r)^{-2} dr^2 + H(r)^2 \left(d\theta + N^\theta(r)\, dt\right)^2 ,$$

Where

$$
\begin{aligned}
G(r)^2 &= \frac{r^2 F(r)^2}{H(r)^2}, \\
F(r)^2 &= \frac{r^2}{\ell^2} - \frac{(1-\alpha)}{\ell^2}\left(\frac{2b^3}{r} + 3b^2\right), \\
H(r)^2 &= r^2 + \frac{(1-\alpha)\,\omega^2}{1-\omega^2}\left(\frac{2b^3}{r} + 3b^2\right), \\
N^\theta(r) &= N^\theta(\infty) + \frac{(1-\alpha)\,\omega}{\ell\,(1-\omega^2)\,H(r)^2}\left(\frac{2b^3}{r} + 3b^2\right).
\end{aligned}
$$

- The event horizon is located at $r_+ = bx_+$, where

$$x_+ = (1-\alpha)^{1/3}\left[\left(1+\sqrt{\alpha}\right)^{1/3} + \left(1-\sqrt{\alpha}\right)^{1/3}\right],$$

provided by $\alpha < 1$ (otherwise there is no horizon) and $b > 0$.

- The scalar field is

$$\phi(r) = \sqrt{\frac{8b}{\kappa(r+b)}}.$$

- The metric is singular at the origin $r = 0$ and its asymptotic behavior fullfills the Brown-Henneaux boundary conditions.

Euclidean action computation

Evaluating the hairy rotating black hole in

$$\delta B=-\frac{k}{2\pi}\int\limits_{\partial\Sigma}\left\langle A_{\tau}\delta A_{\theta}\right\rangle d\theta d\tau-\frac{k}{2\pi}\int\limits_{\partial\Sigma}\left\langle \bar{A}_{\tau}\delta\bar{A}_{\theta}\right\rangle d\theta d\tau\,.$$

we get

$$\delta B=\beta\delta\left(\frac{3\pi(1-\alpha)b^{2}\left(1+\omega^{2}\right)}{\ell^{2}\kappa\left(1-\omega^{2}\right)}\right)+\beta N^{\theta}\left(\infty\right)\delta\left(\frac{6\pi(1-\alpha)b^{2}\omega}{\ell\kappa\left(1-\omega^{2}\right)}\right).$$

Considering that the surface term at infinity is

$$\delta B\left(\infty\right)=\beta\delta M+\beta N^{\theta}\left(\infty\right)\delta J,$$

this immediately yields

$$M=\frac{3\pi\left(1-\alpha\right)b^{2}\left(1+\omega^{2}\right)}{\kappa\ell^{2}\left(1-\omega^{2}\right)},\qquad J=\frac{6\pi\left(1-\alpha\right)b^{2}\omega}{\kappa\ell\left(1-\omega^{2}\right)}.$$

- For obtainig the entropy from $\delta B(r_+)$, the fields have to be smooth at the horizon. The regularity of the connections A and $\bar{A}$ is obtained considering the topology of the euclidean black hole (solid torus).
- The regularity condition dictates that the holonomy along the thermal cycle $0 \leq \tau < \beta$ at the horizon must be trivial at r_+

$$\mathcal{H} = \exp\left[\int A_\mu dx^\mu\right]_{r_+} = \exp\left[\beta A_\tau\right]_{r_+} = -\mathbb{I},$$

- We consider that the Chern-Simons connection of Euclidean three-dimensional gravity with a negative cosmological constant is spanned on $so(3,1)$, which is isomorphic to $sl(2,\mathbb{C})$ where

$$A = \left(\omega^a + \frac{i}{\ell} e^a\right) \hat{J}_a$$

and $\hat{J}$ are the generators of $sl(2,\mathbb{C})$.

.

- The regularity condition for the gravitational part must be imposed on the holonomy $\mathcal{H}$ along $0 \leq \tau < \beta$ in the torus.
This implies that it has to be trivial on the event horizon,

$$\mathcal{H} = \exp\left[\int A_\mu dx^\mu\right]_{r_+} = \exp\left[\beta A_\tau\right]_{r_+} = -\mathbb{I},$$

$$\mathrm{tr}\left[(\beta A_\tau)^2\right] + 2\pi^2 = 0.$$

- For the matter part, there are three diferet cases depending the value of $\bar{\Lambda}$.
  The calculation of the holonomies can be sumarized (and simplified) following an alternative method presented by *J. Matulich, A. Perez, D. Tempo and R. Troncoso, "Higher spin extension of cosmological spacetimes in 3D: asymptotically flat behaviour with chemical potentials and thermodynamics," JHEP (2015)*
- When $\bar{\Lambda} = 0$, it is mandatory to do this procedure due to the lack of a suitable matrix representation from which one can recover the invariant bilinear form.

In general, the holonomy along the thermal cycle is given by

$$\begin{aligned}\mathcal{H} &= \exp\left[\beta \bar{A}_\tau\right]_{r_+} \\ &= \exp\left[\beta\left(\bar{e}^a_\tau P_a + \bar{\omega}^a_\tau J_a\right)\right]_{r_+} \\ &= -\mathbb{I},\end{aligned}$$

The kind of Lie algebras we are treating are

$$[J,J] \sim J \quad , \quad [J,P] \sim P \quad , \qquad [P,P] \sim -\Lambda J \, ,$$

- The group element is

$$g = g_J g_P = e^{\rho^b J_b} e^{p^a P_a}.$$

- The method requires to find a suitable gauge transformation g_P that allows to gauge away the τ- components of the dreibein $\bar{e}_\tau$, so we can partially fix the chemical potentials.
- The remaining conditions can then be implemented through the diagonalization of the holonomy matrix associated to the spin connection along the thermal circle; i.e., without the need of finding the explicit form of g_J .

For our connection, there is no need to find g_P because

$$\begin{aligned}\bar{A}_\tau &= \frac{8(2+3x)(1-\alpha)c^2(N^\theta(\infty)\ell+\omega)}{(1+x)\ell\kappa\left(-1+\omega^2\right)\sqrt{xc^2\left(x^3+\frac{(2+3x)(-1+\alpha)\omega^2}{1-\omega^2}\right)}}P_2 + \\ &\quad \frac{3(1+x)(1-\alpha)c^2(1+N\theta(\infty)\ell\omega)}{\ell^2\left(-1+\omega^2\right)\sqrt{xc^2\left(x^3+\frac{(2+3x)(-1+\alpha)\omega^2}{1-\omega^2}\right)}}J_2\end{aligned}$$

fixes immediately $N^\theta(\infty)$ as

$$N^\theta(\infty) = -\frac{\omega}{\ell}.$$

- Then the regularity condition is stated as

$$\mathrm{tr}\left[(\beta\bar{\omega}_\tau)^2\right] + 2\pi^2 = 0, \tag{3}$$

which is solved for β, yielding

$$\beta = \pm\frac{2\pi x_+^2\ell^2}{3b\,(1+x_+)\,(1-\alpha)\,\sqrt{1-\omega^2}},$$

- The chemical potentials are given by the values that solve the holonomies for A and $\bar{A}$ simultaneously

$$\beta = \frac{2\pi x_+^2 \ell^2}{3b\left(1+x_+\right)\left(1-\alpha\right)\sqrt{1-\omega^2}}, \qquad N^{\theta}\left(\infty\right) = -\frac{\omega}{\ell}.$$

The above result admits $\alpha = 0$.

We verify the general formula for the black hole entropy in terms of the on-shell holonomies proposed in the work of *C. Bunster, M. Henneaux, A. Perez, D. Tempo and R. Troncoso, "Generalized Black Holes in Three-dimensional Spacetime," JHEP, (2014)* .
Here we have to consider the contribution of both gauge connections $A, \bar{A}$ such that

$$
\begin{aligned}
S &= -2k\left[\mathrm{tr}\left(\beta A_\tau A_\theta\right)\right]_{\text{on-shell}} - 2\bar{k}\left[\mathrm{tr}\left(\beta \bar{A}_\tau \bar{A}_\theta\right)\right]_{\text{on-shell}}, \\
&= \frac{4\pi^2 b x_+^2}{\kappa\left(1+x_+\right)\sqrt{1-\omega^2}},
\end{aligned}
$$

which gives the same result that is obtained with the modified Bekenstein-Hawking entropy,

$$
S = \left(1 - \frac{\kappa\phi(r_+)^2}{8}\right)\frac{A}{4G}. \tag{4}
$$

Resume

- We review the standard metric formulation for a scalar field conformally coupled to gravity in three dimensions. A Chern-Simons-like description for this system is given.
- The charges are obtained through the 'usual" boundary term of the Chern-Simons action, by performing the thermodynamic analysis of the system.
- Regularity conditions at the horizon are provided when the holonomies along the thermal cycle are computed. This procedure establishes the value of the chemical potentials and allows to obtain the entropy of the black hole.

Thanks!