Group Theory

in
Particle Physics

DRAFT

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Lecture 2+1

Lie Algebras in Particle Physics
Howard Georgi
(old and new edition)

Lectures on Group Theory for Physicists
A.P. Balachandran, C.G. Trahern

Groups, Representation and Physics
H. F. Jones

Group Theory
Predrag Cvitanovic

Group Representation Theory for Physicists
J-Q. Chen, J. Ping, F. Wang

Why Group Theory?

Group theory is the study of symmetry. It is 'a super-mathematics in which the operations are as unknown as the quantities they operate on'(Arthur S. Eddington).

Group theory was introduced into mathematics in 1810, group representation - in 1920. Applications in many fields of physics $\rightarrow$ rotation group, electron spin, isotopic spin ...
Starting from 1950 a new direction in particle physics: in order to classify elementary particles a search for a higher symmetry group to include both isotopic spin and hypercharge. This led to *The eightfold way (called also unitary spin, flavor SU(3))* - where kaons and pions are members of the same multiplet.

Symmetry plays much more fundamental role in the description of interactions - all fundamental interactions are described by non-Abelian gauge theories. Standard Model = SU(2)xU(1)xSU(3), possible unification in SU(5).

The style of lecture can be formal, however here we follow H. Georgi - who seems to agree with the statement by A. N. Kolmogorov: *"It is better to be right than to be rigorous"*

Content

I. Finite Groups

Here we introduce all basic definitions which applied also to infinite groups.

A group G - a set of elements with a rule (multiplication, composition..) for assigning to every (ordered) pair of elements a third element, satisfying

- If $f, g \in$ G then $h = fg \in G$ (closure)
- For $f, g, h \in G$ $f(gh) = (fg)h$ (associativity)
- There is an identity element e, such that for all $f \in G$, $ef = fe = f$
- Every element $f \in G$ has an inverse, f^{-1}, such that $ff^{-1} = f^{-1}f = e$. (Check that the inverse is unique: if $e = f^{-1}f = \bar{f}f \to \bar{f} = f^{-1}$, since $ef^{-1} = \bar{f}ff^{-1} = e$.)

An **Abelian group** – if the multiplication law is commutative, so

$$g_1 g_2 = g_2 g_1.$$

Multiplication table

If the group elements are discrete, one can write the multiplication table specifying $g_1 g_2$ for each g_1, g_2 (Note order of product (i,j))

	e	g_1	g_2	$\cdots$
e	e	g_1	g_2	$\cdots$
g_1	g_1	$g_1 g_1$	$g_1 g_2$	$\cdots$
g_2	g_2	$g_2 g_1$	$g_2 g_2$	$\cdots$
..	..	..	..	$\cdots$

Order of a group

A group G is **finite** if it has finite number of elements, otherwise it is **infinite**. Elements of infinite **continuous** groups may be labeled by continuous parameters.

A number of elements of G – an **order** of G.

For a finite group G, for each element $c \in G$: $c^n = e$, n – positive integer. The smallest n for which $c^n = e \to$ an **order of an element a**.

Examples of groups

•All the integers under addition constitute an infinite discrete group.
•The integers modulo n (addition mod n)
$0, 1, 2.., n-1 \to Z_n$ group (a=b if a-b=m n (n,m-integer))
•The C_n - the group of rotation of a regular polygon with n *directed* sides about an axis through the centre and normal to the plane of the figure. Rotation can be considered as a symmetry transformation of the polygon into itself through a rotation by an angle $2\pi r/n$, $r = 0, 1, 2, ..., n-1$

C_n^r - el. of the group: $C_n^r = (c)^r, c$isrotationby2π/n
C_n is called a cyclic group of order n
since $C_n = \{c^1, c^2, ...c^{n-1}, c^n(= e)\}$.

Group is generated by el. $c \to$
other name: $C_n = gp(c)$, $c^n = e$.

C_n is Abelian.

C_n isomorphic to Z_n (multiplying elements of C_n equivalent to adding exponents of c modulo n).
isomorphism: $c^r \leftrightarrow r$

Permutation group S_3

Permutations of:

three objects a, b, c in three positions : $1, 2, 3$

Elements of permutation group:

$$(), (12), (23), (13), (123), (321)$$

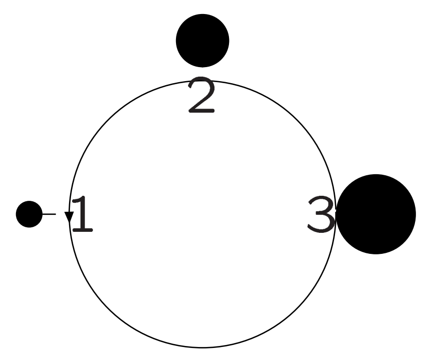
- ( ) - do nothing, so $[a|b|c]$ remains $[a|b|c]$
- (12) - interchange of the objects in positions 1 and 2, so $[a|b|c] \to [b|a|c]$
- (23) - similarly as above
- (13) - similarly as above
- (123)- take object in position 1 and put it into 2, take object in position 2 and put it into 3, object in position 3 into 1, $[a|b|c] \to [c|a|b]$
- (321) - inverse to above..

Group is of order n!.
Multiplication = successive application of permutation
Other name: symmetric group

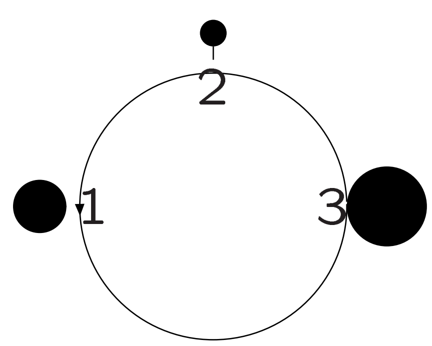
S_3 cont.

Multiplication law: AB -
convention: first B then A

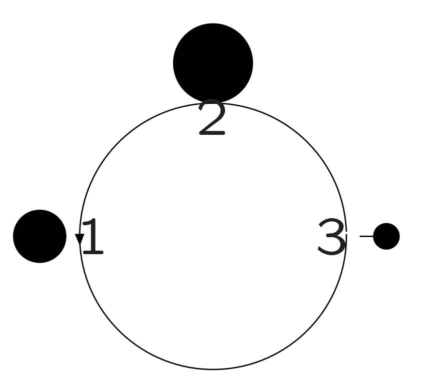
Check if a group: eg. (23)(12)=(321) ...
Start with



First - (12)



then - (23)



and the final results is (321)

Check if Abelian: eg. (12)(23)=(123)$\neq$(321).

Cycle notation

Other notation : An object in the *position* in the upper row is changed to the *position* in the lower row in the same column (position of columns not relevant):

$$(23) = \begin{pmatrix} 1 & 2 & 3 & \ldots \\ 1 & 3 & 2 & \ldots \end{pmatrix},$$

Example AB=C:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

whereas BA=?

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

Moreover, we can simplify notation following what is going on with eg. element “1” to the “end” ie. when we get back to the position 1

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & |3 \\ 2 & 4 & 1 & |3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$= (124)(3)$ and (3) can be omitted.

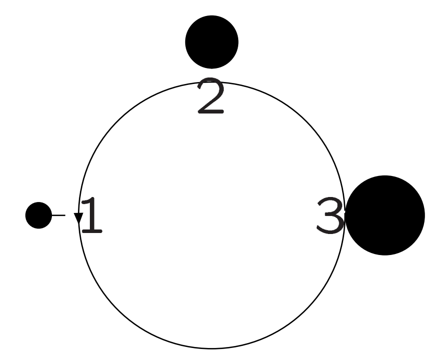
Note, that the bottom row gives no information and can be removed $\rightarrow$ cycle notation.
Permutation can be decomposed into disjoints cycles.

Example

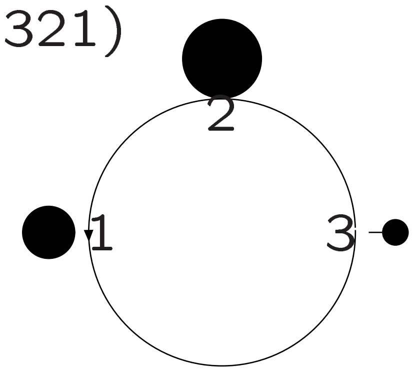
Check for S_3:

$$(23)(132) = (12)$$

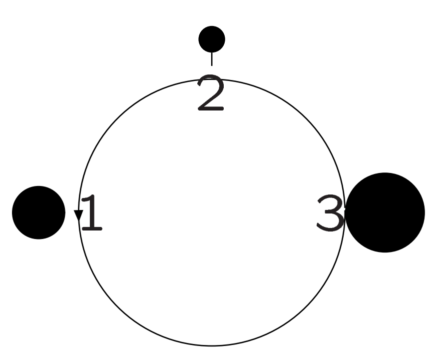
$$(132)(23) = (13)$$



First do (132)=(321)



then (23)



so, the result (12)

$\boxed{\text{Counter examples of groups}}$

•Integers (multiplication) – why not a group?
•Real numbers (multiplication)– why not a group?

Give more examples!

PROBLEM

PROBLEM 1: Write the permutation in cycle notation

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 8 & 5 & 7 & 2 & 3 \end{pmatrix}.$$

What is the order of this permutation?

Homomorphism and isomorphism

A homomorphism in general is a mapping f from one set A to another set B preserving some structure...

Each el. a of A is mapped *into* a unique image, a single el. $b = f(a)$ of B.

The reverse is not true in general:
a given el. $b \in B$ may be the image of many el. of A or may not be image of any el. of A.
(Do mapping *onto* to avoid this case).

(image Imf - all el. of B which are maps of el. of A.)

To be a homomorphism - a structure has to be preserved.

Here we deal with group multiplication, so
the structure to preserve is the group multiplication. The map $f : A \to B$ is a *group homomorphism* if,
for all a_1, a_2 in A,

$$f(a_1 a_2) = f(a_1) f(a_2)$$

Kernel of the homomorphism $Kerf$ consists of those el. of A which are mapped onto the identity el. of B.

Isomorphism Theorem: If $f : G \to G'$ is homomorphism of G into G', with kernel K, then

$$Im(f) \cong G/K$$

($\cong$ denotes isomorphism) G/K - set of left cosets (of el. $g \in G$ - gH, H subgroup of G) $f(g) \leftrightarrow gK$

Note: G' can be G itself (endomorphism and automorphism)

Representations

A representation of the abstract group G - is a mapping D onto a set of linear operators acting in the linear (vector) space V

$$D(g) : V \to V,$$

satisfying the group multiplication

- D(e)=1 - identity operator
- D(g_1)D(g_2)=D($g_1 g_2$)

Dimension of representation – the dimension of the space in which the linear operators act.

Representations - powerful idea since they live in linear space, and we can represent states in a convenient way by making linear transformations..

Example: The additive group of integers

Integers form a group under addition

$$fg \to f + g$$

One of possible representations is:

$$D(n) = e^{in\theta}$$

Is it a representation? Check...

$$e^{in\theta}e^{im\theta} = e^{i(n+m)\theta}$$

Representation of a rotation group C_3

From definition it is related to the rotations with angles: $2\pi r/3,\ r = 0, 1, 2$.
The rotation through an angle θ about the z axis can be represented by the matrix:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, we get the 3 dim representation of C_3:
$D(e) = R(0), D(c) = R(2\pi/3), D(b = c^2) = R(4\pi/3)$

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ D(c) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D(b) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These matrices provide a concrete representation of the *abstract* group C_3 as well.

Representations of an abstract group C_3

C_3 with elements: e,c,b $\{c^1, c^2 = b, c^3 = e\}$
A multiplicative table

	e	c	b
e	e	c	b
c	c	b	e
b	b	e	c

One of possible representation (1 dim !):

$$D(e) = 1, \quad D(c) = e^{2\pi i/3}, \; D(b) = e^{4\pi i/3}.$$

Another representation (3 dim !)

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D(c) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This is a **regular** representation -
constructed directly from multiplication table
(more below)

The regular representation of C_3

Use the group elements as an orthonormal basis for a vector space, eg. for a group C_3

$$|e_1> \equiv |e>, |e_2> \equiv |c>, |e_3> \equiv |b>$$

For each g_j: $\boxed{D(g_1)|g_2> = |g_1 g_2>}$

$$\begin{aligned} D(e)|e> &= |ee> = |e> \\ D(e)|c> &= |ec> = |c> \\ D(e)|b> &= |eb> = |b> \\ \rightarrow D(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{1}$$

$$\begin{aligned} D(c)|e> &= |ce> = |c> \\ D(c)|c> &= |cc> = |b> \\ D(c)|b> &= |cb> = |e> \\ &\rightarrow D(c) = () \end{aligned} \tag{2}$$

$$\begin{aligned} D(b)|e> &= |be> = |b> \\ D(b)|c> &= |bc> = |e> \\ D(b)|b> &= |bb> = |c> \\ &\rightarrow D(b) = () \end{aligned} \tag{3}$$

The regular representation

where

$$|e_1>=\begin{pmatrix}1\\0\\0\end{pmatrix}, |e_2>=\begin{pmatrix}0\\1\\0\end{pmatrix}, |e_3>=\begin{pmatrix}0\\0\\1\end{pmatrix} \quad (4)$$

The dimension of the regular representation
= the order of the group !

Linear operators $\leftrightarrow$ matrices

$$[D(g)]_{ij} =< e_i|D(g)|e_j >$$

Check that:

$$[D(g_1g_2)]_{ij} = \Sigma_k[D(g_1)]_{ik}[D(g_2)]_{kj}.$$

The regular representation for any finite group can be constructed this way.

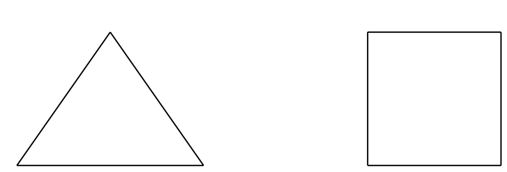
Various

•$C_2 = \{c, e(c^2)\}$ - with c rotation through π. Write the multiplicative table and find representations of dimension 1,2

•S_3 - write the multiplicative table

•The dihedral group D_n (the symmetry group of rotation of a regular polygon with n *undirected* sides). Order $2n$.

D_3 contains $C_3 = \text{gp}(c)$ and additional 3 rotations through π around 3 lines dividing a polygon into two equal parts, called them $b_1, b_2, b_3 (b_i^2 = e)$. $D_3 = \text{gp}(c, b)$. Check that $(bc)^2 = e$.



Cayley's theorem

T: Every finite group of order n can be considered as (is isomorphic to) a subgroup of S_n.

Observe that for a finite group G multiplication of all elements $\{a_i\}$ by a given element g simply permutes them

$$\{ga_1, ga_2, ..., ga_n\} = \{a_{p_1}, a_{p_2}, ..., ap_n\}$$

therefore to each element g of the group there corresponds a permutation $\Pi(g)$ of S_n :

$$g \to \Pi(g) = \begin{pmatrix} 1 & 2 & ... & n \\ p_1 & p_2 & ... & p_n \end{pmatrix}.$$

There are n elements ga_i all distinct, since:
if not, ie. $ga_j = ga_k \to a_j = a_k$ ($g^{-1}ga_j = g^{-1}ga_k$) however all $\{a_i\}$ are distinct. And vice versa if $ga_i = g'a_i \to g = g'$ (by multiplying from right by a_j^{-1}.) so given permutation $\Pi(g)$ can not arises from other g'.

The 1:1 correspondence $g \leftrightarrow \Pi(g)$; it respects the group structure of the group G and of the group S_n. There are n distinct permutations $\Pi(g_j)$, while there are $n!$ elements in S_n - so we deal here only with a subgroup of S_n.

Alternating group

The permutations $\Pi(g)$ can be read off from the multiplicative table of G.
They form an alternating group.

PROBLEM 2:
For C_3 write $\Pi(e)$, $\Pi(c)$, $\Pi(b)$ - an alternating group A_3.

Similarity and equivalent representations

Some states (basis) are more convenient than others - use linear transformation (invertible) $S : V \to V$ such that

$$SS^{-1} = S^{-1}S = 1$$

$$|e_i> \to |e_i'> = S^{-1}|e_i>$$

This produces similarity transformation on the linear operators

$$D(g) \to D'(g) = S^{-1}D(g)S$$

$[D(g)]_{ij} = < e_i|D(g)|e_j >$
$= < e_i|SS^{-1}D(g)SS^{-1}|e_j > = < e_i'|D'(g)|e_j' >$

D' is a representation if D is $\to$
equivalent representations (differ only by a trivial change of basis)

$\boxed{\text{Unitary representations}}$

Unitary operators if $\mathcal{O}^\dagger = \mathcal{O}^{-1}$
they are particularly important in physics

Representation unitary if all the $D(g)$ unitary.

Reducibility and invariant subspace

A representation **reducible** if there is an **invariant** subspace, which means that the result of the action of any $D(g)$ on any vector in the subspace remains in it.

Example: representation of the rotation group C_3 with matrices with element 1 in the (3,3) position. This "block diagonal" form is due to the invariance of z under C_3. The vector space $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ splits up into two decoupled spaces:

$$\vec{r} = \vec{u} + \vec{v},\ \vec{u} = x\vec{i} + y\vec{j},\ \vec{v} = z\vec{k}.$$

Effectively - decomposition into separate representations, $D^{(2)}$ acting on $\vec{u}$ and $D^{(1)}$ acting on $\vec{v}$. They are independent:

$$RR' = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$R(c) = D(c) = D^{(2)}(c) \oplus D^{(1)}(c)$$

Reducible representation

Reducible representation corresponds to

$$D = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

Let D- $m \times n$ representation, with the A m-dim and B n-dim matrices

$$D(gg') = D(g)D(g') \tag{6}$$

$$= \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix} \begin{pmatrix} A(g') & C(g') \\ 0 & B(g') \end{pmatrix}$$

$$= \begin{pmatrix} A(g)A(g') & A(g)C(g') + C(g)B(g') \\ 0 & B(g)B(g') \end{pmatrix}$$

so,

$$A(gg') = A(g)A(g'), \;\; B(gg') = B(g)B(g')$$

and it is clear that A and B are representations

Projection operator

Projector operator P:
projecting onto the invariant subspace V_0:
$PV = V_0$, D$V_0 = V_0$, $P^2 = P$

For reducible representation for each $g \in G$:

$$PD(g)P = D(g)P = P$$

in action on V.

A representation is irreducible (IRR) if not reducible.

PROBLEMS

PROBLEM 3:

Consider regular representation of C_3. Since we can have one-dim. representation of C_3 it is clear that the three-dim. representation should be reducible. Indeed, using

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

check that:

$D(e)P = P, \quad D(c)P = P, \quad D(b)P = P.$

PROBLEM 4:

Calculate the eigenvalues and eigenvectors of the matrix

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Construct the similarity matrix S.

Completely reducible representation

Completely reducible representation -
if equivalent to a representation whose matrix has a **block diagonal** form:

$$\begin{pmatrix} D_1(g) & 0 & .. \\ 0 & D_2(g) & .. \\ 0 & 0 & ... \end{pmatrix}$$

where all $D_j(g)$ are irreducible. The vector space breaks into orthogonal subspaces..

Block diagonal form is said to be the **direct sum** of subrepresentations, symbolically

$$D_1 \oplus D_2 \oplus \cdots$$

A completely reducible representation can be decomposed into a direct sum of irreducible representations

Once more on C_3

Another representation of C_3 which is completely reducible, (3 dim !)

$$D'(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D'(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

$$D'(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

with $\omega = e^{2\pi i/3}$.

PROBLEM 5: Use the similarity transformation

$$S = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

to get this representation from the regular representation.

Example: two-dimensional representation of S_3

Elements of S_3:
$e = (), a_1 = (123), a_2 = (321),$
$a_3 = (12), a_4 = (23), a_5 = (31)$

so, $a_1(a_2)$ - cyclic (anticyclic) permutation,
other a_i realized interchanges
S_3 is non-Abelian, so it is obvious that representations should be more than 1 dim.
Unitary IRR representation of S_3

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D(a_1) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$D(a_2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, D(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D(a_4) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$D(a_5) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

Show that this is representation.

Example: Three-dimensional representation of S_3

There is a 3-dim representation of S_3:

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(a_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$D(a_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, D(a_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D(a_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$D(a_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Is it a regular one?

Example: two-dim repr.of the additive group of integers

Two-dimensional representation of the additive group of integers:

$$D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Reducible, however not completely reducible (and not unitary)– Check!

$D(x)P = P$ with projection op. $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Projection into remaining part by $1 - P$
$D(x)(1 - P) \neq (1 - P)$

Useful theorems

T1. Every representation of a finite group is equivalent to a unitary representation

Let $D(g)$ is a representation of a finite group G. Introducing the operator

$$Y = \Sigma_{g \in G} D(g)^{\dagger} D(g)$$

which is hermitian and positive semidefinite (eigenvalues nonnegative). Diagonalizing it we get y with nonnegative eigenvalues

$$Y = U^{-1} y U$$

with

$$y = \begin{pmatrix} d_1 & 0 & \ldots \\ 0 & d_2 & \ldots \\ 0 & 0 & \ldots \\ . & . & . \end{pmatrix}$$

One can show that that all d_j is positive.
If not - then there is a eigenvector x: $Sx = 0x$, and

$$x^{\dagger} Y x = 0 = \Sigma_{g \in G} [x^{\dagger} D(g)^{\dagger} D(g) x] =$$

$$\Sigma_{g \in G} [D(g) x]^{\dagger} D(g) x] = \Sigma_{g \in G} |D(g) x|^2$$

so for all $g, D(g)$ is zero, however $D(e) = 1$, not 0.

T1 continuation

so, we can construct hermitian and invertible

$$X = Y^{1/2} = U^{-1} \begin{pmatrix} \sqrt{d_1} & 0 & \ldots \\ 0 & \sqrt{d_2} & \ldots \\ 0 & 0 & \ldots \\ . & . & . \end{pmatrix} U$$

Define

$$D'(g) = XD(g)X^{-1}$$

and

$$D'(g)^\dagger D'(g) = X^{-1}D(g)^\dagger Y D(g) X^{-1} = 1$$

since

$$D(g)^\dagger Y D(g) = ..Y = X^2$$

$\boxed{\text{T2}}$

T2. Every representation of a finite group is completely reducible

Proof it.

Subgroups

A subgroup H of a group G is a subset of G which itself forms a group under the composition law of G.

•Use a subgroup H to divide elements of group into subsets called cosets: right- and left-cosets (Hg and gH) for an element g. Every el. of G must belong to one and only one coset.

For finite G the order of H is a factor of order of the G coset-space G/H, with cosets which consist of elements of the space.

•Subgroup H of G - invariant (normal) subgroup if $gH = Hg$

Group is simple -if has no inv.subgroup besides the identity and itself

•For invariant H - coset space is a group called the factor group of G by H - G/H.

•The center of a group - set of all el. of G that commute with all el. of G.

•The condition for an invariant subgroup $gH = Hg$ or $gHg^{-1} = H$ for each g$\in$ G. We can consider sets stisfying the same condition

$$gXg^{-1} = X$$

- conjugacy classes; one-to-one correspondence between these classes and IRR.

Schur's lemma

T: If for two inequivalent irreducible representations $D_1(g)A = AD_2(g)$ for each $g \in G$, then A=0.

T: If for a finite dim. irreducible representations $D(g)A = AD(g)$ for each $g \in G$, then $A \propto I$

Note that finite dimensionality is important here.

So, the basis states of IRR are essentially unique - once D is fixed there is no further freedom to make nontrivial similarity transformation (but a multiplying by a phase factor all states)

$$A^{-1}D(g)A = D(g)$$

for all $g \in G$, and $A \propto I$.

Quantum mechanics

Symmetry group: $g \to D(g)$ for all $g \in G$.
Matrix of any operator $\mathcal{O}$, corresponding to an observable that is invariant under the symmetry transf., $< a, j, x|\mathcal{O}|b, k, y >$, behaves as matrix A:

$$< a, j, x|D(g)|b, k, y >= \delta_{ab}\delta_{xy}[D_a(g)]_{jk},\ j = 1, ...n_a$$

a, b - different representations;
if the same repr. occurs more than once we need x, y to represent some physical parameters;
j to labels the states within the representation.

Symmetry transf. (unitary D)

$$|\alpha >\to D(g)|\alpha >,$$

however since a symmetry operators
$< |\mathcal{O}| >=<' |D(g)\mathcal{O}D(g)^\dagger|' >$,
and $D(g)\mathcal{O}D^{-1}(g) = \mathcal{O} \to [\mathcal{O}, D(g)] = 0$
and

$$< a, j, x|\mathcal{O}|b, k, y >= f_a(x, y)\delta_{ab}\delta_{jk}$$

all physics in $f_a(x, y)$!.

$$\boxed{N = \Sigma_i n_i^2}$$

The order of the group N is equal to the sum of the squares of the dim. of IRR n_i^2

Character theory

Trace of the matrix $TrD(g)$

$$\chi_D(g) \equiv TrD(g) = \Sigma_i [D(g)]_{ii}$$

•All equivalence repr. have the same characters (Tr(ABC)=Tr(BCA)etc); inequivalent representation $\rightarrow$ different characters

$$\frac{1}{N}\Sigma_{g\in G}\ \chi_{D_a}(g)\chi^*_{D_b}(g) = \delta_{ab}$$

•The same characters on conjugacy classes - cc; characters of the independent IRR form a complete, orthonormal basis set for the functions that are constant on cc. Therefore number of IRR is equal to the number of cc.

•Each IRR appears in the regular repr. a number of times equal to its dimension
•Explicitly decompose the reducible repr. into IRR components using a projector op.

$$P_a = \frac{n_a}{N}\Sigma_{g\in G}\ \chi^*_{D_a(g)} D(g)$$

Note: P op. in the initial bases, not diagonal one.

Example - S_n

For n - elements S_n, each element has k_j j-cycles,

$$\Sigma_{j=1}^{n} j k_j = n$$

PROBLEMS

PROBLEM 6

Show that the 3 dim representation of the group of rotation about z axis is completely reducible for axis: $-(x+iy)/\sqrt{2}, z, (x-iy)/\sqrt{2}$.

PROBLEM 7

Verify that

$$D(c) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

generates a 2 dim representation of C_3. Show that the representation is IRR over the field of real numbers.

Transformation groups

The permutation group is an example of transformation group of a physical system.

If transformation corresponds to a symmetry of a physical system then such transformation takes the Hilbert space $\mathcal{H}$ into an equivalent one.

So, for each group element g there is an unitary operator D(g) mapping Hilbert space $\mathcal{H} \to \mathcal{H}'$. These unitary operators form a representation of the transformation (or symmetry) group.

In QM - transf. of a system is described by unitary operator in the Hilbert space (antiunitary operators are rare).

Because the transformed states have the same energy as the original ones, $D(g)$ commute with Hamiltonian H.

This means: we can always choose the energy eigenstates to build IRR of the symmetry group.

Eigenstates

We can divide up the Hilbert space into subspaces with diffrent eigenvalues of hamiltonian H. In each subspace - representation of the symmetry group because $D(g)$, the group representation on the full Hilbert space, cannot change hamiltonian eigenvalue

All of the IRR representations of a finite Abelian group are 1 dimensional.

Parity in QM

Parity transformation is an operation of reflection in a mirror. Reflecting twice gives nothing. If p is a group element representing the parity reflection then $p^2 = e$; p together with $e \to C_2$ group, with table

	e	p
e	e	p
p	p	e

Representation? Only two IRR (1 dim):
trivial one $D(e) = 1,\ D(p) = 1$
second one $D(e) = 1,\ D(p) = -1$.
Others representations are completely reducible, so Hilbert space of any parity invariant system can be decomposed into states that behave like IRR, on which $D(p) = 1$ or $D(p) = -1$.

$D(p)$ commutes with H (Hamiltonian), so they can be diagonalized simultaneously (parity is a good quantum number)– each energy eigenstate has a definite value of $D(p)$: $+1$ or -1

Example

Symmetric (trivial repr.) and antisymmetric (D(p)=-1 repr.) states under $x \to -x$ in nonrel. QM potential.

Puzzle: Why do mirrors reverse left with right, but not up and down? For mirror perpendicular to y axis.

$$(x, y, z) \to (x, -y, z)$$

Lie Group and QM: Michael Weiss

II. Lie group

Lie group - a special kind of continous group with el. $R(\alpha)$ labelled by N real parameters $\alpha_1, ...\alpha_N$. Parameters may vary over a finite or an infinite range: group-parameters space. A group of order (called also dimension) N if
1/identity el. $R(\alpha_0)$ (α_0 usually taken as zero) R(0)
2/ For any α there is $\bar{\alpha}$

$$R(\alpha)R(\bar{\alpha}) = R(\bar{\alpha})R(\alpha) = R(0),$$

so the inverse exists:

$$R(\alpha)^{-1} = R(\bar{\alpha})$$

3/For given α, β we can find γ

$$R(\gamma) = R(\beta)R(\alpha)$$

where γ parameters are real function of the real parameters α, β

$$\gamma = f(\alpha, \beta)$$

4/Associativity $R(\alpha)[R(\beta)R(\gamma)] = [R(\alpha)R(\beta)]R(\gamma)$, $\phi(\phi(\gamma, \beta), \alpha) = \phi(\phi(\gamma, \beta), \alpha))$
5/$\bar{\alpha}$ and γ are analytic functions of $\alpha, (\beta)$
Lie group compact if its parameters are bounded.

Example

•The real linear transf. $GL(2,R)$ in 2 dim. space are described by R(a) (order 4 - 4 real parameters)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

•If $Det(R(\alpha)) = 1 \rightarrow$ a subgroup SL(2,R)

•Complex linear transf.GL(2,C) - order 8

•SU(2) - a special unitary U (2), with 8 complex parameters

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$U^\dagger U = UU^\dagger = 1$ if det U=1 then SU(2)

$$\begin{pmatrix} e^{i\xi} \cos\eta & -e^{i\zeta} \sin\eta \\ e^{-i\zeta} \sin\eta & e^{-i\xi} \cos\eta \end{pmatrix}$$

3 real parameters - ξ, η, ζ.

•Rotation in 3 dimensions: R_3
Consider $R_x(\alpha), R_y(\beta), R_z(\gamma)$ - Euler angles.
$R(\alpha, \beta, \gamma)$

Lie groups

Sophus Lie: it is enough to consider el. of G which differ infinitesimally from the identity. Taylor expansion of the el. of the group.

Compact Lie groups - groups of unitary operators;

any representation of such group is equivalent to a unitary representation.

Any group el. can be obtained from the identity by continuous change of parameters, so

$$\boxed{g(\alpha) = e^{i\alpha_a X_a}, \quad \alpha_a - \text{real},\ a = 1, ...N}$$

Einstein convention = sum over the repeated index

X_a - linearly independent hermitian op.

(since g unitary $g^{-1}=g^{\dagger}$ and $X=X^{\dagger}$)

Set of all linear combinations of $X_a \rightarrow$

N-dim vector space $\mathcal{L_N}$ with $\{X_a\}$ - basis.

Name: X_a - group generator

(also for any el. of vector space)

Generators

N - dimension of the vector space of generators $(a = 1, ..N)$

(not equal to dim. of space in which generators act)

For compact groups we can always take space, on which generators act, to be finite, so think of X_a as finite hermitian matrices.

Generators:

•form a vector space

(unlike group el. - generators can be multiplied by numbers and added to obtain other generators)

•satisfy simple commutation relation

$\rightarrow$ the structure of the group see below

Finite groups versus continuous group

Finite groups are related to rotations through a finite angle $2\pi/n$.
Rotations through an arbitrary angle? - constitute compact groups with angular parameters which vary over finite ranges $[0, \pi)$ or $[0, 2\pi)$.

Lie Algebra

The structure formed by generators is known as an algebra – a vectors space of linear combinations of generators (again generators), and an additional composition law given by commutator (for any two generators the commutator is also a generator.)

Properties:
1. $[x, y] = -[y, x]$

2. Jacobi identity (see below)

Dimension of vector space = dimension of Lie algebra

Multiplication of group elements

Group el. of the form (one parameter family)

$$U(\lambda) = e^{i\lambda\alpha_a X_a}$$

Group multiplication gives simply

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2)$$

In contrast

$$e^{i\alpha_a X_a}e^{i\beta_b X_b} \neq e^{i(\alpha_a+\beta_a)X_a}$$

However (a group !)

$$e^{i\alpha_a X_a}e^{i\beta_b X_b} = e^{i\delta_c X_c}$$

... long calculation with result:

$$\delta_c = \alpha_c + \beta_c - \frac{1}{2}f_{abc}\alpha_a\beta_b + ...$$

go first to the next slide

Commutation relation - simplified case

$$exp(i\lambda X_b)exp(i\lambda X_a)exp(-i\lambda X_b)exp(-i\lambda X_a) =$$

$$= (1+i\lambda X_b - \lambda^2/2X_b^2+...)(1+i\lambda X_a - \lambda^2/2X_a^2+...)$$

$$(1-i\lambda X_b - \lambda^2/2X_b^2+...)(1-i\lambda X_a - \lambda^2/2X_a^2+..)$$

$$= (1+i\lambda(X_b+X_a) - \lambda^2 X_b X_a - \lambda^2/2(X_b^2+X_a^2)...)$$

$$(1-i\lambda(X_b+X_a) - \lambda^2 X_b X_a - \lambda^2/2(X_b^2+X_a^2))...$$

$$= (1+i\lambda(X_b+X_a-X_b-X_a) - 2\lambda^2 X_b X_a +$$

$$\lambda^2(X_b+X_a)^2 - \lambda^2(X_b^2+X_a^2))...$$

$$= 1 + \lambda^2(-2X_bX_a + X_bX_a + X_aX_b) + ... =$$

$$1 + \lambda^2(X_aX_b - X_bX_a) + ...$$

$$= 1 + \lambda^2[X_a, X_b] + ...$$

Check that terms $\lambda^2 X_a^2$ disappear !

Group property: $exp()exp()exp()exp() = exp(i\beta_c X_c)$

$$\lambda^2[X_a, X_b] = i\beta_c X_c$$

sum on the right side !

$\beta_c = \lambda^2 f_{abc}$ and $\boxed{[X_a, X_b] = if_{abc}X_c}$

Lie algebra

In general case we have to calculate

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_c X_c}$$

$$i\delta_c X_c = \ln(1 + e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1)$$

Taylor expansion of $ln(1 + K)$, for small K

$$K = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1 = ...$$

Derive $$\boxed{[X_a, X_b] = i f_{abc} X_c}$$

We got structure constants of the group from the group multiplication rule!

$$f_{abc} = -f_{bac}$$

obviously f_{abc} - antisymmetric in (a, b), more later

The generators form algebra (Lie algebra) under commutation

(commutator in algebra as multiplication law for group)

Note on Lie results

S. Lie: instead of searching for IRR of the Lie group with an infinite number of elements search for IRR of the Lie algebra with a finite number of elements.

Once we know IRR for algebra we know it for a group; we always first find it for corresponding algebra. In physical problems certain Lie algebra occurs naturally while the corresponding Lie group does not have a simple physical meaning.

Lie algebra

Each group repr. $\rightarrow$ repr. of the algebra.

Structure constants - summarize group multiplication law...

Equivalence, reducibility and irreducibility
$\rightarrow$ from group to algebra.

If there is any unitary repr. of algebra
$\rightarrow$ structure constants are real:
For hermitian X fulfilling

$$[X_a, X_b] = i f_{abc} X_c$$

take adjoint of the commutation relation

$$[X_a, X_b]^\dagger = -i f^*_{abc} X_c$$

$$[X_a, X_b]^\dagger = [X_b, X_a] = i f_{bac} X_c = -i f_{abc} X_c$$

and $f^*_{abc} = f_{abc}$.

This is our case, we are interested in groups which have unitary representations.

Jacobi identity

Generators satisfy:

$$[X_a, [X_b, X_c]] + \text{cyclic perm.}(abc \to bca \to cab) = 0$$

Check by expanding...

It is obvious for representation and true for abstract group generators.

From it (sums!):

$$if_{bcd}[X_a, X_d] + .. = -f_{bcd}f_{ade}X_e + ... = 0$$

$$f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0$$

Other form (check !):

$$[X_a, [X_b, X_c]] = [[X_a, X_b], X_c]] + [X_b, [X_a, X_c]]$$

PROBLEM

PROBLEM 8

Derive

$$\delta_c = \alpha_c + \beta_c - \frac{1}{2} f_{abc} \alpha_a \beta_b + ...$$

Adjoint representation

Defining a set of matrices T_a (in X_a basis)

$$\boxed{[T_a]_{bc} \equiv -if_{abc}}$$

From Jacobi identity

$$[T_a, T_b] = if_{abc} T_c$$

structure constants generate a representation of algebra!

Dim. of adjoint representation =
dim. of linear space in which repr. acts =
number of generators (N)

Since f_{abc} real - generators of adjoint representation are pure imaginary!

Structure constants in different basis

Structure constants depend on a basis in the vector space of generators $\mathcal{L}_{\mathcal{N}}$; any vector $X = \alpha_\rho X_\rho$ with α_ρ - coordinates of X. Transformation $\rightarrow$ new coordinates leads to a change of structure constants. If f_{abc} are real, T_a - pure imaginary.

A convenient scalar product: trace of product of T_a, to get a vector space of generators in the adjoint representation.

Take $Tr[T_a T_b]$ - real symmetric matrix (in a,b) -it is symmetric as a property of trace.
We can change form of this matrix by making linear transformation on X_a, which will induces a linear transformation of structure constants, $L^T L = 1$.

Diagonalize it!
(as usual by linear combination of X_a (inducing lin. combination of T_a))
which induces linear transformation on the structure constants.

Our normalization

$$X_a \to X'_a = L_{ab} X_b$$

$$[X'_a, X'_b] = i L_{ad} L_{be} f_{deg} X_g$$

$$= i L_{ad} L_{be} f_{deg} (L^{-1})_{gc} X'_c$$

$$f_{abc} \to f'_{abc} = L_{ad} L_{be} f_{deg} (L^{-1})_{gc}$$

Then

$$[T_a]_{bc} \to [T_a]_{bc} = L_{ad}\ L_{be} [T_d]_{eg} (L^{-1})_{gc}$$

The similarity transf. - trace the same
$Tr[L^{-1} Y L] = Tr[L L^{-1} Y]$: so,

$$Tr[T_a T_b] \to Tr[T'_a T'_b] = L_{ac} L_{bd} Tr[T_c T_d]$$

and we can use a proper L to diagonalized Tr[]

If we do this $\boxed{Tr[T_a T_b] = k_a \delta_{ab} \text{ (no sum)}}$
still freedom to rescale ....

We will consider algebras with

$$Tr[T_a T_b] = \lambda\ \delta_{ab},\ \ \lambda > 0$$

(compact Lie algebras)

Completely antisymmetric f_{abc}

Taking

$$Tr[T_a T_b] = \lambda\ \delta_{ab},\ \ \lambda > 0$$

In this basis f_{abc} is completely antisymmetric, since here

$$f_{abc} = -i\lambda^{-1} Tr([T_a, T_b]T_c)$$

Check!

In this basis generators in the adjoint representation are imaginary.

NOTE:

For the compact Lie group (as for finite group) any representation is equivalent to a representation by hermitian operators, and
all IRR are given by finite hermitian matrices

Simple algebras and groups

Invariant subalgebra - set of generators goes into itself under commutation with any element in the algebra

If X - gen. in the invariant subalgebra
Y - gen. in the whole algebra

[X,Y] - a gen. in the invariant subalgebra

Algebra which has no non-trivial invariant subalgebra
(trivial = whole set or empty set)
is called simple

A simple algebra generates a simple group

Adjoint representation of a simple Lie group (algebra)
with $Tr[T_a T_b] = \lambda \delta_{ab}$ is IRR.
Proof by assuming the contrary.

Abelian invariant subalgebra

A single generator which commutes with all the generators of the group (or subgroup) - a special Abelian invariant subalgebra.

Such algebra is called a U(1) factor of the group

U(1) - the group of the phase transformation;
U(1) - factor does not show up in structure constants $f_{..}$; so structure constants do not tell us anything about the U(1) subalgebras

Algebras without Abelian invariant subalgebras are called semisimple.

Every generator has non-zero commutator with some other generator - so structure constants have a lot of information

use to determine the entire structure of the algebras and its repr.

We will deal mainly with semisimple algebra representation by unitary operators.

States and operators

The generators of a representation - either linear operators or matrices

$$X_a|i>=\Sigma_j|j><j|X_a|i>=\Sigma_j|j>[X_a]_{ji}$$

below we skip Σ-symbol, states $|j>$

as for repr. of finite groups

Representation acts in Hilbert space,
the group elements - transformation of states.
The $g=e^{i\alpha_a X_a}$ acts (transforms) the ket:

$$|i>\to|i'>=e^{i\alpha_a X_a}|i>,\ <i|\to<i'|=<i|e^{-i\alpha_a X_a}$$

and for any op. $\mathcal{O}$

$$\mathcal{O}|i>\to e^{i\alpha_a X_a}\mathcal{O}|i>=e^{i\alpha_a X_a}\mathcal{O}e^{-i\alpha_a X_a}e^{i\alpha_a X_a}|i>$$

$$=\mathcal{O}'|i'>$$

So, any operators transforms as:

$$\mathcal{O}\to\mathcal{O}'=e^{i\alpha_a X_a}\mathcal{O}e^{-i\alpha_a X_a}$$

All matrix el. invariant under this transformation

Action of algebra - infinitesimal transformation

PROBLEMS

PROBLEM 9:
Find all components of the matrix $exp(i\alpha A)$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Hint: Check that $A^2 = B$, $A^3 = A$, etc.

PROBLEM 10:
If $[A, B] = B$, calculate $exp(i\alpha A)Bexp(-i\alpha A)$.

III. SU(2)

The simplest non-Abelian Lie algebra SU(2) consists of three generators $J_a,\ a = 1,2,3$ with $f_{abc} = \epsilon_{abc}$

completely antisymmetric tensor, with $\epsilon_{123} = 1$

The commutation relation is

$$[J_j, J_k] = i\epsilon_{jkl}J_l$$

"angular momentum algebra"

Two techniques: below we use the operator method, known in QM, however we will not use properties of the J^2!
We use $J^{\pm}$ to construct IRR.

Take N dim space of states in the whole Hilbert space and assume it transforms under the N-dim IRR.

We can diagonalize the J_3 (only one operator anyway), and consider its eigenstates, first taking the states with the highest value of j. For them

$$J_3|j,\alpha >= j|j,\alpha >$$

$\alpha = 1,2...$ (other characteristics..) and we can take: $< j,\alpha|j,\beta >= \delta_{\alpha\beta}$

Raising and lowering operators

Definition:

$$J^{\pm} = (J_1 \pm iJ_2)/\sqrt{2}$$

for them (note that $J^+ = (J^-)^\dagger$)

$$[J_3, J^{\pm}] = \pm J^{\pm} \text{ and } [J^+, J^-] = J_3$$

They are called raising and lowering (step, shift) operators since we know how they act on eigen-states for J_3, so (skipping α):

$$J_3|m>= m|m>$$

namely (from eq. above)

$$J_3(J^{\pm}|m) = \pm J^{\pm}|m> + J^{\pm}J_3|m>$$

$$= (m \pm 1)(J^{\pm}|m>).$$

So, for the highest $m = j$ (for all α)

$$J^+|j, \alpha>= 0,$$

and we take

$$J^-|j, \alpha> \equiv N_j(\alpha)|j-1, \alpha>$$

states $j-1$ for various α orthogonal (Check!), can be chosen orthonormal, and we get

$$N_j(\alpha) = \sqrt{j} \equiv N_j$$

$\boxed{\text{what about dependence on } \alpha?}$

$$< j-1,\beta|j-1,\alpha >= \frac{1}{N_j^*(\beta)N_j(\alpha)} < j,\beta|J^+J^-|j,\alpha >=$$

$$\frac{1}{N_j^*(\beta)N_j(\alpha)} < j,\beta|J_3|j,\alpha >= \frac{j}{N_j^*(\beta)N_j(\alpha)} < j,\beta|j,\alpha >$$

$$= j\frac{1}{N_j^*(\beta)N_j(\alpha)}\delta_{\alpha\beta}$$

So, we can take $N_j(\alpha) = \sqrt{j} \equiv N_j$.

From definition of $|j-1,\alpha>$ we get also

$$J^+|j-1,\alpha >= \frac{1}{N_j}J^+(J^-|j,\alpha >) =$$

$$\frac{1}{N_j}[J^+,J^-]|j,\alpha >= \frac{j}{N_j}|j,\alpha >= N_j|j,\alpha >$$

so, J^+ (as J^-) does not change α!

we get orthonormal $j-2$ states, satisfying

$$J^{-}|j-1,\alpha>=N_{j-1}|j-2,\alpha>$$

$$J^{+}|j-2,\alpha>=N_{j-1}|j-1,\alpha>$$

and so on...set of $|j-k,\alpha>$ orthonormal states

$$J^{-}|j-k,\alpha>=N_{j-k}|j-k-1,\alpha>$$

$$J^{+}|j-k-1,\alpha>=N_{j-k}|j-k,\alpha>$$

one can get, choosing real N_j, (from algebra) (Check!)

$$N^2_{j-k}=N^2_{j-k+1}+j-k$$

so (recursion), introducing $m=j-k$ we get

$$N^{(j)}{}_m=\frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$$

Now, since repr. finite dim. $\rightarrow$ max number (nonnegative integer denoted l) of application of J^{-}. The last application gives zero,
this corresponds to $N_{j-l}=0$ ($J^{-}|j-l,\alpha>=0$).
We get zero only if $(j+(j-l))=0$, or

$$l=2j \text{ or } j=l/2$$

j - integer or half-integer.

Dependence on α

We have learned how the generators act on the finite IRR representation. The space of states breaks into $2j + 1$ dim subspaces for each α.

$$\begin{array}{cc} \begin{pmatrix} [J] & 0 & 0 & 0 \\ 0 & [J] & 0 & 0 \\ 0 & 0 & [J] & 0 \end{pmatrix} & \begin{array}{c} \alpha = 1 \\ \alpha = 2 \\ \alpha = 3 \end{array} \end{array}$$

However - original assumption: IRR - so only one α possible ! We have constructed explicitly IRR for j, integer or half-odd-integer
$spin - j$ representation

Value of eigenvalue of J_3 is called weight,
j- the highest weight
state with the highest weight - unique

Method can be used to construct IRR for any compact Lie group..
and bringing a finite dim. repr. into block diagonal form

Notation for $spin - j$ representation

States

$$|j, m> \text{(skip here } \alpha)$$

$Spin - j$ representation od SU(2) group are defined by matrix el. of the generators

$$< j, m'|J_3|j, m >= m \; \delta_{m',m}$$

$$< j, m'|J^+|j, m >= \sqrt{(j + m + 1)(j - m)/2} \; \delta_{m',m+1}$$

$$< j, m'|J^-|j, m >= \sqrt{(j + m)(j - m + 1)/2} \; \delta_{m',m-1}$$

For $a = 1, 2, 3$

$$[J^j_a]_{kl} =< j, j + 1 - k|J_a|j, j + 1 - l >$$

row and columns (k, l)- from 1 to $2j + 1$. or by using $m, m' \rightarrow$ other notation

$$[J^j_a]_{m'm} =< j, m'|J_a|j, m >$$

m, m' from j to $-j$ (step = -1)

Spin-1/2 repr. of SU(2) and Pauli matrices

The simplest nontrivial repr.

$$J_a = \sigma_a/2,\ \sigma_a - \text{ Pauli matrices},\ a = 1,2,3$$

For states $|1/2, m>$

$$J_1^{1/2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2^{1/2} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$J_3^{1/2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin 1/2 representation - the simplest repr. of SU(2) - defining repr. It is responsible for the name SU(2) (special unitary), since exponentation of generators of spin 1/2 repr. $\to$ the finite group elements:

$$\boxed{e^{i\alpha_a \Sigma_a/2}}$$

special (determinant equal 1), unitary 2 $\times$2 matrices called SU(2) group

$J_a^{1/2}$ - generators of "defining" representation of SU(2)

PROBLEM: spin-1 representation of SU(2)

PROBLEM 11

Derive spin-1 representation of SU(2):

$$J_1^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_2^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_3^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\boxed{\text{PROBLEM}}$

PROBLEM 12:

Derive generators for $j = 3/2$ representation of SU(2).

$\boxed{\text{SU(2), SO(2)}}$

Write the most general form of the group elements of SU(2). What is the order of SU(2)?

Arbitrary finite dim. representation

The above procedure can be apply to get block diagonal representation for an arbitrary finite dimensional representation (various α possible)

Steps:
Diagonalize J_3, find state with the highest J_3 eigenvalue - j

Construct the states of IRR spin-j representation by applying lowering operator J^- - (a subspace)

Take now a subspace orthogonal to it, find a state with the next highest j_3 eigenvalue

This way one gets a basis for the Hilbert space states: $|j, m, \alpha>$

satisfying $< j', m', \alpha'|j, m, \alpha >= \delta_{m'm}\delta_{j'j}\delta_{\alpha'\alpha}$

This δ is not a convention, partly from construction.
Check that $< j, j, \alpha|j', j, \beta >$ =0 for say $j' > j$

PROBLEM

PROBLEM 13:
Derive from definition adjoint repr. for SU(2)

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Show that it is equivalent to J^1. Note, that the basis is here $|X_{1,2,3}>=|J_{1,2,3}>$

IV. Weights

The eigenvalues of J_3 are called weights.
The method - the highest weight construction-
we start with unique state with the highest weight.

The highest weight constr. allows to break rep. into IRR ones - obviously the simpler ones

We can also build arbitrary repr. out of the simpler representations...

The direct sum

Take two representations: D_1(dim n) and D_2(dim m) of a group G. We can form other repr. by adding or multiplying them.

Important:basis: $|i>$ and $|x>$

The direct sum: $D_1 \oplus D_2$ with dim $n+m$ in a block-diagonal form

$$\begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}$$

The generators (matrix) of sum of repr.:

$$X_a^{D_1 \oplus D_2} = \begin{pmatrix} X_a^{D_1(g)} & 0 \\ 0 & X_a^{D_2(g)} \end{pmatrix}$$

Direct product

We can form a representation of dim $n \times m$ (tensor product) $|i, x >= |i > |x >$
direct product space, a direct product repr.

$$D_1 \otimes D_2$$

$$(D_1 \otimes D_2)(g)|i > |x > = \{D_1(g)|i > D_2(g)|x >\}$$

and generators add

$$[X_a^{D_1 \otimes D_2}]_{ix,jy} = [X_a^{D_1}]_{ij}\delta_{xy} + \delta_{ij}[X_a^{D_2}]_{xy}$$

Often we write simply (ignoring deltas)

$$J_a^{1\otimes 2} = J_a^1 + J_a^2$$

(eg.adding spin and orbital angular momentum)

Adding J_3 particularly simple in a basis with J_3 diagonal

$1/2 \otimes 1$

Direct product:

$$J_3\,(|j_1,m_1>|j_2,m_2>) = (m_1{+}m_2)\,(|j_1,m_1>|j_2,m_2>)$$

The highest weight method to decompose a product space into IRR's.
The unique highest weight state

$$|\frac{3}{2},\frac{3}{2}>=|\frac{1}{2},\frac{1}{2}>|1,1>$$

Apply J^- to both sides ($j \to j-1$)

$J^-|\frac{3}{2},\frac{3}{2}>= J^-|\frac{1}{2},\frac{1}{2}>|1,1>\sqrt{\frac{3}{2}}\,|\frac{3}{2},\frac{1}{2}>=\sqrt{\frac{1}{2}}|\frac{1}{2},-\frac{1}{2}>$
$|1,1>+|\frac{1}{2},\frac{1}{2}>|1,0>$
so
$|\frac{3}{2},\frac{1}{2}>=\sqrt{\frac{1}{3}}|\frac{1}{2},-\frac{1}{2}>|1,1>+\sqrt{\frac{2}{3}}|\frac{1}{2},\frac{1}{2}>|1,0>$
next
$|\frac{3}{2},-\frac{1}{2}>=\sqrt{\frac{2}{3}}|\frac{1}{2},-\frac{1}{2}>|1,0>+\sqrt{\frac{1}{3}}|\frac{1}{2},\frac{1}{2}>|1,-1>$

and $|\frac{3}{2},-\frac{3}{2}>=|\frac{1}{2},-\frac{1}{2}>|1,-1>$

Remaining states must orthogonal to these, so:

$\sqrt{\frac{2}{3}}|\frac{1}{2},-\frac{1}{2}>|1,1>-\sqrt{\frac{1}{3}}|\frac{1}{2},\frac{1}{2}>|1,0>$
$\sqrt{\frac{1}{3}}|\frac{1}{2},-\frac{1}{2}>|1,0>-\sqrt{\frac{2}{3}}|\frac{1}{2},\frac{1}{2}>|1,-1>$

$\boxed{1/2 \otimes 1 = 3/2 \oplus 1/2}$

So, in this representation two states only.
Apply J^- to these states

$$\rightarrow |1/2, 1/2>, |1/2, -1/2>$$

relative sign of these two states is fixed;

Result:
$1/2 \otimes 1 = 3/2 \oplus 1/2$

Example: direct product of n 2-dim repr. of SU(2)

Let us take two-dim. repr. of SU(2) - direct product of n of them

$$[D \otimes D \otimes ...D]_{i_1...i_n, j_1...j_n} = D_{i_1 j_1}...D_{i_n j_n}$$

acting on object (vector) $u_{j_1...j_n}$.

This is reducible repr., it preserves symmetry of u under permutation of labels 1 to n. It breaks up into IRR, corresponding to subspace with vector of definite symmetry.

The component acting on completely symmetric subspace is $n+1$ dim, ($n+1$ independent components) - this repr is IRR with $j = n/2$.

Tensor operators

In QM quantities $< |O| >$, where O transforms in certain way under eg. the rotation group. Tensor op. O^s_l, $l = 1, ...2s + 1$ (or $-s$ to s) is a set of lin. op. with a simple commutation relation with angular mom. op. J_a

$$[J_a, O^s_l] = O^s_m [J^s_a]_{ml}$$

we say O^s_l transform according to spin-s repr. of SU(2).

Example ($\hbar = 1$):
particle in spherically symmetric potential, so operator acting on state

$$J_a = L_a = \epsilon_{abc} r_b p_c$$

The position vector r_a is a tensor op. since its action on a state

$$[J_a, r_b] = \epsilon_{acd}[r_c p_d, r_b] = -i\epsilon_{acb} r_c = r_c [J^{adj}_a]_{cb}$$

$$p_d = -i\frac{\partial}{\partial x_d}$$

equiv. to spin-1 repr. (better to have a standard form..see NOTE below.)

Note

Sometimes we should find the basis for op. that the conventional spin-s repr. appears in commutation relation (not a adjoint repr.). Let us take set of op. $\Omega_x, x = 1, ..2s+1$, which transforms according to a repr. D equivalent to the $spin-s$ repr. of SU(2):

$$[J_a, \Omega_x] = \Omega_y [J_a^D]_{yx}$$

since by assumption D is equiv to spin-s repr., we can find S:

$$SJ_a^D S^{-1} = J_a^s$$

or

$$[S]_{lx}[J_a^D]_{xy}[S^{-1}]_{yl'} = [J_a^s]_{ll'}$$

then define new set of operators

$$O_l^s = \Omega_y [S^{-1}]_{yl}, \;\; l = -s...s$$

and now

$$[J_a, O_l^s] = [J_a, \Omega_y][S^{-1}]_{yl} =$$

$$\Omega_z [J_a^D]_{zy}[S^{-1}]_{yl} = ... = O_{l'}^s [J_a^s]_{l'l}$$

Note that now $[J_3^s]_{ll'} = l\delta_{ll'}, l = -s, ...s$, and

$$[J_3, O_l^s] = O_{l'}^s [J_3^s]_{l'l} = lO_l^s$$

Position operator - other components

We used: $r_1, r_2, r_3 (x, y, z)$
We want to have r_{-1}, r_0, r_{+1}, like $\mathcal{O}^{(1)}_{m=-1,0,1}$

$$[J_3, r_3] = 0$$

what is r_0?

From definition also $[J_3, r_0] = 0\ r_0$,
so we can take $r_0 = r_3$
Then $[J^{\pm}, r_0] = r_{\pm 1} = \mp (r_1 \pm i r_2)/\sqrt{(}2)$.

Check:
$[J^{\pm}, r_0] = r_m [J^{(1)\pm}]_{m0} = r_{\pm 1}$

More

Take vector $\vec{V}$, we have V_1, V_2, V_3 or V_x, V_y, V_z

$$[J_i, V_j] = i\epsilon_{ijk} V_k$$

Spherical components:
$V_{\pm 1} = \mp (V_x \pm i V_y)/\sqrt{(}2)$

$V_0 = V_z$

For $\vec{r}$ analogously; in polar coordinates:
$z = rcos\theta,\ x = rsin\theta cos\phi,\ y = rcos\theta sin\phi$

So, $r_{\pm} = \mp r/2 sin\theta e^{\pm\phi}$, $\ r_0 = rcos\theta$

Or $r_l = \sqrt{4\pi/3} r Y_{1l}(\theta, \phi)$, where Y spherical harmonics for $l = 1$

Clebsch-Gordon coefficients

How the object $O^s_l|j,m,\alpha>$ transforms
(product of two object: tensor and ket)?
Take:
$J_a O^s_l|j,m,\alpha> =$
$= [J_a, O^s_l]|j,m,\alpha> + O^s_l J_a|j,m,\alpha>$
$= O^s_{l'}|j,m,\alpha> [J^s_a]_{l'l} + O^s_l|j,m',\alpha> [J^j_a]_{m'm}$
transformation law for tensor product of spin-s and spin-j: $s \otimes j$.
In the standard basis - J_3 is diagonal, so we have here simple

$$J_3 O^s_l|j,m,\alpha>= (l+m) O^s_l|j,m,\alpha>$$

So: J_3- eigenvalue of a product of tensor op. and a state (ket) is sum of eigenvalues J_3 of the op. and state

C-G continuation

Using the h.w decomposition one can show that for tensor product

$$s \otimes j = (s + j) \oplus ... \oplus |s - j|$$

Clebsch-Gordon coeff. - group theory

Here take

$$O_s^s|j, j, \alpha >$$

with $J_3 = j + s$, as the highest weight state. One can lower it to construct the rest of the spin $j + s$ representation.

$$\boxed{\text{PROBLEM: } 1/2 \otimes 1/2 = 1 \oplus 0}$$

PROBLEM 14:
Derive that $1/2 \otimes 1/2 = 1 \oplus 0$.

PROBLEM

PROBLEM 15:

For product of two tensor op. $O^{s_1}_{m_1}$ and $O^{s_2}_{m_2}$ show that it transforms under $s_1 \otimes s_2$ representation.

V. Internal symmetry- Isospin

Idea - nuclear physics in 1934 (Heisenberg).
Nucleon=doublet of proton and neutron

$$N = \begin{pmatrix} p \\ n \end{pmatrix}$$

Charge independence

charge independence:
forces between pp, nn, pn similar
SU(2) (Pauli matrices) to describe isobaric spin
(isobars - with the same number of barions)
(Wigner called this isotopic (???) spin - in short
isospin)

Creation operators for nucleon N:

$$|p, \alpha >= a^{\dagger}_{N,+1/2,\alpha}|0>$$

$$|n, \alpha >= a^{\dagger}_{N,-1/2,\alpha}|0>$$

$$|0> -\text{vacuum state}$$

Properties of operators

Creation op. not-hermitian, their adjoints - annihilation op.: $a_{N,\pm1/2,\alpha}$

$$a_{N,\pm1/2,\alpha}|0>=0$$

Denote for simplicity op. P, N (for neutron state- this is not consistent notation, since N=nucleon)

$$P_\alpha^\dagger \equiv a^\dagger_{N,+1/2,\alpha};\ P_\alpha \equiv a_{N,+1/2,\alpha}$$

and similarly $N^\dagger, N\ldots$; states $|P>, |N>, |0>$ protons, neutrons are fermions, Pauli exclusion:

$$P_\alpha^\dagger P_\alpha^\dagger = P_\alpha P_\alpha = 0$$

$$\boxed{P_\alpha^\dagger P_\alpha + P_\alpha P_\alpha^\dagger = 1}$$

since acting on states
$P^\dagger P|P>=1|P>, PP^\dagger|P>=$
$PP^\dagger P^\dagger|0>=0|P>$ or
$P^\dagger P|0>=0, PP^\dagger|0>=P|P>=1|0>$
We can choose $\{P_\alpha^\dagger, P_\beta\} = (P_\alpha^\dagger P_\beta + P_\beta P_\alpha^\dagger) = \delta_{\alpha\beta}$

$$\{P_\alpha^\dagger, P_\beta^\dagger\} = \{P_\alpha, P_\beta\} = 0$$

and similarly for neutrons ...,
State with n protons: $P^\dagger_{\alpha_1}\ldots P^\dagger_{\alpha_n}|0>=|\alpha_1,..\alpha_n>$ - antisymmetric under $\alpha_i \leftrightarrow \alpha_j$ (Fermi-Dirac statistics)

$$\{P_\alpha^\dagger, N_\beta^\dagger\} = 0 - why??$$

Number operators

$P^\dagger_\alpha P_\alpha$ counts the number of protons in state α

Consider op. $\Sigma_\alpha P^\dagger_\alpha N_\alpha$ - it replaces n by p

Derive op. which counts neutrons and nucleons.

Isospin symmetry of strong interaction

In the limit where weak and e-m interaction are turned off

$$[H_s, \Sigma_\alpha P^\dagger_\alpha N_\alpha] = 0$$

similarly for $\Sigma_\alpha N^\dagger_\alpha P_\alpha$

Generators replacing n and p

Define

$$T^+ = \frac{1}{\sqrt{2}} \Sigma_\alpha P^\dagger_\alpha N_\alpha$$

$$T^- = \frac{1}{\sqrt{2}} \Sigma_\alpha N^\dagger_\alpha P_\alpha$$

and introduce $[T^+, T^-] \equiv T_3$

Check that

$$T_3 = \frac{1}{2}(\Sigma_\alpha P^\dagger_\alpha P_\alpha - \Sigma_\alpha N^\dagger_\alpha N_\alpha)$$

counts the number of p minus number of n (times 1/2)

T_3 commutes with H_{strong} (also H_{em})

Check: $[T_3, T^\pm] = \pm T^\pm$

$\rightarrow$ SU(2)$_{isospin}$ algebra

Note: Group elements - useless

States

Consider subspace of eigenstates of H_{strong} with eigenvalue h. Isospin generators $T_3, T^{\pm}$ acting on this states give another state in this subspace (they commute with H $[T_a, H] = 0$)

decomposition into IRR

so states can be labeled

$$|I, I_3, h, \alpha >$$

I - the highest T_3 eigenvalue (I_3)

$2I + 1$ states called isospin multiplet

Isospin is conserved by the strong interaction, also the charge Q and barion B numbers

$$Q = \Sigma_\alpha P^\dagger_\alpha P_\alpha, \;\; B = \Sigma_\alpha P^\dagger_\alpha P_\alpha + \Sigma_\alpha N^\dagger_\alpha N_\alpha$$

so we have (operators!operators!)

$$\boxed{Q = T_3 + \frac{1}{2}B}$$

Isospin for pions

For pions (mass $\sim$ 140 MeV, spin 0, B=0) isospin generators are

$$T_3 = a^\dagger_{\pi^+} a_{\pi^+} - a^\dagger_{\pi^-} a_{\pi^-}$$

$$T^+ = a^\dagger_{\pi^+} a_{\pi^0} + a^\dagger_{\pi^0} a_{\pi^-}$$

$$T^- = a^\dagger_{\pi^-} a_{\pi^0} + a^\dagger_{\pi^0} a_{\pi^+}$$

Check if eigenvalues of T_3 are equal:

$$\begin{array}{rl} +1 & for\, \pi^+ \\ 0 & for\, \pi^0 \\ -1 & for\, \pi^- \end{array} \tag{7}$$

$$T_3|\pi^+> = a^\dagger_{\pi^+} a_{\pi^+}|\pi^+> = a^\dagger_{\pi^+}|0> = +1|\pi^+>$$

Note:

- Yukawa hypothesis: nuclear force by exchange of (massive) pions
- π^0 decay by H_em (10^{-16} sec), $\pi^\pm$ by H_w (10^{-8} sec).
- $H = H_{strong} + H_{em} + H_w$, where H_em, H_w small pert. since coupling constants small
- Mass splitting?

$\boxed{\pi p \text{ -scattering}}$

$D^1 \otimes D^{\frac{1}{2}} = D^{3/2} \oplus D^{\frac{1}{2}}$ so any state = lin. comb. of isospin-3/2 and 1/2 repr.
A unique highest weight state

$$|3/2, 3/2 >= |1, 1 > |1/2, 1/2 >$$

using notation as for spin

Apply J^-

$J^-|3/2, 3/2 >= |1, 1 > |1/2, 1/2 > |3/2, 3/2 >= |1, 1 > |1/2, 1/2 > |3/2, 3/2 >= |1, 1 > |1/2, 1/2 >$

etc

Production of resonances with isospin 3/2 and 1/2

PROBLEM

PROBLEM 15b:
Scattering processes $\pi N \to \pi N$, for states with definite isospin. Write $|\pi^0 p>=|1,0>|1/2,1/2>$ and $|\pi^+ n>=|1,1>|1/2,-1/2>$ in terms of states with definite total isospin of the πN system.
Express scattering amplitudes $<\pi^+ n|S|\pi^+ n>$, $<\pi^0 p|S|\pi^+ n>$ by amplitudes with define total isospin $a_{3/2}$ and $a_{1/2}$

$$<\pi^+ n|S|\pi^+ n>= 2/3 a_{1/2} + 1/3 a_{3/2}.$$

PROBLEM: Symmetry of tensor products

PROBLEM 16:
Tensor products of two identical spin (isospin)-1/2 representations $\rightarrow$ decomposition into spin-1 and spin-0 representation

Check that: spin-1 - symmetric representation
spin-0 - antisymmetric representation

Isospin generators

Isospin: I, I_3
- I is the highest eigenvalue of op. T_3 (analog of j for J_3),
- I_3 - eigenvalue of T_3 (analog of m for J_3)

T_3 for p and n (sum over m, m', α)

$$T_3 = \sum a^{\dagger}_{N,m',\alpha} [\sigma_3/2]_{m'm} a_{N,m,\alpha}$$

(Note that $[\sigma_3/2]_{m'm} = m\delta_{m'm}$) similarly

$$T_a = \sum a^{\dagger}_{N,m',\alpha} [J_a^{1/2}]_{m'm} a_{N,m,\alpha}$$

We can introduce similar op. also for other particles (not only p and n)

PROBLEM

PROBLEM 17:
Check, that $a^{\dagger}_{N,m,\alpha}$ are tensor operators in spin-1/2 representation by calculating

$$[T_a, a^{\dagger}_{N,m,\alpha}].$$

Generalized exclusion principle

Particles in a single isospin multiplet we treat as one particle in various states

That why we take creation and annihilation op. for p and n to anticommute

$$a^{\dagger}_{N,1/2,\alpha} \equiv P^{\dagger}_{\alpha}$$

$$a^{\dagger}_{N,-1/2,\alpha} \equiv N^{\dagger}_{\alpha}$$

I skip below N (Nucleon) as a subscript

$$\{a^{\dagger}_{m,\alpha}, a_{m',\beta}\} = \delta_{m'm}\delta_{\alpha\beta}$$

$$\{a^{\dagger}_{m,\alpha}, a^{\dagger}_{m',\beta}\} = \{a_{m,\alpha}, a_{m',\beta}\} = 0$$

PROBLEM: Deuteron

PROBLEM 18:

Deuteron: state with one p and one n. We can build such state as linear combination of states with definite J_3, T_3 and position r

$$|m, I_3, r; m', I_3', r' >= -|m', I_3', r'; m, I_3, r >$$

Pauli exclusion - antisym. state!

Total spin and isospin of such state: 1 or 0!

But J or $I = 1$ symmetric under exchange of spin or isospin states while J or I =0 - antisymmetric..

If the spatial wave f. symmetric (s-wave) - then only J=1 I=0 or J=0 I=1 possible

Deuteron is only one, so...

VI. ROOTS and WEIGHTS

Generalization of SU(2) to an arbitrary simple Lie algebra (simple - no non-trivial invariant subalgebra)

Idea: divide generators into set of (analog of J_3) - hermitian operators to be diagonalized and raising and lowering operators (analog of $J^{\pm}$)

Suppose X_a - generators of a simple Lie group in IRR D:
First step - take the largest set of commuting herm. generators: m=max number of them and diagonalize them.

We take m lin. indep. combinations of X_a:

$$H_i, i = 1, ..m, \; H_i = C_{ia} X_a$$

CARTAN subalgebra (the largest possible subset..), Cartan generators, m = rank of algebra,...

$$C_{ia} - real, H_i \; hermitian(H_i = H_i^{\dagger}), [H_i, H_j] = 0$$

They form a linear space; taking basis such that

$$Tr(H_i H_j) = k_D \delta_{ij}$$

Examples

Rank of SO(2) - 1

Rank of SO(3) - 1

Weights

Now diagonalize H_i,
take $|\mu, (x), D>$ as a state of repr. D
(x- other quantum numbers, often skipped)

$$H_i|\mu, D >= \mu_i|\mu, D >$$

Def.: (real) eigenvalues μ_i are called weights $\rightarrow$ weight vector

$$\mu = (\mu_1, ...\mu_m)$$

We define the weights for any repr. using the same generators H_i, so, in particular ...

Roots

We can do the same for adjoint representation weight vector of adj. repr. $\to$ roots vector. States of adjoint repr. correspond (are labeled) by generators themselves $|X_a>$.
Linear combination of states $\to$ linear combination of generators

$$\alpha|X_a> + \beta|X_b> = |\alpha X_a + \beta X_b>,$$

scalar product can be introduced ($\lambda \equiv k_{D=adj}$),

$$< X_a|X_b> = \lambda^{-1} Tr(X_a^\dagger X_b)$$

Action on state

$$X_a|X_b> = |X_c>< X_c|X_a|X_b> = |X_c> [T_a]_{cb} =$$

$$-if_{acb}|X_c> = |if_{abc}X_c> = |[X_a, X_b]>$$

States of zero roots (weights) -
correspond to the Cartan generators

$$H_i|H_j> = 0$$

Cartan states are orthonormal

$$< H_i|H_j> = \lambda^{-1} Tr(H_i H_j) = \delta_{ij}$$

States E_α

Consider the rest of space of generators $(n-m)$, not Cartan states, they have non-zero weights - states $|E_\alpha>$

For them

$$H_i|E_\alpha>= \alpha_i|E_\alpha>$$

or

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

(take states of both sides)

E_α are not hermitian ! Check, that $E_\alpha^\dagger = E_{-\alpha}$ (like for $(J^+)^\dagger = J^-$)

Root vector

$$\alpha = (\alpha_1, ..., \alpha_m)$$

Without proof: non-zero roots uniquely specify states, no extra (x) needed.

Normalization, etc

States with different weights - orthogonal, we can normalize as follows

$$< E_\alpha | E_\beta >= \lambda^{-1} Tr(E_\alpha^\dagger E_\beta) = \delta_{\alpha\beta} (= \Pi_i \delta_{\alpha_i \beta_i})$$

$E_{\pm\alpha}$ - raising and lowering operators for states with weights μ, indeed

$$H_i (E_{\pm\alpha} | \mu, D >) = ... = (\mu \pm \alpha)_i (E_{\pm\alpha} | \mu, D >)$$

since $E_\alpha | E_{-\alpha} >$ has zero weight, it must correspond to a linear combination of $| H_i >$ states

$$E_\alpha | E_{-\alpha} >= \beta_i | H_i >= | \beta_i H_i >= | [E_\alpha, E_{-\alpha}] >$$

To find β_i we start as follows:

$$E_\alpha | E_{-\alpha} >= | H_i >< H_i | E_\alpha | E_{-\alpha} >= \beta_i | H_i >$$

$\beta_i =< H_i | E_\alpha | E_{-\alpha} >=< H_i | [E_\alpha, E_{-\alpha}] >$, ...
so $\beta_i = \alpha_i$, and

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H$$

(as for $[J^+, J^-] = J_3$)

Many SU(2)'s

FOR EACH NON-ZERO PAIR
OF ROOT VECTORS $\pm\alpha$
$\rightarrow$ SU(2) subalgebra with generators:

$$E^{\pm} = |\alpha|^{-1} E_{\pm\alpha}$$

$$E_3 = |\alpha|^{-2} \alpha \cdot H$$

Check:

$$[E_3, E^{\pm}] = \pm E^{\pm},\ [E^{+}, E^{-}] = E_3$$

Important p

For any weight μ of repr. D (Check!)

$$E_3|\mu, x, D> = \frac{\alpha \cdot \mu}{\alpha^2}|\mu, x, D>$$

where eigenvalues are integer or half-integer!
So $2\frac{\alpha \cdot \mu}{\alpha^2}$ integer!

states are unique - no dependence on x - without proof

The general state $|\mu, x, D>$ - a linear combination of states according to definite repr. of SU(2). If the highest spin state in this combination is j, then there is some non-negative p

$$(E^+)^p|\mu, x, D> \neq 0$$

with weight $\mu + p\alpha$ - the highest weight of E_3 state of the $spin - j$ repr., while

$$(E^+)^{(p+1)}|\mu, x, D> = 0$$

Eigenvalue of E_3:

$$\frac{\alpha \cdot (\mu + p\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} + p = j$$

$\boxed{\text{Important } q}$

Likewise, for the lowest state, with non-negative q

$$(E^-)^{(q+1)}|\mu, x, D >= 0$$

and

$$\frac{\alpha \cdot (\mu - q\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} - q = -j$$

We can p times go up and q times go down over states

We get by adding the above equations we get a master formula:

$$\frac{\alpha \cdot \mu}{\alpha^2} = -\frac{1}{2}(p - q)$$

Geometric classification

For D=adj repr. we have $\mu = \beta$, and using $|E_\alpha >$

$$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{1}{2}(p-q) = \frac{m}{2}$$

using SU(2) algebra with E_β

$$\frac{\beta \cdot \alpha}{\beta^2} = -\frac{1}{2}(p'-q') = \frac{m'}{2}$$

So,

$$\cos^2\theta_{\alpha\beta} = (p-q)(p'-q')/4$$

since $(p-q)(p'-q')$ integer, and non-negative we have only n=0,1,2,3 cases:

$$\theta_{\alpha\beta} = 90^o, 60^o(120^o), 45^o(135^o), 30^o(150^o)$$

angles between root vectors: $\vec{\alpha}, \vec{\beta}$

The possibility 4 corresponds to 0^0 or 180^0: 0 is ruled out (theorem of uniqueness), 180 is trivial (roots in pairs always)

PROBLEMS

PROBLEM 19

Show that $[E_\alpha, E_\beta]$ must be proportional to $E_{\alpha+\beta}$.
What happens if $\alpha + \beta$ is not a root?

VII. SU(3)

SU(3)- the group of 3x3 unitary matrices with determinant 1. Generators - 3x3 hermitian (traceless) matrices $T_a, a = 1, ..8$. Gell-Mann matrices λ_a, $T_a = \lambda_a/2$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Normalization

$Tr[T_a T_b] = 1/2\delta_{ab}$.

$T_{1,2,3}$ - SU(2) (isospin) subgroup of SU(3)

There are other SU(2) subgroups here
T-spin algebra, U-spin algebra, V-spin algebra with corresponding shift operators; what quantum numbers are involved?

Cartan subalgebra

Convenient to take (already diagonal !)

$$H_1 = T_3, \; H_2 = T_8$$

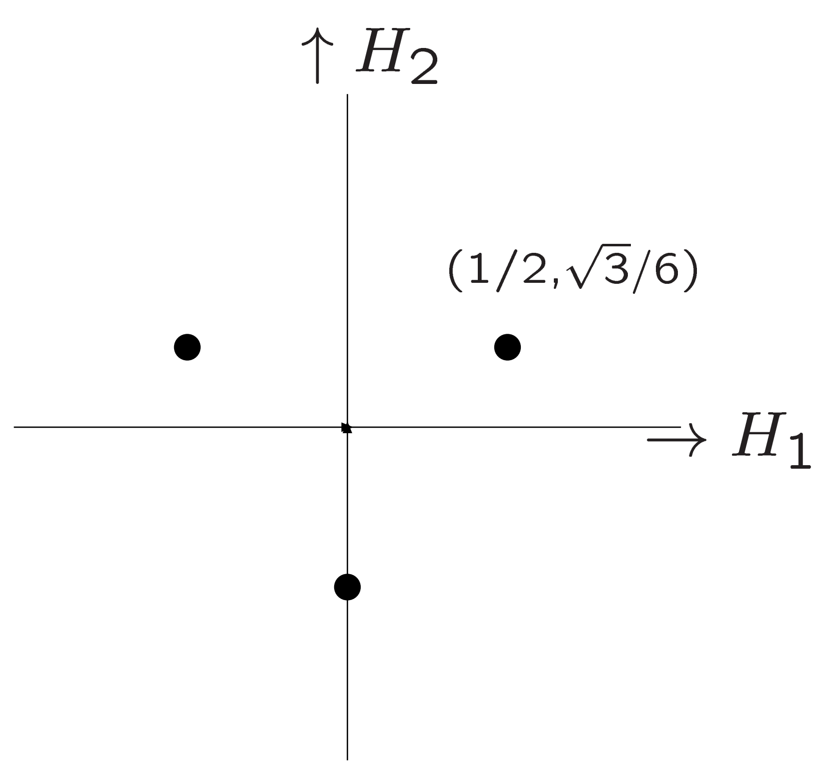
Weights and roots of SU(3)

Weights vectors of: $H_1 = T_3,\ H_2 = T_8$. Weights for eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to (1/2, \sqrt{3}/6) \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to (-1/2, \sqrt{3}/6),$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to (0, -\sqrt{3}/3)$$

3 (μ_1, μ_2) weights-vectors of H_1, H_2 for 3 eigen-vectors

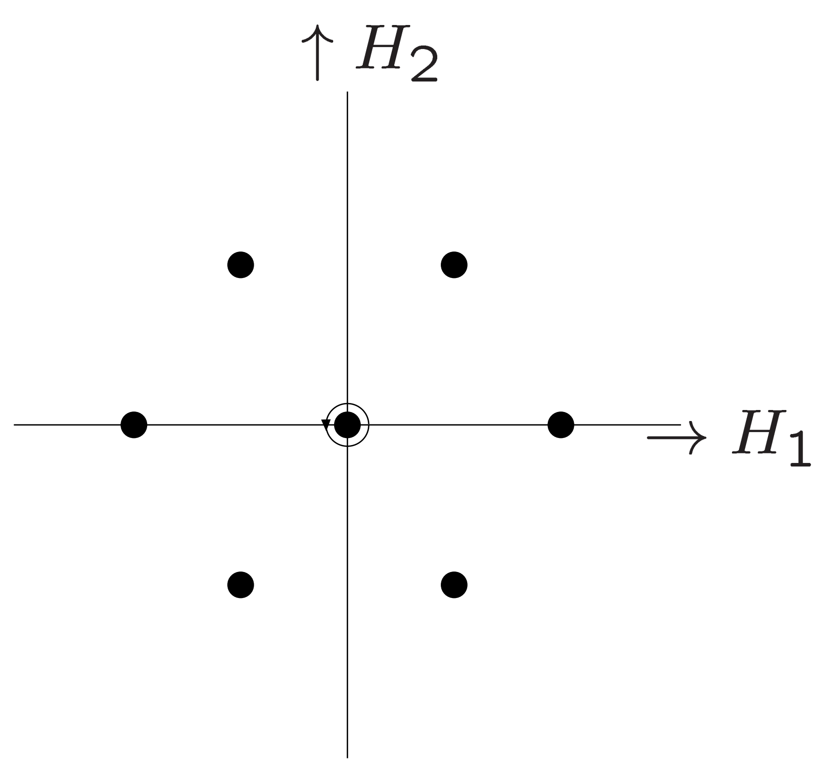


Roots

E_α (combinations of $T_j, j \neq 3, 8$) take us from one weight to other so from one eigenstate to other, eg. from (0,1,0) to (1,0,0) $E_{1,0} = ...$

$\frac{1}{\sqrt{2}}(T_1 \pm iT_2) = E_{\pm 1,0}$
$\frac{1}{\sqrt{2}}(T_4 \pm iT_5) = E_{\pm 1/2, \pm\sqrt{3}/2}$
$\frac{1}{\sqrt{2}}(T_6 \pm iT_7) = E_{\mp 1/2, \pm\sqrt{3}/2}$

There are 8 roots μ: 2 of them zero
$(\pm 1, 0), (\pm 1/2, \pm\sqrt{3}/2), (\mp 1/2, \pm\sqrt{3}/2), (0, 0), (0, 0)$



PROBLEMS

PROBLEM 20:
Derive $E_{\pm 1/2, \pm\sqrt{3}/2}$ in terms of generators T_a.

PROBLEM 21:
How E_α look like? Write $E_{1,0}$ and check how it acts on eigenvectors of T_3, T_8.

PROBLEM 22:
Calculate $\mathrm{Tr}(H_1^2)$ and $\mathrm{Tr}(E_{-\alpha}E_\alpha)$ for SU(3).

VIII. Simple Roots

Concept of the highest weight for arbitrary Lie group $\rightarrow$ positivity for weight vectors

We fix temporally a basis in the Cartan subalgebra $H_1, H_2, ...$ so the components of weight vectors $\mu_1, \mu_2 ..$ are fixed

A weight is said to be "positive" if its first non-zero component is positive, similarly "negative", and "zero"(=all μ_i equal 0).

Every weight vector is positive, negative or zero.

Now we can define ordering, eg $x > y$ if $x - y > 0$ (positive)

The highest weight state (unique!) in IRR - if weight greater than others. If we act on state with highest weight in any representation with E_α, for positive α, we must get zero

A <u>simple</u> root is a positive root which can not be expressed as the sum of two positive roots; they determinate a whole structure of group

Positive and negative roots for SU(3)

On weight diagram for SU(3)
only one weight vector is positive $(1/2, \sqrt{3}/6)$

We can define ordering - taking on the highest weight

In adj. representation
positive roots correspond to the raising op. and vice versa for negative roots.

In SU(3) adj. rep positive roots on the right

More

If α, β are different simple roots then
$\beta - \alpha$ is not a root!
Proof:
a/ if $\beta - \alpha$ is a root and $\beta - \alpha > 0$ then β is not simple since then $\beta = \alpha + (\beta - \alpha)$

b/ if $\beta - \alpha$ is a root and $\beta - \alpha < 0$ then α is not simple since then $\alpha = \beta + (\alpha - \beta)$

Contradictions!

Angles between simple roots

If $\beta - \alpha \neq$ root, then $E_{-\alpha}|E_{\beta} >= E_{-\beta}|E_{\alpha} >= 0$ like for the lowest state...

Since lowering not possible, the corresponding $q = 0$,
therefore from the master formulae

$$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{1}{2}(p - q) \ , \ \frac{\beta \cdot \alpha}{\beta^2} = -\frac{1}{2}(p' - q')$$

we get $q = q' = 0$ and

$$\cos\theta_{\alpha\beta} = -\frac{\sqrt{pp'}}{2} \quad \beta^2/\alpha^2 = p/p'$$

$$\boxed{\pi/2 \leq \theta < \pi}$$

Derive!

Without proof
Any set of vectors satisfying the above condition is linearly independent

Simple roots - linear indep. complete set

Simple roots - linearly indep., complete set so their number $= m$

Any positive root ϕ can be written as a sum of simple roots with non-negative integer coefficients

$$\phi = \Sigma_{\alpha,simple} k_\alpha \alpha$$

Construction a whole algebra (all roots)

Let us introduce $k = \Sigma k_\alpha$ and denote:

$$\phi_k = \Sigma_{\alpha,simple} k_\alpha \alpha$$

Dynkin diagrams

Dynkin diagrams - a short-hand notation for simple roots

Each simple root - an open circle
Angle - denoted by line:
no line if angle 90^o
one line - 120^o
two lines - 135^o
three lines 150^o

Dynkin diagrams: circles connected by lines

SU(3) : O-O
Check what is the Dynkin diagram for SU(2)

Dynkin showed that every simple Lie algebra has a unique representative diagram.

PROBLEM

PROBLEM 23:
Prove that simple roots are linearly independent.
Hint: Assume contrary, and therefore $\Sigma\kappa_\alpha\alpha = 0$, with some constants κ_α.

PROBLEM 24:
Prove that number of simple roots is m.
Hint: Consider two cases - number of simple roots is smaller and larger than m.

Example

Determining of all roots or in other words constructing the whole algebra from the simple roots.

Question:
which combination $\Sigma_{\alpha,simple} k_\alpha \alpha$ is actually a root?

One can use induction to answer this question:
$\phi_1, \phi_2, ...$

Example: SU(3)

3 positive roots; 2 simple roots

$$\alpha = (1/2, \sqrt{3}/2) \quad \beta = (1/2, -\sqrt{3}/2)$$

Calculate $\alpha^2 = 1, \beta^2 = 1,\ \alpha \cdot \beta = -1/2$

so, $p = 1$ and $\beta + \alpha$ - OK;
however $\beta + 2\alpha$, etc - not roots.

Constructing algebra from simple roots

By subtracting two egs instead of adding to get the master equation check that

$$p + q = 2j,$$

so if we know p and q, we know what is the highest spin representation.

For for μ being a root adj repr we can conclude that $|\beta> = |j, \alpha \cdot \beta/\alpha^2>$, with E_3 eigenvalue equal to $\alpha \cdot \beta/\alpha^2$. (Up to phases)

Fundamental weights

Take simple roots of Lie algebra α^j $(j = 1, ..m)$. If μ the highest weight of an arbitrary IRR D then $\mu + \phi$ is not a weight for any positive root ϕ. Enough that $\mu + \alpha^j$ is not a weight for any α^j

$$E_{\alpha^j}|\mu >= 0$$

so, for every α^j, $p = 0$, thus

$$\frac{2\alpha^j \cdot \mu}{(a^j)^2} = q^j,$$

where q^j are non-negative integers. The α^j-lin. indep. so the set of $\{q^j\}$ completely determines μ - which is the highest weight of some IRR. We can construct the entire representation by acting on μ with lowering op. $E_{-\alpha^j}$.
We can label IRR of rank m by set of m non-negative integers q^j - Dynkin coefficients.

NOTE: For arbitrary μ : $\frac{2\alpha^j \cdot \mu}{(a^j)^2} = l^j = q^j - p^j$

Fund. weights - cont'n

Now consider weight vectors μ^k satisfying

$$\frac{2\alpha^j \cdot \mu^k}{(a^j)^2} = \delta_{jk},$$

μ^k is the highest weight of repr. with $q^k = 1, q^j = 0$ for $j \neq k$

every highest weight μ can be uniquely expressed by μ^j

$$\mu = \Sigma_{j=1}^m q^j \mu^j$$

We can build the repr. with highest weight μ by constructing a tensor product of q^1 repr. of highest weight μ^1, q^2 with h.w. μ^2, etc (such repr is generally reducible)

The vectors μ^j - fundamental weights
the m IRR repr. that have these f.w μ^i as highest weights are called fundamental repr. D^j

Fundamental weights form a complete set,
we can expand any weight of any representations in terms of them.

$\boxed{\text{Theorem}}$

The trace of any generator of any representation of a compact simple Lie group is zero.

Proof: The trace is invariant under similarity trans. In the weight basis every generator is a lin. combination of Cartan generators and and raising and lowering op. The trace of raising and lowering op. $E_{..}$ are zero (off-diagonal), Cartan gen. can be written as lin. combination of $\alpha^i \cdot H$. But each $\alpha^i \cdot H$ is prop. to generators of an SU(2) subalgebra and trace is zero ...

IX. More SU(3)

Other IRR repr... Fundamental repr. of SU(3):
SU(3) simple roots

$$\alpha^1 = (1/2, \sqrt{3}/2) \qquad \alpha^2 = (1/2, -\sqrt{3}/2)$$

We can find μ^i, the highest weights of fund. repr. by $\mu^i \cdot \alpha^j = 0$, for $i \neq j$, and normalizing (master formula)

$$\mu^1 = (1/2, \sqrt{3}/6) \qquad \mu^2 = (1/2, -\sqrt{3}/6)$$

we know that μ^1 - the highest weight generated by Gell-Mann matrices (defining representation)

μ^2 - other repr. , for it $q^1 = 0, q^2 = 1$ so (0,1)

Acting with lowering operators we got three weights: $\mu^2, \mu^2 - \alpha^2, \mu^2 - \alpha^2 - \alpha^1$ Check !

Second fund. repr. is three dim. their weights are negatives of the first (defining) repr.

Constructing states

To be more general let us start with IRR with the highest weight μ; all the states in the representation can be written as

$$E_{\phi_1}E_{\phi_2}...E_{\phi_n}|\mu>$$

(ϕ_i - any roots).

We can take all ϕ_i to be negative. Check.
However negative roots can be presented as a sum over simple roots and non-negative integer coefficients.

etc.

Weyl reflection

Introduction:

1/ If μ is a weight, then

$$\mu - \frac{2\alpha\cdot\mu}{\alpha^2}\alpha$$

is also a weight for any weight.

This is since $\frac{2\alpha\cdot\mu}{\alpha^2} = (q-p)$ so we have $\mu - (q-p)\alpha$.

2/The vector $\mu - \frac{2\alpha\cdot\mu}{\alpha^2}\,\frac{\alpha}{2}$ is perpendicular to the vector α

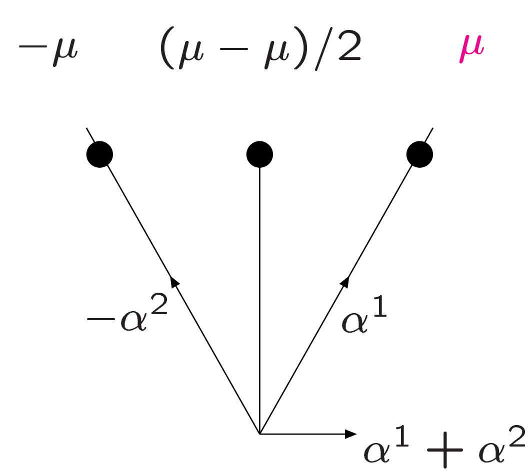
3/ The origin of symmetry of representations - for each root direction there is SU(2) group, with symmetry of eigenvalues of E_3 (like for j-spin repr, symmetry of J_3 eigenvalues $m \leftrightarrow -m$)

$$E_3|\mu> = \frac{\alpha\cdot\mu}{\alpha^2}|\mu>$$

we are looking for states such that $E_3|\mu_w> = -\frac{\alpha\cdot\mu}{\alpha^2}|\mu_w>$, and find that

$$\mu_w = \mu - (q-p)\alpha$$

Weyl reflection in the hyperplane perpendicular to α. Any representation is invariant under all such reflections (ie. for all roots, not necessary simple)



Complex representation

T_a from repr. D and $(-T_a^*)$ satisfy the same commutation relations,

$$([T_a, T_b])^* = (if_{abc}T_c)^* = [T_a^*, T_b^*] = -if_{abc}T_c^*$$

and .. complex conjugate of D, denoted $\bar{D}$.

Repr. real if equiv. to c.c, if not - complex. If μ is a weight, $-\mu$ is a weight of the c.c. repr.

(Cartan gen $H \to -H^*$, eigenvalues of H^* are equal to eigenvalues of H.)

Notation of SU(3) repr.

Ordered pair of (q^1, q^2) or dimension of repr.

(1,0)$\equiv$ 3 and (0,1)$\equiv$ $\bar{3}$

(n, m) and c.c. (m, n)

this gives a clear content for the highest and lowest weights

repr (n, n) are real.

Representations of SU(3): (2,0)

Repr. (2,0), here the highest weight is equal to (2 times μ^1)

$$\mu = 2\mu^1 = (1, 1/\sqrt{3})$$

unique states, obtained from the highest one

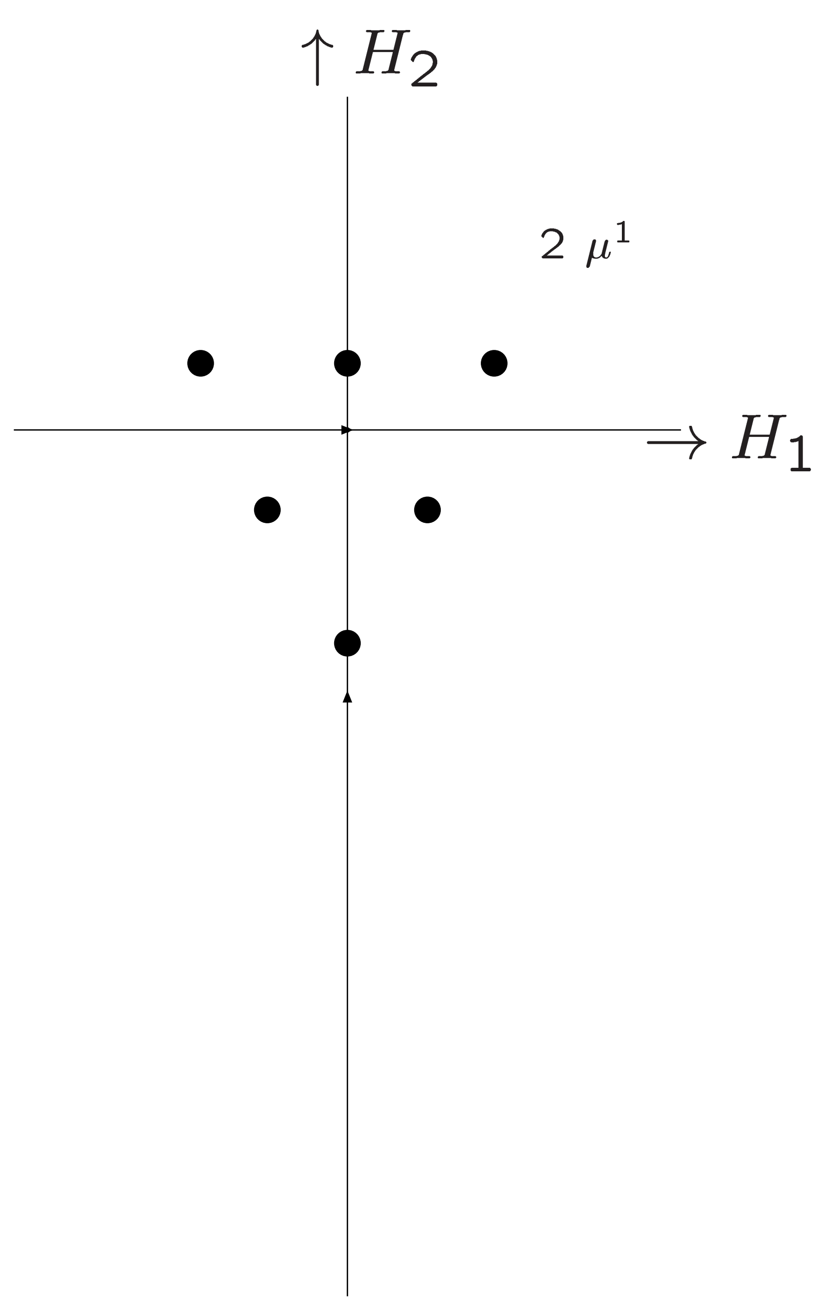
$$2\mu^1, 2\mu^1 - \alpha^1,\ 2\mu^1 - 2\alpha^1,$$

other states...

$$2\mu^1 - \alpha^1 - \alpha^2, etc$$

by eg. Weyl reflection

6-dim rep $(2,0) \equiv 6 \qquad (0,2) \equiv \bar{6}$



$\boxed{\text{Repr. of SU(3): (1,1)}}$

$$\mu^1 + \mu^2 = \alpha^1 + \alpha^2$$

RHS=highest weight of adjoint repr. and Cartan diagram we know: two degenerate zero states and 6 others.

so $(1,1) \equiv 8$

Representation of SU(3): (3,0)

(3,0)=10 and $(0,3)=\overline{10}$
The highest weight of (3,0) $\mu = 3\mu^1$ one can go down 3 steps by using α^1

other states by eg. Weyl reflection

PROBLEMS

PROBLEM 25: For repr. (1,1) of SU(3) show that states $A = E_{-\alpha^1}E_{-\alpha^2}|\mu >$ and $B = E_{-\alpha^2}E_{-\alpha^1}|\mu >$ where μ is the highest weight, are linearly independent.
Hint: consider $< A|A >< B|B >$ and $< A|B ><$ $B|A >$

PROBLEM 26: For repr. (3,0) of SU(3) show that
states $E_{-\alpha^1}E_{-\alpha^2}E_{-\alpha^1}|\mu >$, and $E_{-\alpha^2}E_{-\alpha^1}E_{-\alpha^2}|\mu >$ where μ is the highest weight, are linearly dependent. Consider also state $E_{-\alpha^2}E_{-\alpha^1}E_{-\alpha^1}|\mu >$.

X. Tensor Methods

Introducing three states
(see eigenvectors (1,0,0);(0,1,0);(0,0,1) of H_1, H_2):
of a **3** as follows

$$|1/2, 1/2\sqrt{3} > \equiv |_1 > \tag{8}$$
$$| - 1/2, 1/2\sqrt{3} > \equiv |_2 >$$
$$|0, -1/\sqrt{3} > \equiv |_3 >$$

with transformation $T_a|_i > = |_j > (T_a)^j_i$

and for $\bar{\mathbf{3}}$

$$| - 1/2, -1/2\sqrt{3} > \equiv |^1 > \tag{9}$$
$$|1/2, -1/2\sqrt{3} > \equiv |^2 > \tag{10}$$
$$|0, 1/\sqrt{3} > \equiv |^3 > \tag{11}$$
$$\tag{12}$$

with transformation $T_a|^i > = -|^j > (T_a)^i_j$,
we can define a state in this tensor product space

$$|^{i_1 \dots i_m}_{j_1 \dots j_n} >$$

Let a tensor v in this tensor product space

$$|v>=|^{i_1 \ldots i_m}_{j_1 \ldots j_n}> v^{j_1 \ldots j_n}_{i_1 \ldots i_m}>$$

is characterized by its components $v^{j_1 \ldots j_n}_{i_1 \ldots i_m}$

IRR (n, m)

The state with the highest weight is

$$|^{222...}_{111...}>$$

or v_H, symmetric in upper and lower indices, and traceless. All states in (n,m) can be obtained by acting on v_H with lowering operators; these states preserve these properties (symmetry in upper/lower indices and tracelessness)

Tracelessness:

$$\delta^{i_1}_{j_1} v^{j_1 \ldots j_n}_{i_1 \ldots i_m} = 0$$

Decomposition

Tensor method for $3 \otimes 3$:
So, take v^i, u^j from 3,3 and study $v^i u^j$.

$$v^i u^j = 1/2(v^i u^j + v^j u^i) + 1/2\epsilon^{ijk}\epsilon_{klm} v^l u^m$$

(summetrizing and using ϵ)
This is 6 (the highest state with $u^1 v^1$ or (2,0))
and $\bar{3}$ ($w_k = \epsilon_{klm} v^l u^m$)

$$\boxed{3 \otimes 3 = 6 \oplus \bar{3}}$$

For $3 \otimes \bar{3}$ with v^i and u_j,
the product $v^i u_j$ is equal to

$$(v^i u_j - 1/3\delta^i_j v^k u_k) + 1/3\delta^i_j v^k u_k$$

first traceless object $= 8$, second $(0,0) \equiv 1$

$$3 \otimes \bar{3} = 8 \oplus 1$$

Check that: $3 \otimes 8 = 15 \oplus \bar{6} \oplus 3$

Dimension of (n, m)

$$D(n,m) = \frac{(n+1)(m+1)(n+m+2)}{2}$$

PROBLEMS

PROBLEM 27:

Decompose the product of tensor components $u^i v^{jk}$, where u^i transforms like a **3** of SU(3), and $v^{jk} = v^{jk}$ like a **6** of SU(3).

XI. Hypercharge and Strangeness

In years 1950 - discovery of strange particles (the isospin was established already as an approximate symmetry of strong intr. Experiments: strange particles produced in pairs in scattering of hadrons, like nucleons and pions, and decayed much slower than typical for strong intr. (10^{-8} s vs 10^{-24})

Additive quantum number - strangeness S (conserved by strong intr. and e-m, not by weak intr.); strange particles: $S \neq 0$.

The lightest strange particles are K- mesons mass around 495 MeV, spin 0, baryon number B=0 (superscripts - electric charge)

$\boxed{S = +1}$

$$K^+ \quad I_3 = +1/2$$

$$K^0 \quad I_3 = -1/2$$

and their antiparticles $\boxed{S = -1}$:

$$\bar{K}^0 \quad I_3 = +1/2$$

$$K^- \quad I_3 = -1/2$$

Strange baryons ($B = +1$)

$\boxed{S = -1}$

$$\Sigma^+ \;\; I_3 = +1$$

$$\Sigma^0 \;\; I_3 = 0$$

$$\Sigma^- \;\; I_3 = -1$$

$$\Lambda \;\; I_3 = 0$$

$\boxed{S = -2}$

$$\Xi^0 \;\; I_3 = +1/2$$

$$\Xi^- \;\; I_3 = -1/2$$

All these particles satisfy

$$\boxed{Q = I_3 + \frac{Y}{2}}$$

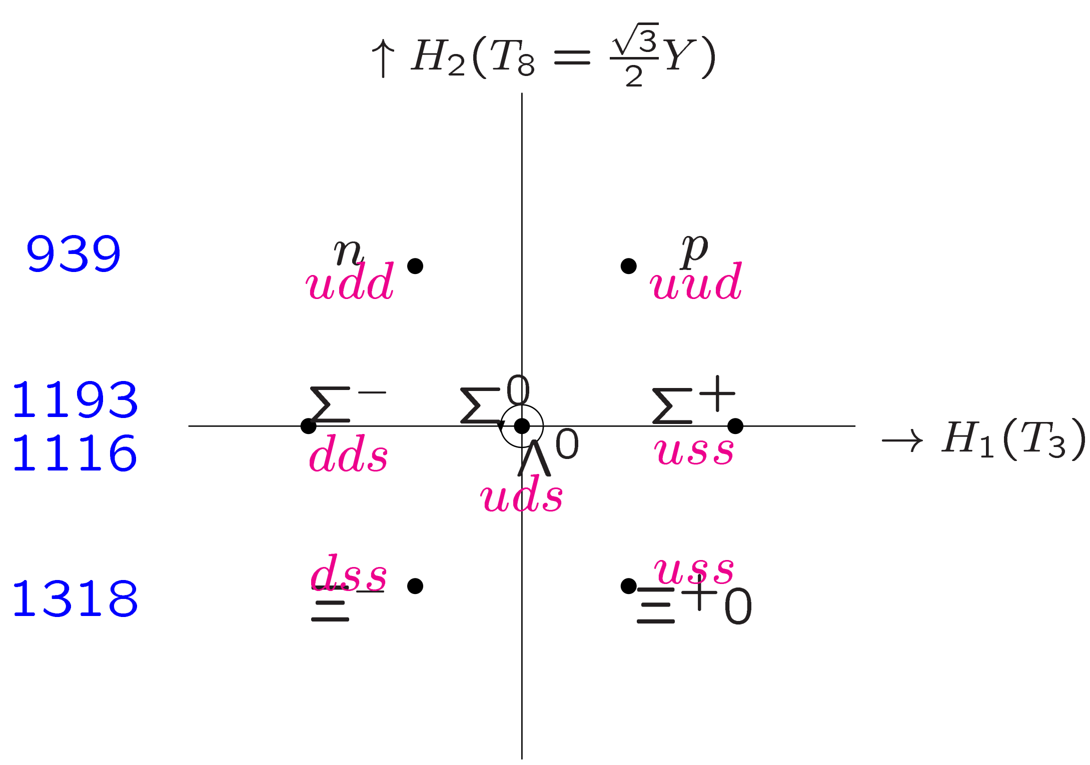
Q- el. charge in units of proton charge

Hypercharge Y = B+S

Note: states $|I, I_3>$ (eigenvalues of op. T_3)

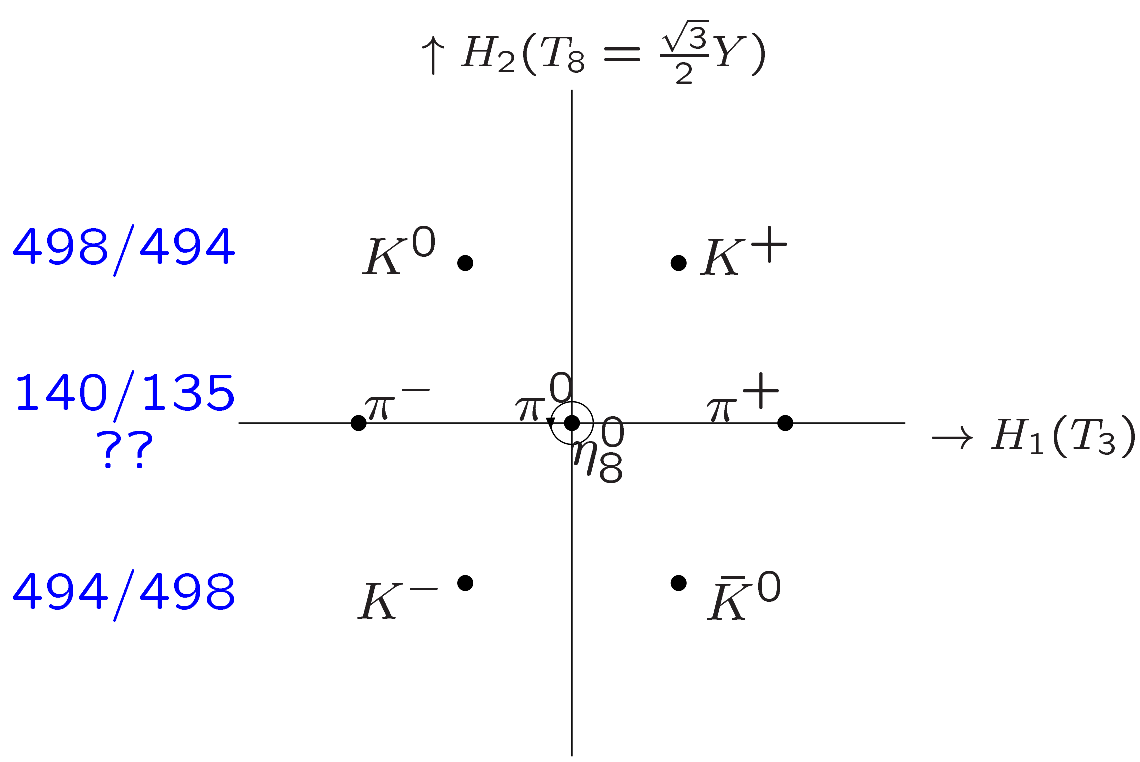
The baryon octet with spin-parity $J^P = \frac{1}{2}^+$

Light baryons with mass around 1 GeV
(*mass in MeV*)



The light meson octet (nonet)with $J^P = 0^-$

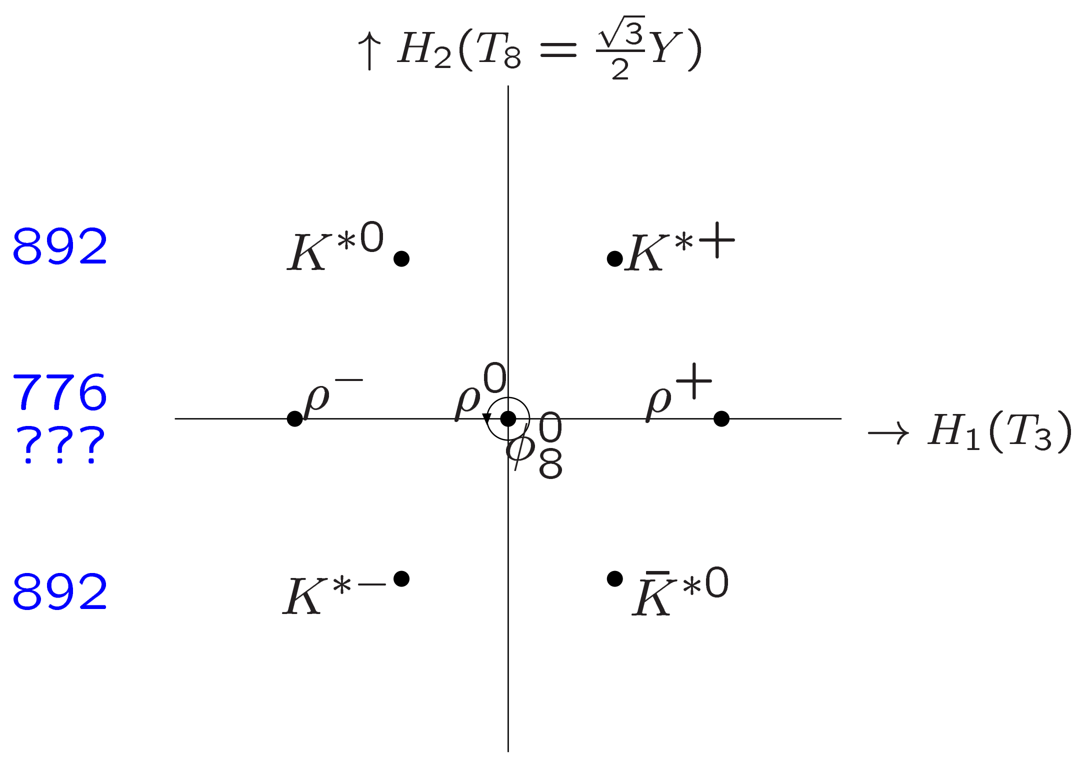
Pseudoscalar mesons (spin-parity) with mass below 550 MeV
(*mass in MeV*)



In addition there is $\eta_0'^0$ - isosinglet, state which is orthogonal to η_8^0 from the octet. Physical particles $\eta(548)$ and $\eta'(958)$ are mixtures of these octet and singlet states.

The light meson octet (nonet) with $J^P = 1^-$

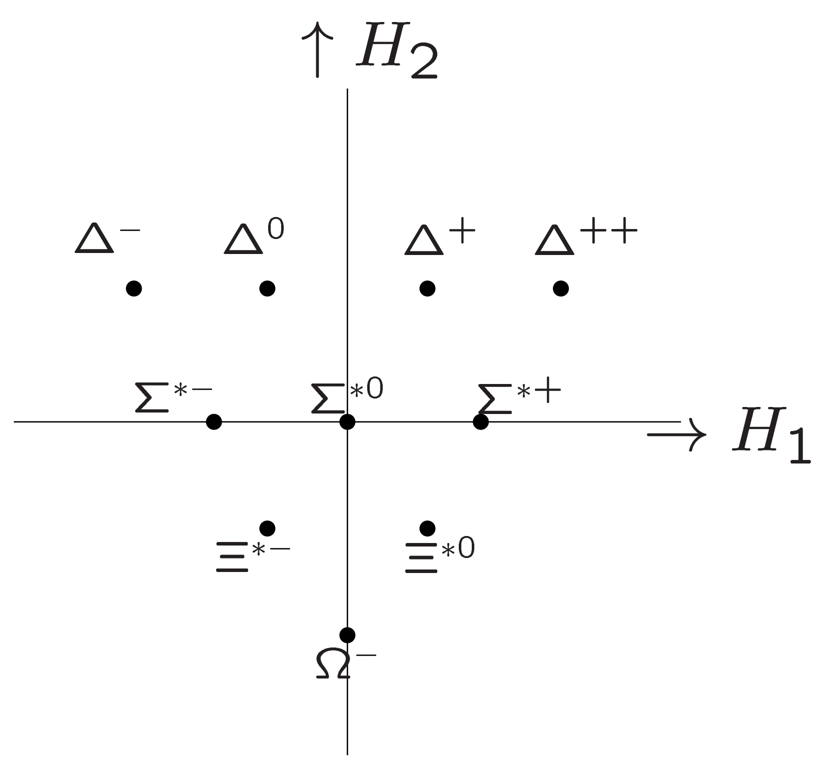
Vector mesons (spin-parity 1^-) with mass around 1000 MeV
(*mass in MeV*)



In addition there is $\phi_0'^0$ - isosinglet state which is orthogonal to ϕ_8^0. Strong mixing and physical particles: $\phi(1019)$ and $\omega(783)$

The heavy baryon octet with $J^P = \frac{3}{2}^+$

Baryons with mass above 1 GeV
(*mass in MeV*)



NOTE: Δ - first resonance discovered in πp scattering - mass 1230 MeV; other masses Σ^* - 1385 MeV, Ξ^* - 1530 MeV , Ω - 1672 MeV.

Flavour SU(3)

The existence in nature of octets and decouplets suggest the SU(3) symmetry of the strong interaction, with isospin generator T_a and hypercharge generator equal to $2T_8/\sqrt{3}$.

Flavour SU(3) symmetry approximate as the masses in multiplets are not exactly the same.

PROBLEM 28:

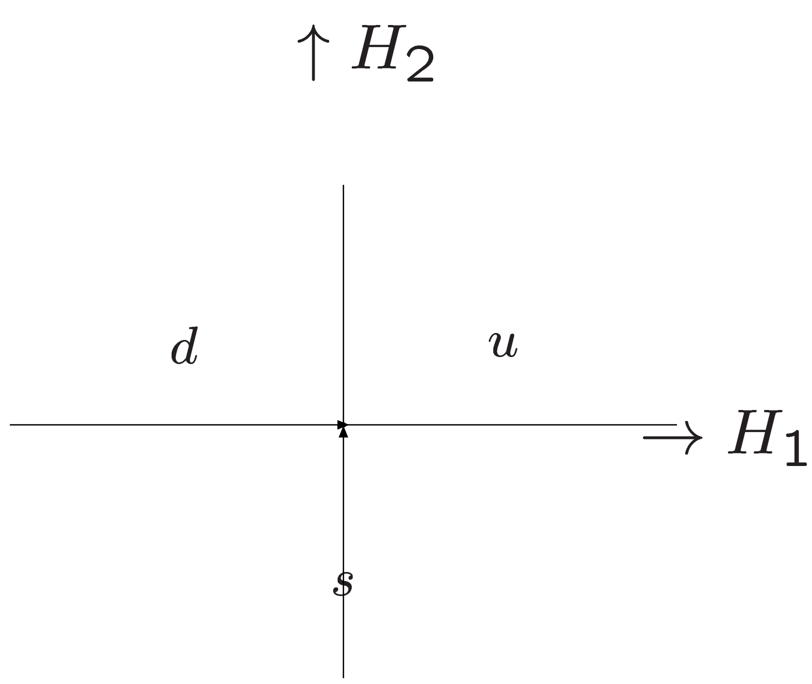
Write matrix for barions with spin 1/2, using states defined as

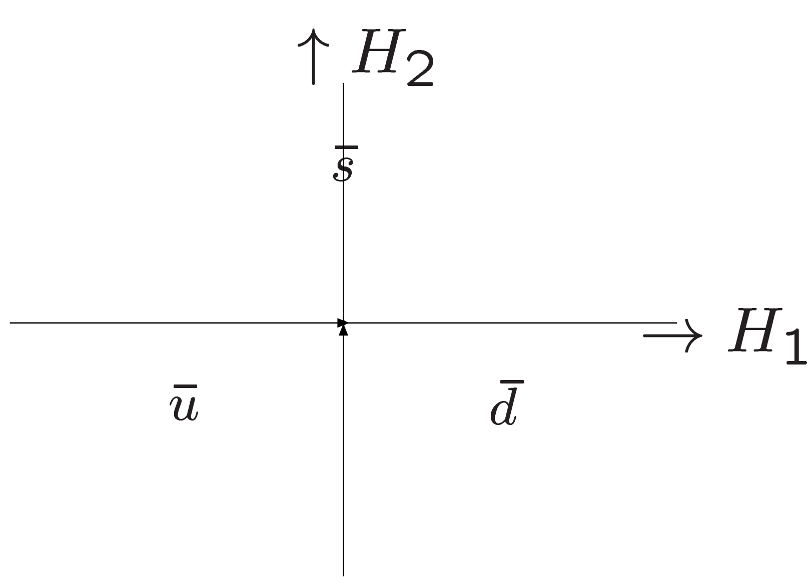
$$B^i_j|^j_i>,$$

where B^i_j labels the particular barions.

Quarks

Light quarks u, d, s - fund. repr. of SU(3) - **3** ;
$\bar{u}, \bar{d}, \bar{s}$ - second fund. repr. of SU(3) - $\bar{3}$ (complex repr of 3); spin 1/2





Quarks inside hadrons

The baryons = three quarks:

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

The mesons = quark-antiquark:

$$3 \otimes \bar{3} = 8 \oplus 1$$

Quarks are spin 1/2 particles; B=1/3; u, d have $S = 0$, while s has $S = -1$

Electric charge: for operators $Q = T_3 + Y/2$

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} =$$

$$\begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$$

PROBLEM 29:

Write the quark content of light meson octets for $J^P = 0^+, J^P = 1^-$.

XII. Young Tableaux*

XIII. SU(N)*

XIV. SU(6) and Quark Model

Including quarks
SU(3) - favours of quarks
SU(2) - spin quarks

6 - dim representation of SU(6) acting on 6- dim tensor space

$$(|u>, |d>, |s>)(|1/2> | -1/2>)$$

$1/\lambda_a \;\; 1/2\sigma_j$, with $a = 1, ...8; j = 1, 2, 3$

enlarged with I op. in spin space, and flavour space, respectively

SU(6) has $SU(3) \otimes SU(2)$ subgroup

The low-lying baryons tranform like 56 dim representation of SU(6), the completely symmetric combination of three 6.

Example

Decuplet

$$|\Delta^{++}, 3/2 >= |uuu > |\uparrow\uparrow\uparrow>$$

$$I_{-}|uuu >= N(|duu > +|udu > +|uud >)$$

Heavy Quarks

Heavy quarks discovered after 1974 with new additive quantum numbers, conserved by strong interaction

charm - mass around 1.5 GeV , CC=+1

bottom - mass around 5 GeV, BB=-1 (1977)

top - mass around 174 GeV, T=+1 (1995)

There is no approximate flavour SU(4) etc symmetry...

Quantum number relation holds:

$$Q = I_3 + (B + S + CC + BB + T)/2$$

Problems of Quark Model

1/ Quarks do not exist in Nature

2/ Why only $q\bar{q}$ and qqq states in Nature?

3/ Fermi-Dirac statistic violated in states eg.

$$|\Delta^{++}, J_3 = 3/2 >= |uuu > | \uparrow\uparrow\uparrow>$$

since the lowest state ($l = 0$) is described by symmetric function, we have the totally symmetric function under the exchange of quarks u, in contradiction to the fact that Δ^{++} is a fermion.

PROBLEM 30:
Calculate Casimir operator $T^2 = \Sigma_a T_a T_a$ for representation D = 3, 8, 10 and 6 of SU(3)
Hint:

$$TrT^2 = dimDT^2 = \Sigma_a k_D,$$

since $Tr(H_i H_j) = k_D \delta_{ij}$

Note, that for $SU(N)$

$$(T^a T^a)_{ij} = \frac{N^2 - 1}{2N} \delta_{ij}$$

XV. Color

Color is a recipe for problems of Quark Model

Introducing LOCAL $SU(3)_c$ to describe interaction of strong interaction as a result of interaction of quarks with "charge", which is called color, and carriers of such interaction - gluons, also being charged (colored).

Postulated number of colors: $N_c = 3$
Support from the measurements:

• $R_{e^+e^-} = N_c \sum e_q^2$

• Decay of π^0 into two photons,
is realized via quark loops, and the decay width for this decay is proportional to N_c^2

• Another argument from cancelation of anomaly in SM

There are 8 gluons.

$\boxed{SU(3)_c}$

Quarks with specific flavour exist in various color states (repr. 3 of $SU(3)_c$). Hadrons = singlets of$SU(3)_c$.

$$q^i, i = 1, 2, 3$$

Barions = antisymmetric state of three quarks:

$$\epsilon_{ijk} q^i q^j q^k$$

There are antiquarks as well $\bar{q}_i, i = 1, 2, 3$

Mesons = quark-antiquark state

$$q^i \bar{g}_i$$

Argument why such states arises from analogy with QED, and attractive forces between object A and B for $Q_A Q_B < 0$;
here in QCD $\Sigma_{a=1}^{8} T_a^A T_a^B < 0$

Colorless hadrons

$q\bar{q}$ $\quad 3 \otimes \bar{3} = 1 \oplus 8$

qqq $\quad 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$

Note that

qq $\quad 3 \otimes 3 = \bar{3} \oplus 6$

$qqqq$ $\quad 3 \otimes 3 \otimes 3 \oplus 3 = 3 \oplus 3 \oplus 3 \oplus \bar{6} \oplus 15 \oplus 15 \oplus 15$

Only color singlet in Nature - confinement of quarks

XVII. SU(2)xU(1) of Standard Model

Introduction

• If spin 1/2 $\hbar$ particles are moving their state can be characterized by helicity - the component of the spin in the direction of motion

• Eg. spin 1/2 $\hbar$: $h = \pm 1/2$
Right-handed (RH): helicity = 1/2
Left-handed (LH): helicity $= -1/2$

(Antiparticle of RH particle is LH, and vice versa).
Massive particle have to have both components.

• Weak inter. violates parity i.e. - treats LH and RH differently

The Glashow-Salam-Weinberg theory of electroweak interaction $SU(2)_{I_w} \times U(1)_{Y_w}$

Creation and annihilation operators in SM

Creation and annihilation op. are tensor operators under these groups (as for isospin..).

For simplicity (!) we consider only first generation u, d, e and their antiparticles plus $\bar{\nu}$, and use only RH components:

so we consider $u^\dagger$, $d^\dagger$, $e^\dagger$, $\bar{u}^\dagger$, $\bar{d}^\dagger$, $\bar{e}^\dagger$, $\bar{\nu}^\dagger$
(All other indexes are skipped)

Let us denote:
SU(2) with generators R_a, $a = 1, 2, 3$
U(1) with generator S

$$Q = R_3 + S$$

Under SU(2) algebra:
- positron and antineutrino (both RH) form a doublet (as p and n) $\rightarrow \bar{l}$
- $\bar{d}^\dagger$ and $\bar{u}^\dagger$ as well $\rightarrow \bar{\psi}$

$$\bar{l} = \begin{pmatrix} \bar{e}^\dagger \\ \bar{\nu}^\dagger \end{pmatrix} \; \bar{\psi} = \begin{pmatrix} \bar{d}^\dagger \\ \bar{u}^\dagger \end{pmatrix}$$

Commutation relation

Since $u^\dagger, d^\dagger, e^\dagger$ are singlets (spin-0 representation)

$[R_a, u^\dagger] = [R_a, d^\dagger] = [R_a, e^\dagger] = 0$

for spin-1/2 representation (σ_a- Pauli matrices):

$[R_a, \overline{\psi}_r^\dagger] = \overline{\psi}_s^\dagger [\sigma_a]_{sr}/2$
$[R_a, \overline{l}_r^\dagger] = \overline{l}_s^\dagger [\sigma_a]_{sr}/2$

Using these and relation $Q = R_3 + S$ we obtain:

$[S, u^\dagger] = s u^\dagger = 2/3\ u^\dagger$
$[S, d^\dagger] = -1/3\ d^\dagger$
$[S, e^\dagger] = -1\ e^\dagger$
$[S, \overline{\psi}_r^\dagger] = -1/6\ \overline{\psi}_r^\dagger$
$[S, \overline{l}_r^\dagger] = 1/2\ \overline{l}_r^\dagger$

All fields scalars only $\bar{l}$ and $\bar{\psi}$ - doublets.

Spontaneous symmetry breaking

Higgs mechanism in SM

We add doublet of scalar fields ϕ, with potential V respecting SU(2)xU(1) symmetry

and assume that symmetry is not satisfied in states

This is called spontaneous symmetry breaking - SSB

Symmetry of the vacuum

Vacuum:

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

op. SU(2) :$R_i = \sigma_i/2$

op. U(1)$_{weak\ hypercharge}$:

$$S = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

op. Q (el. charge):

$$\hat{Q} \equiv \sigma_3/2 + S = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 1/2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 1/2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

$$\hat{Q} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$vacuum invariant versus U(1)$_{QED}$
since

$$e^{i\hat{Q}\alpha}\phi_0 = \phi_0.$$

XVII. SU(5)

Using $(D,d)_s$ for describing representation of $SM = SU(3) \times SU(2) \times U(1)$ (s - is a S quantum number)

derive full representation of SM for RH fields (*)

$(3,1)_{2/3} \oplus (3,1)_{-1/3} \oplus (1,1)_{-1} \oplus (\bar{3},2)_{-1/6} \oplus (1,2)_{1/2}$

We want to find an algebra G with contains SM=... as a subgroup, and which has a representation transforming like above under this subgroup

Rank- at least four (T_3, T_8, R_3, S)

The simplest is SU(5)

5-dim. representation

Question:
can we find an SU(3)xSU(2)xU(1) subgroup of SU(5) such that 5 transforms like some 5 dim. subset of creation op.
From (*) : only possibility 5-dim subset

$$(3,1)_{-1/3} \oplus (1,2)_{1/2}$$

(or $(3,1)_{2/3} \oplus (1,2)_{1/2}$? S not traceless on it...)
Generators for these states (all traceless):

$$SU(3): \left(\begin{array}{c|c} T_a & 0 \\ \hline 0 & 0 \end{array}\right).$$

$$SU(2): \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & R_a \end{array}\right).$$

$$S: \left(\begin{array}{ccc|cc} -1/3 & & & 0 & 0 \\ & -1/3 & & 0 & 0 \\ & & -1/3 & 0 & 0 \\ \hline & 0 & & 1/2 & 0 \\ & & & 0 & 1/2 \end{array}\right).$$

There is also $\bar{5}, 10, \overline{10}$ representations.