

Virtual Photon Structure functions to NNLO in QCD

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Introduction

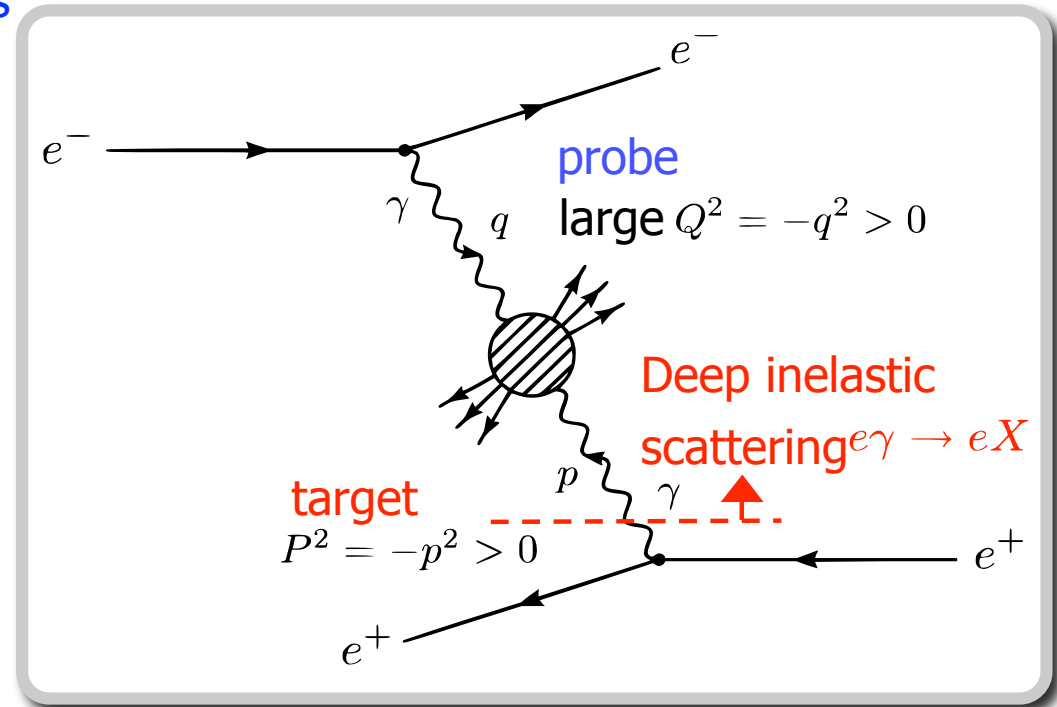
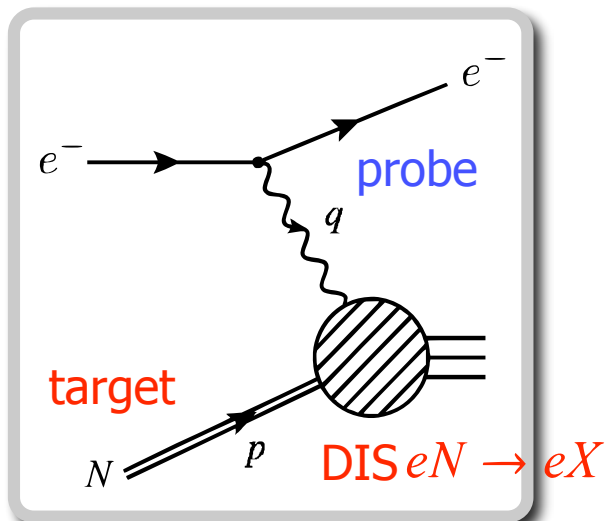
- Future linear collider experiment (e.g. **ILC**)

Two-photon process $e^+e^- \rightarrow (e^+e^-\gamma\gamma) \rightarrow e^+e^-X$

Viewed as a **deep-inelastic electron-photon scattering**

We can study **the structures of photon**

⇒ F_2^γ and F_L^γ



Two-photon Process

- Differential cross section

$$d\sigma = \frac{d^3l'_1 d^3l'_2}{2E'_1 2E'_2 (2\pi)^5} \frac{(4\pi\alpha)^3}{p^2 q^2} \frac{1}{4\sqrt{(l_1 \cdot l_2)^2 - m_e^2}} \rho_1^{\mu\nu} \rho_2^{\rho\tau} W_{\mu\nu\rho\tau}$$

- Leptonic part $\rho_1^{\mu\nu}$ and $\rho_2^{\rho\tau}$

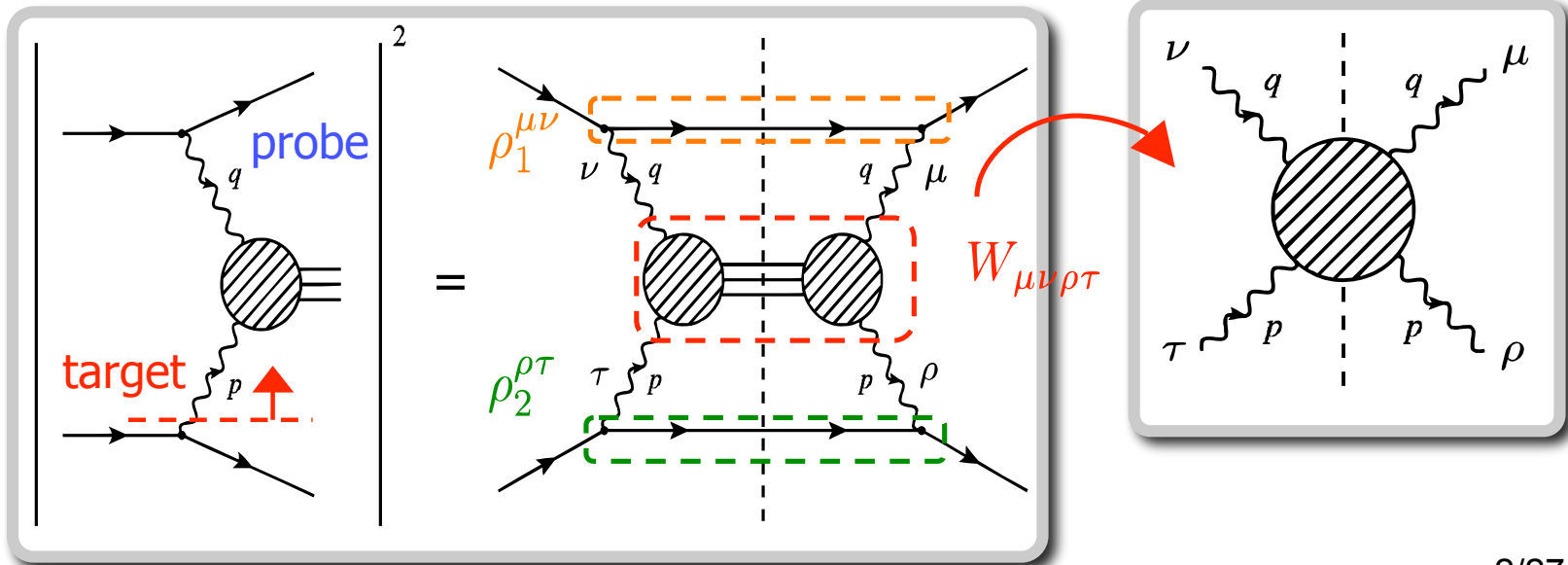
$$e \rightarrow e + \gamma$$

QED vertex

- Hadronic part

$W_{\mu\nu\rho\tau}$ photon structure tensor

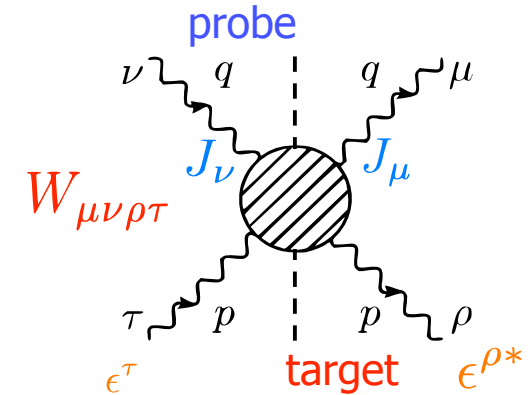
$$\gamma\gamma \rightarrow X$$



Spin-averaged Virtual Photon Structure Funcs.

● Spin-averaged structure tensor

$$\begin{aligned}
 W_{\mu\nu}^\gamma(p, q) &= \frac{1}{2} \sum_{\lambda} \epsilon_{(\lambda)}^{\rho*}(p) W_{\mu\nu\rho\tau} \epsilon_{(\lambda)}^\tau(p) \\
 &= \frac{1}{2\pi} \int d^4x e^{iq \cdot x} \langle \gamma(p) | J_\mu(x) J_\nu(0) | \gamma(p) \rangle_{\text{spin ave.}}
 \end{aligned}$$



● Structure functions F_2^γ and F_L^γ

$$F_L = F_2 - xF_1$$

$$W_{\mu\nu}^\gamma = e_{\mu\nu} \frac{1}{x} F_L^\gamma(x, Q^2, P^2) + d_{\mu\nu} \frac{1}{x} F_2^\gamma(x, Q^2, P^2) \quad \text{In analogy with the nucleon}$$

$$e_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \quad d_{\mu\nu} = -g_{\mu\nu} + \frac{q_\mu p_\nu + p_\mu q_\nu}{p \cdot q} - \frac{p_\mu p_\nu q^2}{(p \cdot q)^2}$$

$$x = \frac{Q^2}{2p \cdot q} \quad \text{:Bjorken variable} \quad 0 \leq x \leq 1$$

$$-Q^2 = q^2 \leq 0 \quad \text{:mass squared of the probe photon}$$

$$-P^2 = p^2 \leq 0 \quad \text{:mass squared of the target photon}$$

F_2^γ in Perturbative QCD

- For **real** photon target ($P^2 \approx 0$) $P^2 \ll Q^2$

$$F_2^\gamma(x, Q^2) = \alpha \left[\frac{1}{\alpha_s(Q^2)} A + B + B' + \mathcal{O}(\alpha_s) \right]$$

OPE+RGE
lowest order in α

$$\sim \ln \frac{Q^2}{\Lambda^2}$$

(LO)

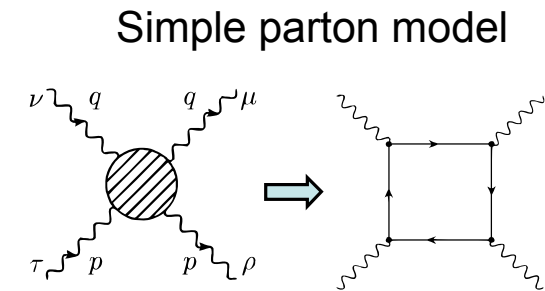
(NLO)

Hadronic piece

Witten (1977)

Bardeen-Buras (1979)

⇒ NNLO extension Moch-Vermaseren-Vogt (2002, 2006)



Point-like contribution
dominates $\sim \ln Q^2$

Walsh-Zerwas (1973)

- For highly **virtual** photon target ($\Lambda^2 \ll P^2 \ll Q^2$)

$$F_2^\gamma(x, Q^2, P^2) = \alpha \left[\frac{1}{\alpha_s(Q^2)} \tilde{A} + \tilde{B} + \mathcal{O}(\alpha_s) \right] \quad \Lambda : \text{QCD scale parameter}$$

(LO)

(NLO)

Uematsu-Walsh (1981, 1982)

Hadronic piece can also be dealt with **perturbatively**

Definite prediction of F_2^γ , **its shape and magnitude**, is possible

- We extend the analysis to **NNLO** ($\alpha\alpha_s$)

Motivated by the calculation of 3-loop anomalous dimensions
Vogt-Moch-Vermaseren (2004, 2006)

Overview of the method

- $$W_{\mu\nu}^\gamma(p, q) = \frac{1}{2\pi} \int d^4z e^{iq \cdot z} \langle \gamma(p) | \underbrace{J_\mu(z) J_\nu(0)}_{\text{red wavy line}} | \gamma(p) \rangle_{\text{spin ave.}}$$

$$= e_{\mu\nu} \frac{1}{x} F_L^\gamma(x, Q^2, P^2) + d_{\mu\nu} \frac{1}{x} F_2^\gamma(x, Q^2, P^2)$$

- The OPE near the light-cone** $Q^2 \rightarrow \infty \iff$ light-cone

$$J(z)J(0) \sim \sum_i C_i(z) O_i(0) \quad i : \text{over relevant ops.}$$

$$\implies \langle \gamma(p) | J(z)J(0) | \gamma(p) \rangle \sim \sum_i C_i(z) \langle \gamma(p) | O_i | \gamma(p) \rangle$$

- Spin- n twist-2 operators** (hadronic ops. + photon op.)

(n -th moment) (dominant) $\tau = d_O - n$

quark : $O_\psi^{\mu_1 \dots \mu_n}$ (flavor singlet) gluon : $O_G^{\mu_1 \dots \mu_n}$

$O_{NS}^{\mu_1 \dots \mu_n}$ (flavor non-singlet) photon : $O_\gamma^{\mu_1 \dots \mu_n}$

QCD with massless quarks with n_f flavors

Overview of the method

- Moment sum rule of F_2^γ

$$M_2^\gamma(n, Q^2, P^2) = \int_0^1 dx x^{n-2} F_2^\gamma(x, Q^2, P^2)$$

we take $\mu^2 = -p^2 = P^2$

$$= \sum_{i=\psi, G, NS, \gamma} C_{2,n}^i \left(\frac{Q^2}{P^2}, \bar{g}(P^2), \alpha \right) A_n^i(\bar{g}(P^2), \alpha) \quad \text{for even } n$$

RG improved coefficients

Photon matrix element
 $\langle \gamma(p) | O_i^n | \gamma(p) \rangle_{\text{spin ave.}}$

Perturbatively calculable when $\Lambda^2 \ll P^2$

$$C_{2,n}^i \left(\frac{Q^2}{P^2}, \bar{g}(P^2), \alpha \right) = \left(T \exp \left[\int_{\bar{g}(Q^2)}^{\bar{g}(P^2)} dg \frac{\gamma_n(g, \alpha)}{\beta(g)} \right] \right)_{ij} C_{2,n}^j(1, \bar{g}(Q^2), \alpha)$$

Solution of RG eq. for coefficient functions

$\beta(g)$: QCD beta function

Expand $M_2^\gamma(n, Q^2, P^2)$ up to NNLO ($\alpha\alpha_s$)

γ_n : anomalous dimensions

- Inverse Mellin transformation: n -space \Rightarrow x -space

$$F_2^\gamma(x, Q^2, P^2) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dn x^{1-n} M_2^\gamma(n, Q^2, P^2)$$

numerically inverted

Anomalous dimension matrix

- To the lowest order in α

$$\gamma_n(g, \alpha) = \left(\begin{array}{c|c} \hat{\gamma}_n(g) & \mathbf{0} \\ \hline \mathbf{K}_n(g, \alpha) & 0 \end{array} \right)$$

- 3×3 anomalous dimension matrix in the hadronic sector

$$\hat{\gamma}_n(g) = \begin{pmatrix} \gamma_{\psi\psi}^n(g) & \gamma_{G\psi}^n(g) & 0 \\ \gamma_{\psi G}^n(g) & \gamma_{GG}^n(g) & 0 \\ 0 & 0 & \gamma_{NS}^n(g) \end{pmatrix} \quad \hat{\gamma}_n^{(0)} = \sum_{i=+,-,NS} \lambda_i^n P_i^n$$

$$\sum_i P_i = 1 \quad P_i P_j = \delta_{ij} P_i$$

- Mixing anomalous dimensions between photon operator and three hadronic operators

$$\mathbf{K}_n(g, \alpha) = \left(K_{\psi}^n(g, \alpha) \quad K_G^n(g, \alpha) \quad K_{NS}^n(g, \alpha) \right)$$

Moment Sum Rule for F_2^γ

- Summarized as

$$\begin{aligned}
 & \int_0^1 dx x^{n-2} F_2^\gamma(x, Q^2, P^2) && \text{LO } (\alpha\alpha_s^{-1}) && \text{for even } n \\
 &= \frac{\alpha}{4\pi} \frac{1}{2\beta_0} \left\{ \underbrace{\frac{4\pi}{\alpha_s(Q^2)} \sum_i \mathcal{L}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{d_i^n + 1} \right]}_{\text{NLO } (\alpha)} \right. \\
 & \quad + \underbrace{\sum_i \mathcal{A}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{d_i^n} \right] + \sum_i \mathcal{B}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{d_i^n + 1} \right] + \mathcal{C}^n}_{\text{NLO } (\alpha)} \\
 & \quad + \frac{\alpha_s(Q^2)}{4\pi} \left(\sum_i \mathcal{D}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{d_i^n - 1} \right] + \sum_i \mathcal{E}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{d_i^n} \right] \right. \\
 & \quad \left. + \underbrace{\sum_i \mathcal{F}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{d_i^n + 1} \right] + \mathcal{G}^n}_{\text{NNLO } (\alpha\alpha_s)} \right) + \mathcal{O}(\alpha_s^2) \left. \right\} \\
 & && \text{NNLO } (\alpha\alpha_s)
 \end{aligned}$$

Moment Sum Rule for F_2^γ

LO :

$$\mathcal{L}_i^n = \mathbf{K}_n^{(0)} P_i^n \mathbf{C}_{2,n}^{(0)} \frac{1}{d_i^n + 1} \quad d_i^n = \frac{\lambda_i^n}{2\beta_0} \quad i = +, -, NS$$

NLO :

$$\begin{aligned} \mathcal{A}_i^n = & -\mathbf{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \mathbf{C}_{2,n}^{(0)} \frac{1}{d_i^n} - \mathbf{K}_n^{(0)} P_i^n \mathbf{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1 - d_i^n}{d_i^n} \\ & + \mathbf{K}_n^{(1)} P_i^n \mathbf{C}_{2,n}^{(0)} \frac{1}{d_i^n} - 2\beta_0 \tilde{\mathbf{A}}_n^{(1)} P_i^n \mathbf{C}_{2,n}^{(0)} \end{aligned}$$

$$\begin{aligned} \mathcal{B}_i^n = & \mathbf{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \mathbf{C}_{2,n}^{(0)} \frac{1}{1 + d_i^n} \\ & + \mathbf{K}_n^{(0)} P_i^n \mathbf{C}_{2,n}^{(1)} \frac{1}{1 + d_i^n} - \mathbf{K}_n^{(0)} P_i^n \mathbf{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{d_i^n}{1 + d_i^n} \end{aligned}$$

$$\mathcal{C}^n = 2\beta_0 (\mathbf{C}_{2,n}^{\gamma(1)} + \tilde{\mathbf{A}}_n^{(1)} \cdot \mathbf{C}_{2,n}^{(0)})$$

Moment Sum Rule for F_2^γ

NNLO :

$$\begin{aligned}
 \mathcal{D}_i^n = & -\mathbf{K}_n^{(0)} P_i^n \mathbf{C}_{2,n}^{(0)} \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \frac{1}{1-d_i^n} \right) \left(1 - \frac{d_i^n}{2} \right) \\
 & -\mathbf{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \mathbf{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1-d_j^n}{1-d_i^n} \\
 & -\mathbf{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 4\beta_0} \mathbf{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \left(\frac{1-d_i^n + d_j^n}{1-d_i^n} \right) \\
 & +\mathbf{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(2)} P_i^n}{\lambda_j^n - \lambda_i^n + 4\beta_0} \mathbf{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} \\
 & -\mathbf{K}_n^{(0)} \sum_{j,k} \frac{P_k^n \hat{\gamma}_n^{(1)} P_j^n \hat{\gamma}_n^{(1)} P_i^n}{(\lambda_j^n - \lambda_i^n + 2\beta_0)(\lambda_k^n - \lambda_i^n + 4\beta_0)} \mathbf{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} \\
 & +\mathbf{K}_n^{(1)} P_i^n \mathbf{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} + \mathbf{K}_n^{(1)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \mathbf{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} \\
 & -\mathbf{K}_n^{(2)} P_i^n \mathbf{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} + 2\beta_0 \tilde{\mathbf{A}}_n^{(1)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \mathbf{C}_{2,n}^{(0)} \\
 & -2\beta_0 \tilde{\mathbf{A}}_n^{(1)} P_i^n \mathbf{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} d_i^n - 2\beta_0 \tilde{\mathbf{A}}_n^{(2)} P_i^n \mathbf{C}_{2,n}^{(0)} ,
 \end{aligned}$$

Moment Sum Rule for F_2^γ

NNLO :

$$\begin{aligned}
 \mathcal{E}_i^n = & -K_n^{(0)} P_i^n C_{2,n}^{(1)} \frac{\beta_1}{\beta_0} \frac{1 - d_i^n}{d_i^n} - K_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} C_{2,n}^{(1)} \frac{1}{d_i^n} \\
 & + K_n^{(1)} P_i^n C_{2,n}^{(1)} \frac{1}{d_i^n} + K_n^{(0)} P_i^n C_{2,n}^{(0)} \frac{\beta_1^2}{\beta_0^2} (1 - d_i^n) \\
 & - K_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} C_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1 - d_i^n}{d_i^n} + K_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} C_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \\
 & - K_n^{(0)} \sum_{j,k} \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n \hat{\gamma}_n^{(1)} P_k^n}{(\lambda_i^n - \lambda_k^n + 2\beta_0)(\lambda_j^n - \lambda_i^n + 2\beta_0)} C_{2,n}^{(0)} \frac{1}{d_i^n} \\
 & - K_n^{(1)} P_i^n C_{2,n}^{(0)} \frac{\beta_1}{\beta_0} + K_n^{(1)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} C_{2,n}^{(0)} \frac{1}{d_i^n} \\
 & - 2\beta_0 \tilde{A}_n^{(1)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} C_{2,n}^{(0)} + 2\beta_0 \tilde{A}_n^{(1)} P_i^n C_{2,n}^{(0)} \frac{\beta_1}{\beta_0} d_i^n - 2\beta_0 \tilde{A}_n^{(1)} P_i^n C_{2,n}^{(1)}
 \end{aligned}$$

Moment Sum Rule for F_2^γ

NNLO :

$$\begin{aligned}
 \mathcal{F}_i^n = & K_n^{(0)} P_i^n C_{2,n}^{(2)} \frac{1}{1+d_i^n} - K_n^{(0)} P_i^n C_{2,n}^{(1)} \frac{\beta_1}{\beta_0} \frac{d_i^n}{1+d_i^n} \\
 & + K_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} C_{2,n}^{(1)} \frac{1}{1+d_i^n} \\
 & + K_n^{(0)} P_i^n C_{2,n}^{(0)} \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \frac{1}{1+d_i^n} \right) \frac{d_i^n}{2} \\
 & - K_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} C_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{d_j^n}{1+d_i^n} \\
 & - K_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 4\beta_0} C_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1+d_i^n - d_j^n}{1+d_i^n} \\
 & + K_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(2)} P_j^n}{\lambda_i^n - \lambda_j^n + 4\beta_0} C_{2,n}^{(0)} \frac{1}{1+d_i^n} \\
 & + K_n^{(0)} \sum_{j,k} \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n \hat{\gamma}_n^{(1)} P_k^n}{(\lambda_j^n - \lambda_k^n + 2\beta_0)} C_{2,n}^{(0)} \left(\frac{1}{\lambda_i^n - \lambda_j^n + 2\beta_0} - \frac{1}{\lambda_i^n - \lambda_k^n + 4\beta_0} \right) \frac{1}{1+d_i^n} ,
 \end{aligned}$$

$$\mathcal{G}^n = 2\beta_0(C_{2,n}^{\gamma(2)}) + \tilde{\mathbf{A}}_n^{(1)} \cdot C_{2,n}^{(1)} + \tilde{\mathbf{A}}_n^{(2)} \cdot C_{2,n}^{(0)} .$$

Needed parameters in $\overline{\text{MS}}$

- QCD β -function

Tarasov-Vladimirov-Zarkov (1980)
Larin-Vermaseren (1993)

$$\beta(g) = -\frac{g^3}{16\pi^2}\beta_0 - \frac{g^5}{(16\pi^2)^2}\beta_1 - \frac{g^7}{(16\pi^2)^3}\beta_2 + \mathcal{O}(g^9)$$

1-loop 2-loop 3-loop

- Anomalous dimensions of hadronic operators

Moch-Vermaseren-Vogt (2004)

$$\hat{\gamma}^n(g) = \frac{g^2}{16\pi^2}\hat{\gamma}^{(0),n} + \frac{g^4}{(16\pi^2)^2}\hat{\gamma}^{(1),n} + \frac{g^6}{(16\pi^2)^3}\hat{\gamma}^{(2),n} + \mathcal{O}(g^8)$$

1-loop 2-loop 3-loop

- Hadronic coefficient functions

van Neerven-Zijlstra (1991,1992)

$$C_2^n(g) = C_2^{(0),n} + \frac{g^2}{16\pi^2}C_2^{(1),n} + \frac{g^4}{(16\pi^2)^2}C_2^{(2),n} + \mathcal{O}(g^6)$$

tree 1-loop 2-loop

Needed parameters in $\overline{\text{MS}}$

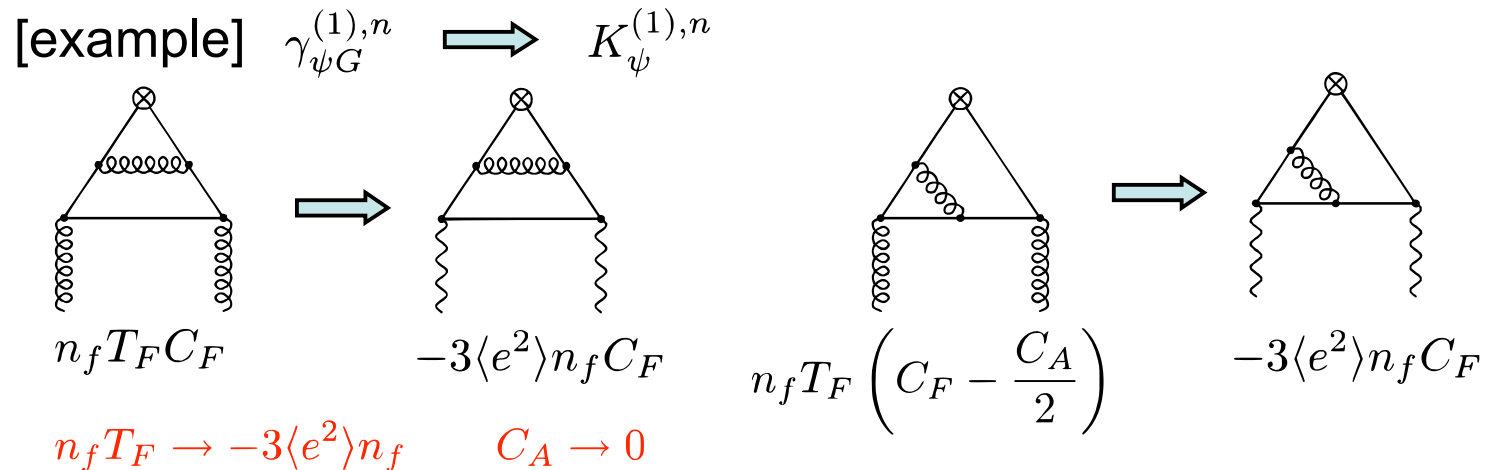
- Mixing anomalous dimensions

$$K^n(g, \alpha) = -\frac{\alpha}{4\pi} \left[\underbrace{K^{(0),n}}_{\text{1-loop}} + \frac{g^2}{16\pi^2} \underbrace{K^{(1),n}}_{\text{2-loop}} + \frac{g^4}{(16\pi^2)^2} K^{(2),n} + \mathcal{O}(g^6) \right]$$

- Photon coefficient function

$$C_{2,\gamma}^n(g, \alpha) = \frac{\alpha}{4\pi} \left[\underbrace{C_{2,\gamma}^{(1),n}}_{\text{1-loop}} + \frac{g^2}{16\pi^2} \underbrace{C_{2,\gamma}^{(2),n}}_{\text{2-loop}} + \mathcal{O}(g^4) \right]$$

$K^{(1),n}$, $C_{2,\gamma}^{(2),n}$ are obtained by the replacement of group factors



Needed parameters in \overline{MS}

- Mixing anomalous dimensions

$$K^n(g, \alpha) = -\frac{\alpha}{4\pi} \left[K^{(0),n} + \frac{g^2}{16\pi^2} K^{(1),n} + \frac{g^4}{(16\pi^2)^2} K^{(2),n} + \mathcal{O}(g^6) \right]$$

3-loop

No exact expressions for 3-loop $K_\psi^{(2),n}$, $K_{NS}^{(2),n}$, $K_G^{(2),n}$ yet

But the compact parametrization of

3-loop splitting functions $P_{ns\gamma}^{(2),\text{approx}}(x)$, $P_{G\gamma}^{(2),\text{approx}}(x)$

(and exact $P_{ps\gamma}^{(2)}(x)$) exist

(the deviation < 0.1 %)

A.Vogt, S.Moch and J.Vermaseren
Acta Phys.Polon. B37,683(2006)

- We use

$$K_{\psi,\text{approx}}^{(2),n} = \int dx x^{n-1} \left[P_{ns\gamma}^{(2),\text{approx}}(x) + P_{ps\gamma}^{(2)}(x) \right]$$

$$K_{NS,\text{approx}}^{(2),n} = \int dx x^{n-1} \left[P_{ns\gamma}^{(2),\text{approx}}(x) \right]$$

$$K_{G,\text{approx}}^{(2),n} = \int dx x^{n-1} \left[P_{G\gamma}^{(2),\text{approx}}(x) \right]$$

Needed parameters in $\overline{\text{MS}}$

- Photon matrix elements Perturbatively calculable when $\Lambda^2 \ll P^2$

$$A^n(g, \alpha) = \frac{\alpha}{4\pi} \left[\underbrace{A^{(1),n}}_{\text{1-loop}} + \frac{g^2}{16\pi^2} \underbrace{A^{(2),n}}_{\text{2-loop}} + \mathcal{O}(g^4) \right]$$

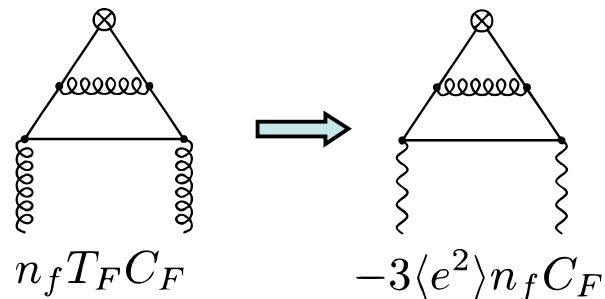
- Unrenormalized gluon matrix elements of hadronic operators were calculated in x -space

Matiounine-Smith-van Neerven (1998)

- We obtain $A_\psi^{(2),n}$, $A_{NS}^{(2),n}$, $A_G^{(2),n}$, after taking moments,

$$\hat{A}_{qg}^k \left(n, \frac{-p^2}{\mu^2}, \frac{1}{\epsilon} \right) = \int_0^1 dx x^{n-1} \hat{A}_{qg}^k \left(x, \frac{-p^2}{\mu^2}, \frac{1}{\epsilon} \right)$$

renormalization and the replacement of group factors



Numerical Inversion

- Inverse Mellin transformation

$$F_2^\gamma(x, Q^2, P^2) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dn x^{1-n} M_2^\gamma(n, Q^2, P^2)$$

over complex n -plane
numerically inverted

- Analytic continuation from even- n
- **Harmonic sums** appear in the expressions of **coefficient functions**, **anomalous dimensions** and **photon matrix elements**

$$S_k(n) = \sum_{j=1}^n \frac{1}{j^k} \quad S_{-k}(n) = \sum_{j=1}^n \frac{(-1)^j}{j^k} \quad k = 1, 2, \dots$$

$$S_{k, \vec{m}}(n) = \sum_{j=1}^n \frac{1}{j^k} S_{\vec{m}}(j) \quad S_{-k, \vec{m}}(n) = \sum_{j=1}^n \frac{(-1)^j}{j^k} S_{\vec{m}}(j)$$

[example] $S_{1, -2, 2}(n) = \sum_{j=1}^n \frac{1}{j} \sum_{k=1}^j \frac{(-1)^k}{k^2} \sum_{l=1}^k \frac{1}{l^2}$ (up to weight-5)

- We need to evaluate **harmonic sums** numerically on the **complex n -plane**

up to NLO: we only need $S_1, S_2, S_{-2}, S_3, S_{-3}, S_{-2,1}$

Evaluation of the Harmonic Sums

● [Example] $\gamma_{qg}^{(2)}(n)$

$$\begin{aligned} \gamma_{qg}^{(2)}(N) = & 16C_A C_F n_f \left((N_- + 4N_+ - 2N_{+2} - 3) \left[\frac{31}{2} S_1 \zeta_3 - \frac{3997}{96} S_1 - \frac{11}{2} S_{1,-4} + 6S_{1,-3,1} \right. \right. \\ & - \frac{3}{2} S_{1,-3} - \frac{9}{2} S_{1,-2} - 3S_{1,-2,2} - \frac{5}{2} S_{1,-2,1} - 2S_{1,-2,1,1} + 2S_{1,-2,2} - \frac{2405}{216} S_{1,1} + 6S_{1,1,-3} \\ & + 3S_{1,1} \zeta_3 + \frac{5}{2} S_{1,1,-2} - 6S_{1,1,-2,1} - \frac{128}{9} S_{1,1,1} - 6S_{1,1,1,-2} - \frac{13}{3} S_{1,1,1,1} - 4S_{1,1,1,1,1} - 3S_{1,1,1,2} \\ & - \frac{35}{12} S_{1,1,2} + 3S_{1,1,2,1} + S_{1,1,3} + \frac{53}{8} S_{1,2} + 3S_{1,2,-2} + \frac{15}{4} S_{1,2,1} + 6S_{1,2,1,1} - 6S_{1,3,1} - \frac{2833}{216} S_2 \\ & + \frac{3}{2} S_{1,4} + 3S_2 \zeta_3 - 6S_{2,-3} - \frac{5}{2} S_{2,-2} + 6S_{2,-2,1} + \frac{49}{4} S_{2,1} + 6S_{2,1,-2} - 6S_{2,1,1} + 3S_{2,1,2} - S_{2,2,1} \\ & + 2S_{2,1,1,1} + \frac{49}{4} S_{2,2} - 3S_{2,3} - \frac{551}{72} S_3 + \frac{173}{12} S_{3,1} - 2S_{3,1,1} - \frac{79}{6} S_4 + 2S_{4,1} \left. \right] + (N_- - N_+) \left[\frac{55}{12} S_1 \right. \\ & - 4S_1 \zeta_3 - \frac{371}{108} S_{1,1} + \frac{23}{9} S_{1,1,1} - \frac{2}{3} S_{1,1,1,1} + \frac{4}{3} S_{1,1,2} - \frac{23}{9} S_{1,2} + \frac{2}{3} S_{1,3} \left. \right] + (N_- - N_+) \left[\frac{8543}{192} S_1 \right. \\ & - \frac{71}{2} S_1 \zeta_3 - S_{1,-3} + 23S_{1,-2} - \frac{9}{2} S_{1,-2,1} + \frac{1301}{216} S_{1,1} + \frac{13}{2} S_{1,1,-2} + \frac{109}{18} S_{1,1,1} - \frac{5}{2} S_{1,2,1} + 4S_{1,3,2} \\ & + \frac{55}{6} S_{1,3} + \frac{23}{6} S_{1,1,1,1} + \frac{4}{3} S_{1,1,2} - \frac{235}{72} S_{1,2} + \frac{55}{8} S_2 + 9S_2 \zeta_3 - \frac{21}{2} S_2 - \frac{269}{36} S_{2,1} - 4S_{2,1,2} \\ & + 2S_{2,-3} + \frac{83}{12} S_{2,1,1} + \frac{3}{2} S_{2,1,1,1} - 3S_{2,1,2} - \frac{41}{4} S_{2,2} + S_{2,2,1} - \frac{5}{2} S_{2,3} - \frac{55}{48} S_3 + 3S_{3,-2} - \frac{143}{12} S_{3,1} \\ & - 2S_{3,1,1} + \frac{49}{4} S_4 + 4S_{4,1} - 2S_5 \left. \right] + (1 - N_-) \left[\frac{145}{2} S_1 \zeta_3 - \frac{3571}{64} S_1 + 2S_{1,-3} - \frac{58}{3} S_{1,3} - \frac{25}{9} S_{1,1,1} \right. \\ & + \frac{23}{2} S_{1,-2,1} + \frac{335}{216} S_{1,1} - \frac{31}{2} S_{1,1,-2} - \frac{11}{3} S_{1,1,1,1} - \frac{5}{3} S_{1,1,2} + \frac{245}{72} S_{1,2} + \frac{3}{2} S_{2,1,1,1} + 8S_{4,1} - 2S_5 \\ & + \frac{1}{2} S_{1,2,1} - \frac{83}{2} S_{1,-2} + 27S_2 \zeta_3 - 8S_{2,-3} + \frac{3}{2} S_{2,-2} + 8S_{2,-2,1} - \frac{183}{4} S_4 + 8S_{2,1,-2} - \frac{117}{4} S_{2,1,1} \\ & - 3S_{2,1,2} + \frac{157}{4} S_{2,2} - 3S_{2,2,1} - \frac{9}{2} S_{2,3} - \frac{581}{16} S_3 - S_{3,-2} + \frac{237}{4} S_{3,1} - 8S_{3,1,1} + 8S_{3,2} + \frac{73}{3} S_{2,1} \\ & - \frac{4319}{48} S_2 \left. \right] + 16C_A n_f^2 \left(\frac{1}{6} (N_- + 4N_+ - 2N_{+2} - 3) \left[\frac{175}{27} S_1 - 2S_{1,-3} + \frac{7}{3} S_{1,-2} - \frac{7}{9} S_{1,1} + \frac{4}{3} S_3 \right. \right. \\ & + \frac{7}{3} S_{1,1,1} - S_{1,1,1,1} + S_{1,1,2} - S_{1,2,1} - S_{1,3} + \frac{229}{18} S_2 \left. \right] + \frac{1}{6} (N_- - 1) \left[S_{1,-2} - \frac{4}{3} S_{1,1} + \frac{4}{3} S_{1,1,1} \right] \\ & - \frac{53}{162} (N_- - 1) S_1 - (N_- - N_+) \left[\frac{149}{648} S_1 + \frac{7}{4} S_2 - \frac{2}{9} S_3 - \frac{1}{3} S_4 \right] - (1 - N_+) \left[\frac{473}{648} S_1 - \frac{169}{36} S_2 \right. \\ & + \frac{1}{6} S_{2,1} - \frac{43}{18} S_3 + \frac{5}{3} S_4 \left. \right] + 16C_F^2 n_f \left((N_- + 4N_+ - 2N_{+2} - 3) \left[\frac{3220}{27} S_1 - \frac{3}{2} S_{1,-4} + \frac{377}{2} S_{1,-3} \right. \right. \\ & - \frac{31}{2} S_1 \zeta_3 + \frac{61}{6} S_{1,-3} + 2S_{1,-3,1} + 3S_{1,-2,1} - \frac{8}{3} S_{1,-2,1,1} - 2S_{1,-2,1,1,1} - 2S_{1,-2,1,2} + 6S_{1,-2,1,2,1} \\ & - \frac{95}{54} S_{1,1} - 3S_{1,1} \zeta_3 + 2S_{1,1,-3} + \frac{20}{3} S_{1,1,-2} + \frac{47}{8} S_{1,1,1} + \frac{4}{3} S_{1,1,1,1} + 2S_{1,1,1,1,1} - S_{1,1,2} + \frac{37}{6} S_{1,3} \\ & + 4S_{1,1,2,1} + \frac{21}{4} S_{1,1,2} + 2S_{1,1,2,1} + \frac{69}{8} S_{1,2} - S_{1,2,2} + \frac{23}{12} S_{1,2,1} - S_{1,2,1,1} + 2S_{1,2,1,2} - \frac{5}{2} S_{1,4} + 9S_2 \\ & - 3S_2 \zeta_3 - S_{2,-3} - \frac{21}{2} S_{2,-2} + 2S_{2,-2,1} - \frac{155}{72} S_{2,1} + \frac{53}{6} S_{2,1,1} + 3S_{1,3,1} - \frac{5}{12} S_{2,2} + \frac{31}{12} S_{3,1} - 3S_4 \\ & + \frac{2561}{72} S_3 - 2S_{1,2,2} \left. \right] + (N_- - 1) \left[4S_1 \zeta_3 - \frac{2351}{108} S_1 - \frac{8}{3} S_{1,-3} - \frac{4}{3} S_{1,1,2} - \frac{52}{9} S_{1,-2} + \frac{4}{3} S_{1,-2,1} \right. \\ & + \frac{161}{36} S_{1,1} - \frac{4}{3} S_{1,1,-2} - \frac{10}{9} S_{1,1,1} + \frac{2}{3} S_{1,1,1,1} - \frac{3}{2} S_{1,2} + \frac{56}{27} S_2 - \frac{20}{9} S_{2,1} - 2S_{1,3} - \frac{2}{3} S_{2,1,1} \left. \right] \\ & - (N_- - 1) S_{1,2,1} + (N_- - N_+) \left[22S_1 \zeta_3 - \frac{1759}{24} S_1 - \frac{13}{6} S_{1,-3} - \frac{799}{36} S_{1,-2} - \frac{8}{3} S_{1,-2,1} - \frac{21}{2} S_{1,3} \right. \\ & - \frac{37}{3} S_{1,1,1} - \frac{425}{72} S_{1,1,1,1} - \frac{7}{12} S_{1,1,1,1,1} - \frac{35}{6} S_{1,1,2} - \frac{217}{24} S_{1,2} - \frac{1385}{18} S_2 + \frac{593}{36} S_{1,1} - \frac{49}{6} S_{2,1,1} \\ & + \frac{5}{2} S_{2,-3} - 8S_{2,-2} - \frac{209}{24} S_{2,1} + 3S_{2,1,-2} - S_{2,1,1,1} + 2S_{2,1,2} + \frac{17}{12} S_{2,2} - 6S_2 \zeta_3 + \frac{13}{4} S_{2,3} + \frac{9}{4} S_{4,1} \\ & - \frac{1363}{72} S_3 + \frac{9}{2} S_{3,-2} + \frac{1}{6} S_{3,1} + 3S_{3,1,1} + \frac{25}{6} S_4 + 4S_5 \left. \right] + (1 - N_+) \left[\frac{15}{4} S_{2,2} + \frac{1783}{24} S_1 - 41S_1 \zeta_3 \right. \\ & + \frac{4}{3} S_{1,-3} + \frac{995}{36} S_{1,-2} + \frac{16}{3} S_{1,-2,1} - \frac{2731}{72} S_{1,1} + \frac{62}{3} S_{1,1,-2} + \frac{319}{72} S_{1,1,1} - \frac{7}{12} S_{1,1,1,1} + \frac{49}{6} S_{1,1,2} \\ & + \frac{287}{24} S_{1,2} + \frac{79}{4} S_{1,3} + \frac{73141}{216} S_2 - 24S_2 \zeta_3 + \frac{17}{2} S_{2,-3} + \frac{93}{2} S_{2,-2} - \frac{1567}{72} S_{2,1} - \frac{34}{3} S_4 - \frac{15}{4} S_{4,1} \\ & + 7S_{2,1,-2} + \frac{167}{6} S_{2,1,1} - 3S_{2,1,1,1} + 6S_{2,1,2} + \frac{53}{4} S_{2,3} + \frac{7385}{72} S_3 - \frac{7}{2} S_{3,-2} + \frac{47}{4} S_{3,1} + 5S_{3,1,1} \\ & - 19S_5 \left. \right] + 16C_F n_f^2 \left((N_- + 4N_+ - 2N_{+2} - 3) \left[\frac{2303}{324} S_1 + \frac{7}{54} S_{1,1} - \frac{7}{18} S_{1,1,1} - \frac{1}{6} S_{2,1,1} - S_4 \right. \right. \\ & + \frac{4}{9} S_{1,2} + \frac{1}{6} S_{1,1,1,1} - \frac{1}{3} S_{1,3} + \frac{35}{18} S_2 + \frac{7}{18} S_{2,1} - \frac{11}{9} S_3 \left. \right] - \frac{1}{6} (N_- - 1) \left[S_{1,1,1} + S_{1,2} - S_{2,1} \right] \\ & - (N_- - N_+) \left[\frac{59963}{2592} S_1 - \frac{7}{18} S_{1,1} - \frac{251}{27} S_2 + \frac{199}{24} S_3 - \frac{25}{6} S_4 + 2S_5 \right] + (1 - N_+) \left[\frac{163}{24} S_2 + 6S_5 \right. \\ & + \frac{96277}{2592} S_1 - \frac{17}{36} S_{1,1} - \frac{7}{24} S_3 - \frac{19}{2} S_4 \left. \right] + \frac{77}{81} (N_- - 1) S_1 + 16C_F^2 n_f \left((N_- - 1) \left[4S_{2,1,-2} \right. \right. \end{aligned}$$

$$\begin{aligned} & - (1 - N_+) \left[\frac{473}{648} S_1 \right. \\ & - (N_- - 3) \left[\frac{3220}{27} S_1 - \frac{3}{2} S_{1,-4} + \right. \\ & 2S_{1,-2,1,1} - 2S_{1,1,-2,1} + 6S_{1,1,1,1,1} + 2S_{1,1,1,1,1} - S_{1,1,1,2} \\ & \left. \left. + 2S_{1,1,1,2,1} + 2S_{1,1,2,1} + 2S_{1,1,2,1,1} - S_{1,1,2,2} + \frac{31}{12} S_{3,1} - 3S_4 \right. \right. \end{aligned}$$

A. Vogt, S. Moch, J.A.M. Vermaseren
Nucl.Phys.B691,129-181(2004)

Evaluation of the Harmonic Sums

- Our strategy

- For large $|n| > n_0$, we use **asymptotic expansions** of harmonic sums
- For small $|n| < n_0$, we use **translation relations** and shift the argument n , as $n \rightarrow n + 1 \rightarrow \dots \rightarrow \tilde{n}$, till $|\tilde{n}| > n_0$ so that the asymptotic expansion can be used ($n_0 = 16$)

[Example]
$$S_1(n) = \sum_{j=1}^n \frac{1}{j} = \psi(n+1) + \gamma_E$$

- Asymptotic expansion:

$$S_1(n) = \ln(n) + \gamma_E + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \mathcal{O}\left(\frac{1}{n^6}\right)$$

Euler-Maclaurin formula

- Translation relation: **shift** $n \rightarrow n + 1$

$$S_1(n) = S_1(n+1) - \frac{1}{n+1}$$

Evaluation of the Harmonic Sums

● [Another example] $S_{1,1,-2,1}(n) = \sum_{i=1}^n \frac{1}{i} \sum_{j=1}^i \frac{1}{j} \sum_{k=1}^j \frac{(-1)^k}{k^2} \sum_{l=1}^k \frac{1}{l}$

● Asymptotic expansion:

$$S_{1,1,-2,1}^{\text{even}}(n) = c_{0,2} \ln^2(n) + c_{0,1} \ln(n) + c_{0,0} + \frac{c_{1,1} \ln(n)}{n} + \frac{c_{1,0}}{n} \\ + \frac{c_{2,1} \ln(n)}{n^2} + \frac{c_{2,0}}{n^2} + \frac{c_{3,0}}{n^3} + \frac{c_{4,1} \ln(n)}{n^4} + \frac{c_{4,0}}{n^4} + \frac{c_{5,1} \ln(n)}{n^5} + \frac{c_{5,0}}{n^5} + \mathcal{O}\left(\frac{1}{n^6}\right)$$

Euler-Maclaurin formula
(straightforward but tedious)

$$c_{0,2} = c_{1,1} = -\frac{5}{16}\zeta(3) \quad c_{0,1} = -\frac{5}{8}\gamma_E\zeta(3) - \frac{3}{40}\zeta^2(2)$$

$$c_{0,0} = -\frac{3}{40}\gamma_E\zeta^2(2) - \frac{5}{16}\gamma_E^2\zeta(3) - \ln(2)\text{Li}_4\left(\frac{1}{2}\right) - \frac{7}{16}\ln^2(2)\zeta(3) + \frac{1}{6}\ln^3(2)\zeta(2)$$

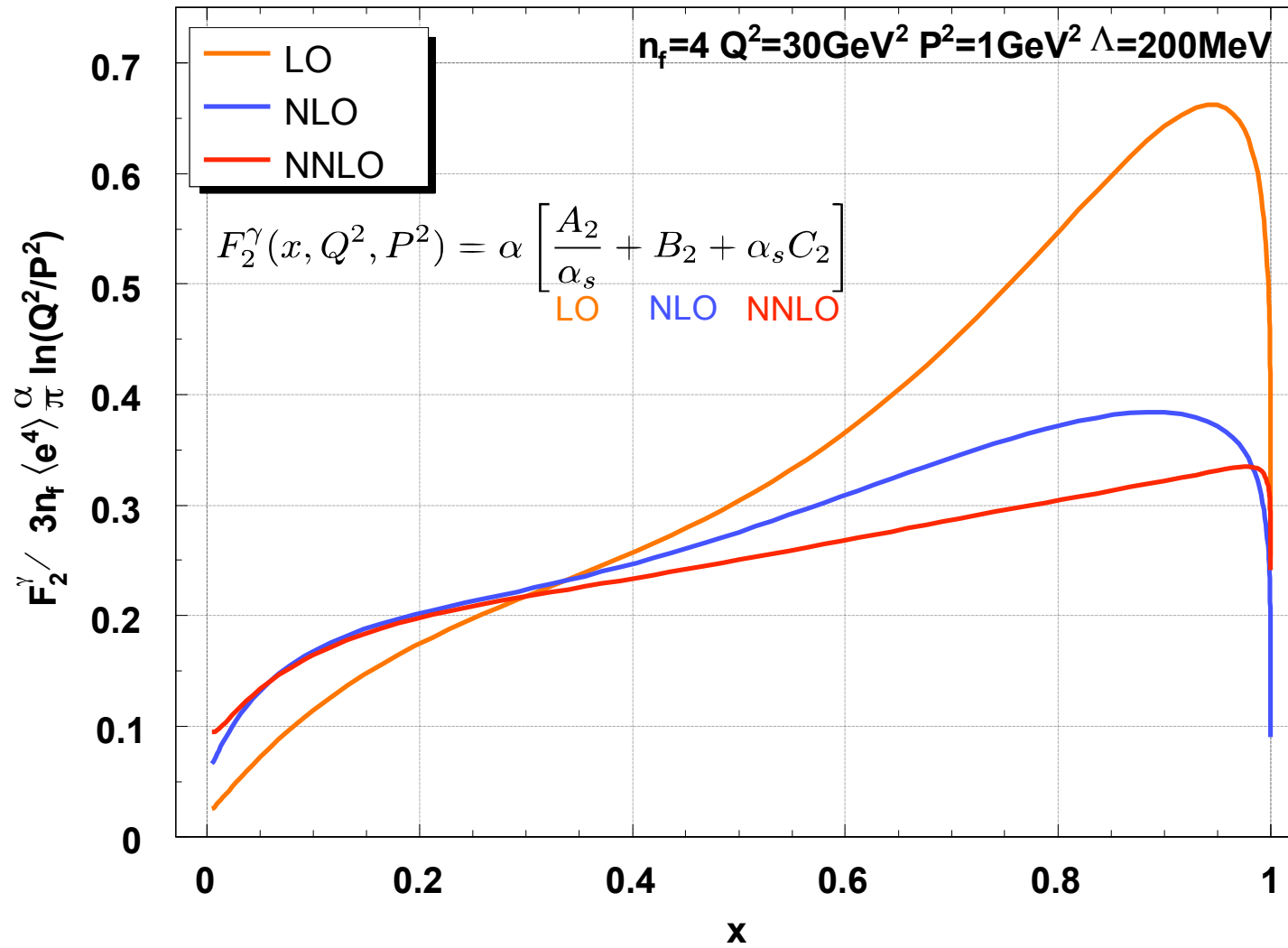
$$-\frac{1}{30}\ln^5(2) + \frac{1}{8}\zeta(2)\zeta(3) + \frac{1}{8}\zeta(5) - \text{Li}_5\left(\frac{1}{2}\right) \quad c_{1,0} = -\frac{5}{16}\gamma_E\zeta(3) - \frac{3}{80}\zeta^2(2) + \frac{5}{16}\zeta(3)$$

$$c_{2,1} = \frac{5}{96}\zeta(3) \quad c_{2,0} = \frac{5}{96}\gamma_E\zeta(3) + \frac{1}{160}\zeta^2(2) - \frac{15}{64}\zeta(3) \quad c_{3,0} = \frac{5}{64}\zeta(3) \quad c_{4,1} = \frac{1}{8} - \frac{1}{192}\zeta(3)$$

● Translation relation: shift $n \rightarrow n + 2$

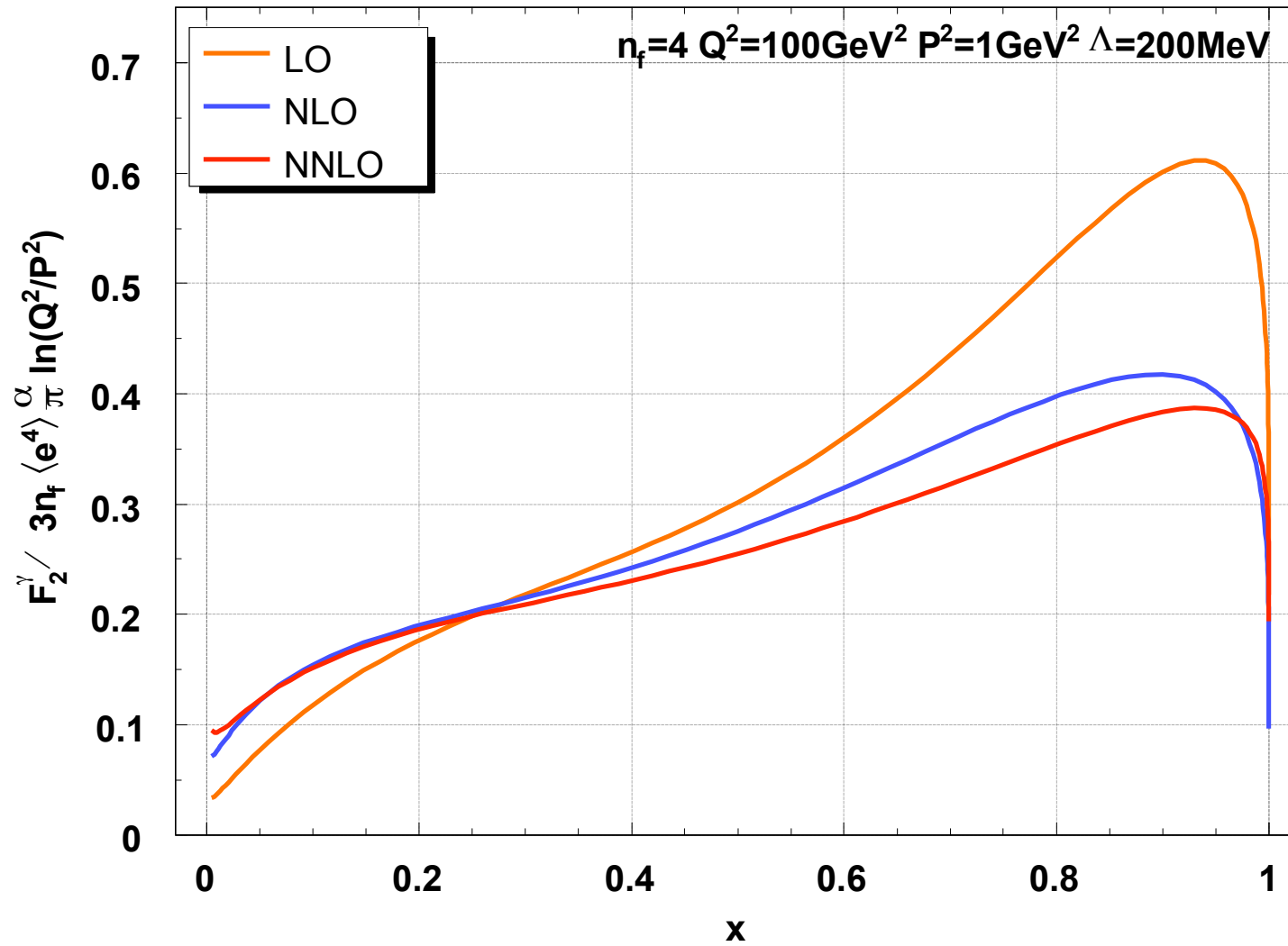
$$S_{1,1,-2,1}^{\text{even}}(n) = S_{1,1,-2,1}^{\text{even}}(n+2) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right) S_{1,-2,1}^{\text{even}}(n+2) + \frac{1}{(n+1)(n+2)} S_{-2,1}^{\text{even}}(n+2)$$

Numerical Plot $F_2^\gamma(x, Q^2, P^2)$

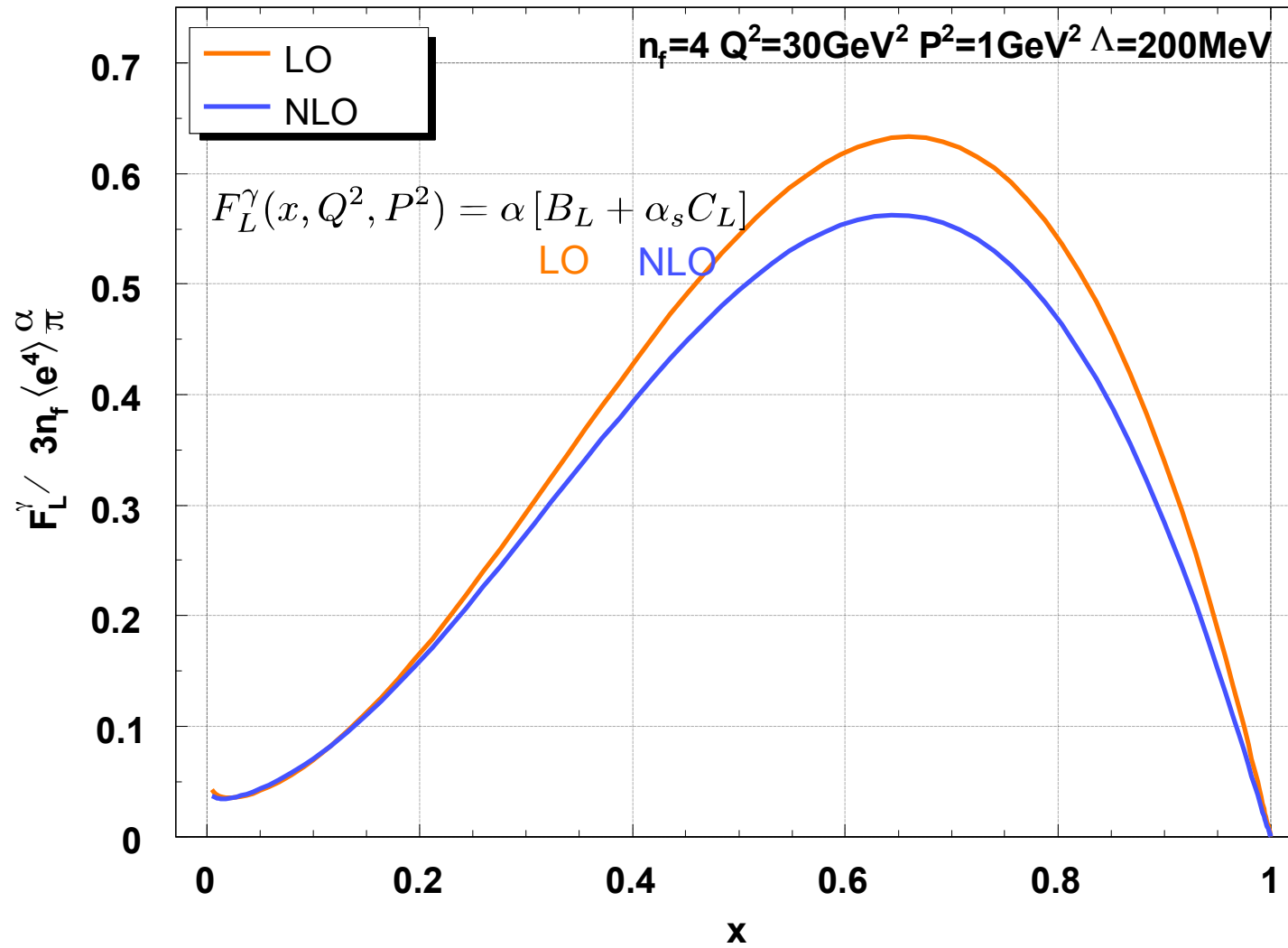


Result

Numerical Plot $F_2^{\gamma}(x, Q^2, P^2)$

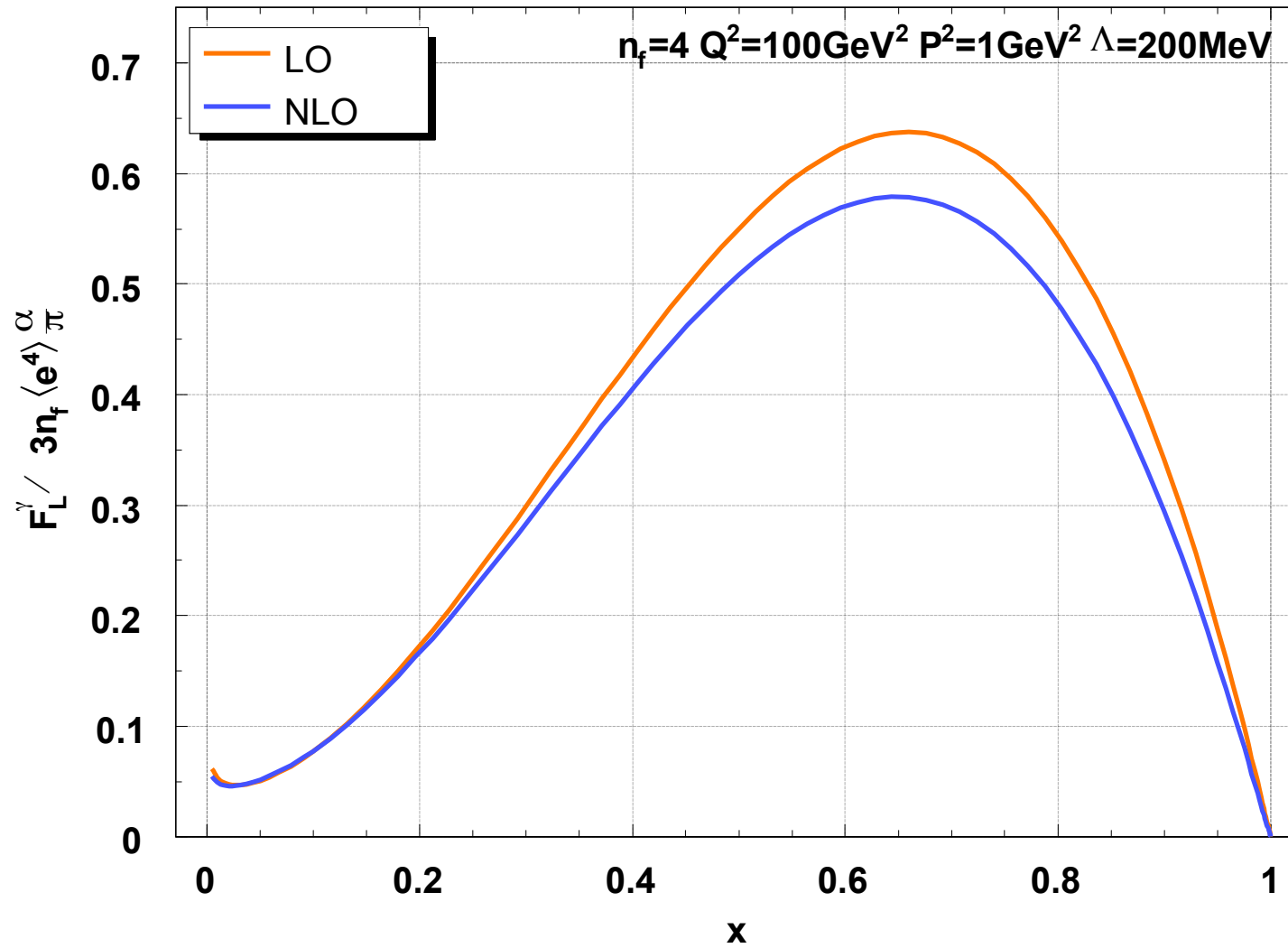


Numerical Plot $F_L^\gamma(x, Q^2, P^2)$



Result

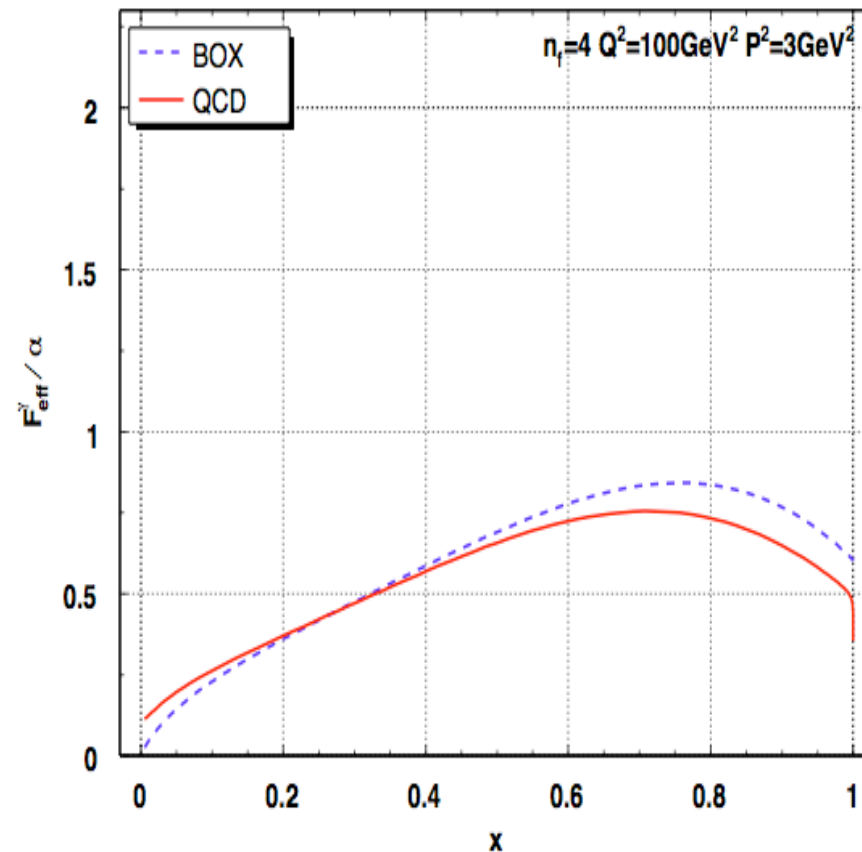
Numerical Plot $F_L^\gamma(x, Q^2, P^2)$



Numerical Plot F_{eff}^γ

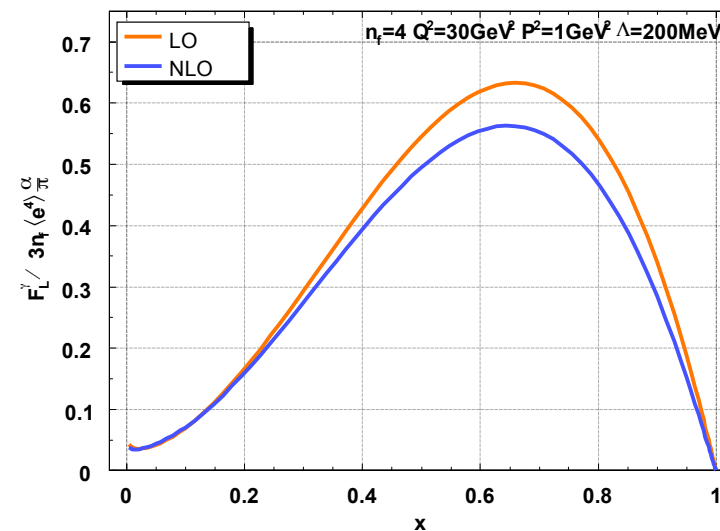
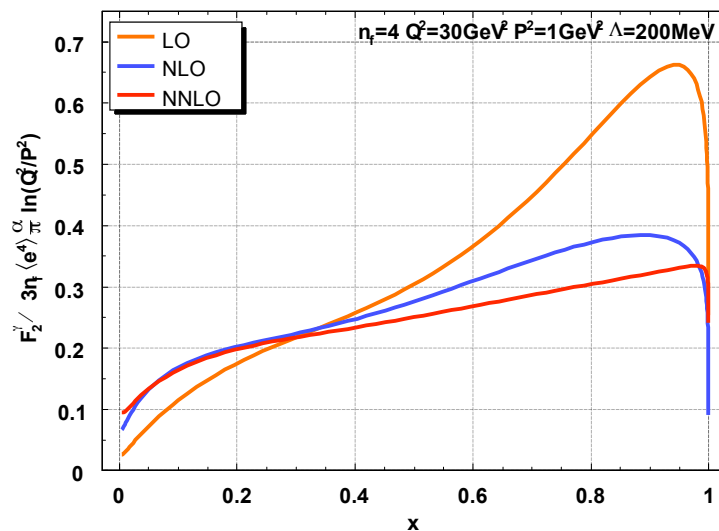
● $F_{\text{eff}}^\gamma \approx F_2^\gamma + \frac{3}{2}F_L^\gamma$

directly accessible in the experiment



Summary

- The virtual photon structure function $F_2^\gamma(x, Q^2, P^2)$ was investigated in the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$
- Definite predictions were made perturbatively up to NNLO ($\alpha\alpha_s$)
- NNLO ($\alpha\alpha_s$) corrections appear at large x
- We also analyzed the virtual photon structure function $F_L^\gamma(x, Q^2, P^2)$



Future works

- Quark and gluon distribution functions in the virtual photon up to NNLO

T. Ueda, T. Uematsu, K.S. in preparation

- Phenomenological analysis of the virtual photon structure functions including target mass effects up to NNLO

Y. Kitazono, T. Ueda, T. Uematsu, K.S. in preparation