Virtual Photon Structure functions to NNLO in QCD

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Introduction

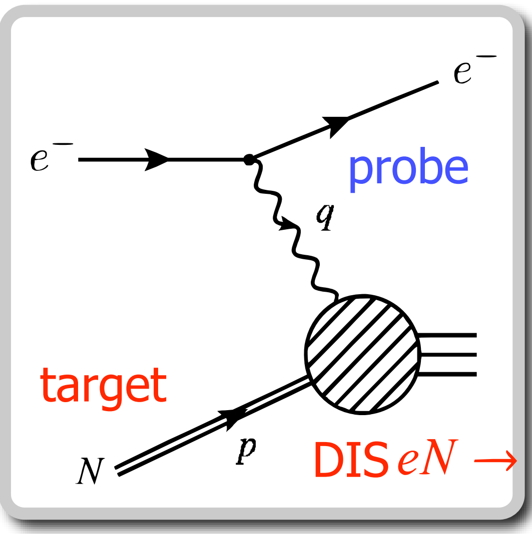
- Future linear collider experiment (e.g. ILC)

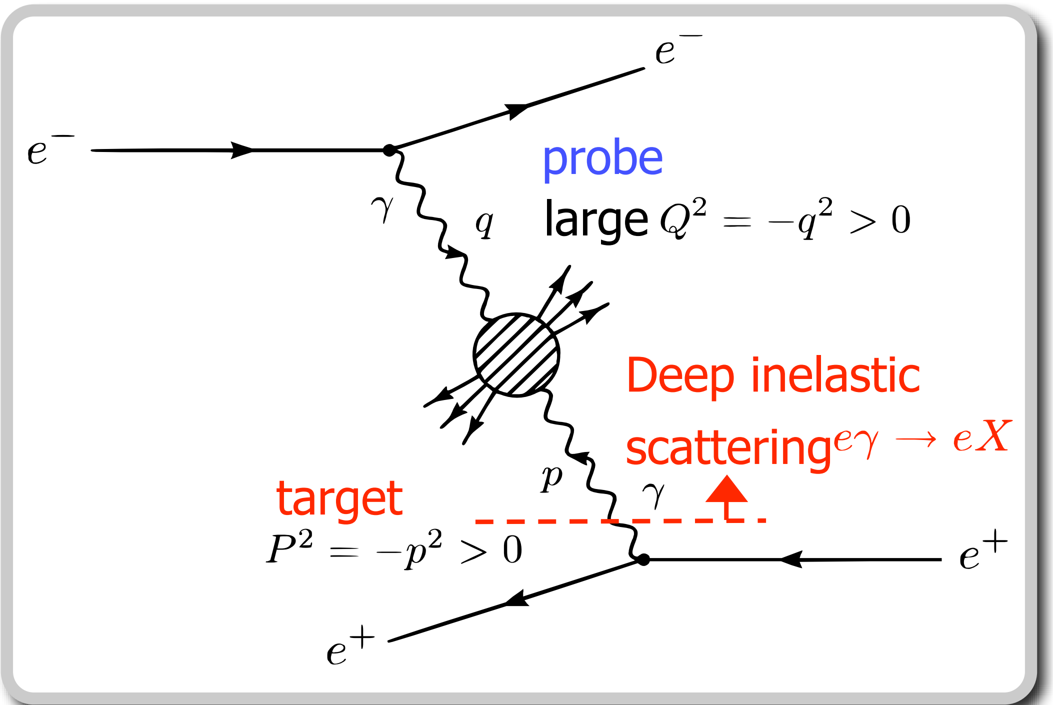
  Two-photon process $e^+e^- \to (e^+e^-\gamma\gamma) \to e^+e^-X$

  Viewed as a deep-inelastic electron-photon scattering

We can study the structures of photon

⇨ F_2^γ and F_L^γ





Two-photon Process

- Differential cross section

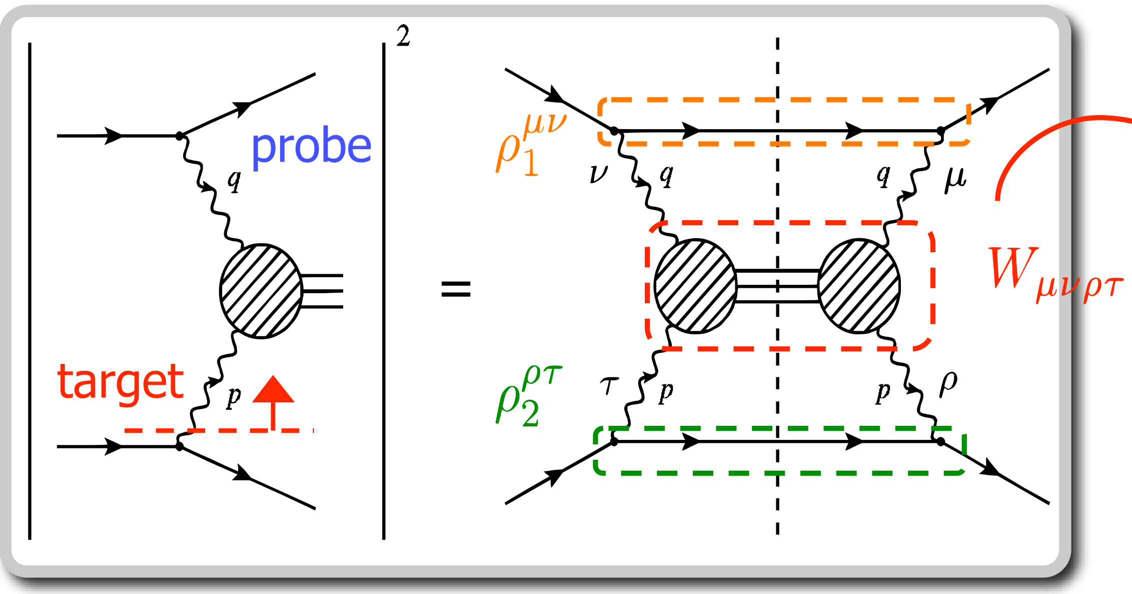
$$d\sigma = \frac{d^3 l_1' d^3 l_2'}{2E_1' 2E_2' (2\pi)^5} \frac{(4\pi\alpha)^3}{p^2 q^2} \frac{1}{4\sqrt{(l_1 \cdot l_2)^2 - m_e^2}} \rho_1^{\mu\nu} \rho_2^{\rho\tau} W_{\mu\nu\rho\tau}$$

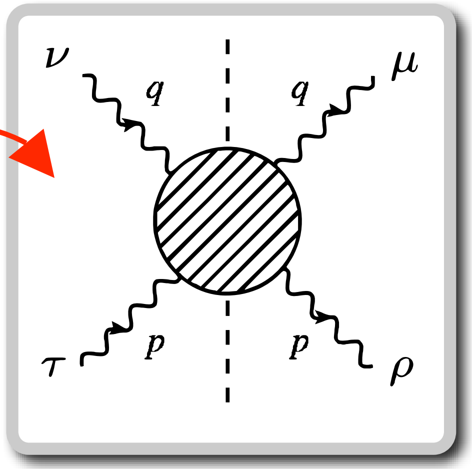
- Leptonic part $\rho_1^{\mu\nu}$ and $\rho_2^{\rho\tau}$

  $e \to e + \gamma$ QED vertex

- Hadronic part $W_{\mu\nu\rho\tau}$ photon structure tensor

  $\gamma\gamma \to X$

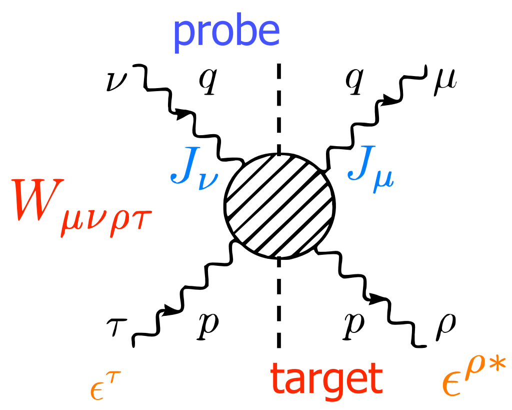




Spin-averaged Virtual Photon Structure Funcs.

- Spin-averaged structure tensor

$$W^{\gamma}_{\mu\nu}(p,q) = \frac{1}{2}\sum_{\lambda} \epsilon^{\rho *}_{(\lambda)}(p) W_{\mu\nu\rho\tau} \epsilon^{\tau}_{(\lambda)}(p)$$
$$= \frac{1}{2\pi}\int d^4x\, e^{iq\cdot x} \langle \gamma(p)| J_\mu(x) J_\nu(0) |\gamma(p)\rangle_{\text{spin ave.}}$$



- Structure functions F_2^{γ} and F_L^{γ} $F_L = F_2 - xF_1$

$$W^{\gamma}_{\mu\nu} = e_{\mu\nu}\frac{1}{x}F^{\gamma}_{L}(x,Q^2,P^2) + d_{\mu\nu}\frac{1}{x}F^{\gamma}_{2}(x,Q^2,P^2)$$

In analogy with the nucleon

$$e_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \qquad d_{\mu\nu} = -g_{\mu\nu} + \frac{q_\mu p_\nu + p_\mu q_\nu}{p\cdot q} - \frac{p_\mu p_\nu q^2}{(p\cdot q)^2}$$

$x = \frac{Q^2}{2p\cdot q}$:Bjorken variable $0 \le x \le 1$

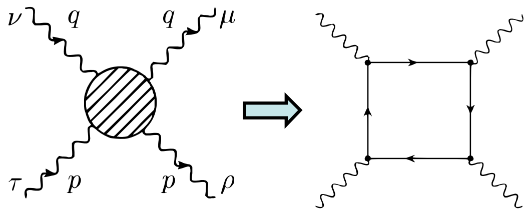
$-Q^2 = q^2 \le 0$:mass squared of the probe photon

$-P^2 = p^2 \le 0$:mass squared of the target photon

F_2^γ in Perturbative QCD

- For real photon target $(P^2 \approx 0)$ $P^2 \ll Q^2$

Simple parton model



$$F_2^\gamma(x,Q^2) = \alpha\left[\frac{1}{\alpha_s(Q^2)}A + B + B' + \mathcal{O}(\alpha_s)\right]$$

(LO) (NLO) ← Hadronic piece

$\sim \ln \frac{Q^2}{\Lambda^2}$

OPE+RGE
lowest order in α

Witten (1977)
Bardeen-Buras (1979)

Point-like contribution dominates $\sim \ln Q^2$

Walsh-Zerwas (1973)

⇒ NNLO extension Moch-Vermaseren-Vogt (2002, 2006)

- For highly virtual photon target $(\Lambda^2 \ll P^2 \ll Q^2)$

$$F_2^\gamma(x,Q^2,P^2) = \alpha\left[\frac{1}{\alpha_s(Q^2)}\tilde{A} + \tilde{B} + \mathcal{O}(\alpha_s)\right]$$

Λ : QCD scale parameter

(LO) (NLO) Uematsu-Walsh (1981,1982)

Hadronic piece can also be dealt with perturbatively
Definite prediction of F_2^γ, its shape and magnitude, is possible

- We extend the analysis to NNLO ($\alpha\alpha_s$)

Motivated by the calculation of 3-loop anomalous dimensions
Vogt-Moch-Vermaseren (2004,2006)

Overview of the method

- $$W^{\gamma}_{\mu\nu}(p,q) = \frac{1}{2\pi}\int d^4z\, e^{iq\cdot z}\langle\gamma(p)|J_\mu(z)J_\nu(0)|\gamma(p)\rangle_{\text{spin ave.}}$$
$$= e_{\mu\nu}\frac{1}{x}F^{\gamma}_{L}(x,Q^2,P^2) + d_{\mu\nu}\frac{1}{x}F^{\gamma}_{2}(x,Q^2,P^2)$$

- The OPE near the light-cone $Q^2 \to \infty$ ⟺ light-cone

$$J(z)J(0) \sim \sum_i C_i(z)O_i(0)$$

i : over relevant ops.

⇒ $$\langle\gamma(p)|J(z)J(0)|\gamma(p)\rangle \sim \sum_i C_i(z)\langle\gamma(p)|O_i|\gamma(p)\rangle$$

- Spin-n twist-2 operators (hadronic ops. + photon op.)

(n-th moment) (dominant) $\tau = d_O - n$

quark : $O_\psi^{\mu_1\cdots\mu_n}$ (flavor singlet) gluon : $O_G^{\mu_1\cdots\mu_n}$

$O_{NS}^{\mu_1\cdots\mu_n}$ (flavor non-singlet) photon : $O_\gamma^{\mu_1\cdots\mu_n}$

QCD with massless quarks with n_f flavors

Overview of the method

- Moment sum rule of F_2^γ

we take $\mu^2 = -p^2 = P^2$

$$M_2^\gamma(n,Q^2,P^2) = \int_0^1 dx\, x^{n-2} F_2^\gamma(x,Q^2,P^2)$$

$$= \sum_{i=\psi,G,NS,\gamma} C_{2,n}^i\left(\frac{Q^2}{P^2},\bar{g}(P^2),\alpha\right) A_n^i\left(\bar{g}(P^2),\alpha\right)$$

for even n

RG improved coefficients

Photon matrix element $\langle\gamma(p)|O_i^n|\gamma(p)\rangle_{\text{spin ave.}}$

Perturbatively calculable when $\Lambda^2 \ll P^2$

$$C_{2,n}^i\left(\frac{Q^2}{P^2},\bar{g}(P^2),\alpha\right) = \left(T\exp\left[\int_{\bar{g}(Q^2)}^{\bar{g}(P^2)} dg \frac{\gamma_n(g,\alpha)}{\beta(g)}\right]\right)_{ij} C_{2,n}^j\left(1,\bar{g}(Q^2),\alpha\right)$$

Solution of RG eq. for coefficient functions

$\beta(g)$: QCD beta function

γ_n : anomalous dimensions

Expand $M_2^\gamma(n,Q^2,P^2)$ up to NNLO ($\alpha\alpha_s$)

- Inverse Mellin transformation: n -space ⇨ x -space

$$F_2^\gamma(x,Q^2,P^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{1-n} M_2^\gamma(n,Q^2,P^2)$$

numerically inverted

Anomalous dimension matrix

- To the lowest order in α

$$\gamma_n(g,\alpha) = \left(\begin{array}{c|c} \hat{\gamma}_n(g) & \mathbf{0} \\ \hline \boldsymbol{K}_n(g,\alpha) & 0 \end{array}\right)$$

- 3×3 anomalous dimension matrix in the hadronic sector

$$\widehat{\gamma}_n(g) = \begin{pmatrix} \gamma^n_{\psi\psi}(g) & \gamma^n_{G\psi}(g) & 0 \\ \gamma^n_{\psi G}(g) & \gamma^n_{GG}(g) & 0 \\ 0 & 0 & \gamma^n_{NS}(g) \end{pmatrix}$$

$$\widehat{\gamma}^{(0)}_n = \sum_{i=+,-,NS} \lambda^n_i \; P^n_i$$

$$\sum_i P_i = 1 \qquad P_i P_j = \delta_{ij} P_i$$

- Mixing anomalous dimensions between photon operator and three hadronic operators

$$\boldsymbol{K}_n(g,\alpha) = \left(K^n_\psi(g,\alpha) \;\; K^n_G(g,\alpha) \;\; K^n_{NS}(g,\alpha)\right)$$

Moment Sum Rule for F_2^γ

- Summarized as

LO $(\alpha\alpha_s^{-1})$

for even n

NLO (α)

$$
\begin{aligned}
&\int_0^1 dx\, x^{n-2} F_2^\gamma(x, Q^2, P^2) \\
&= \frac{\alpha}{4\pi}\frac{1}{2\beta_0}\Bigg\{ \frac{4\pi}{\alpha_s(Q^2)} \sum_i \mathcal{L}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n+1}\right] \\
&\quad + \sum_i \mathcal{A}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n}\right] + \sum_i \mathcal{B}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n+1}\right] + \mathcal{C}^n \\
&\quad + \frac{\alpha_s(Q^2)}{4\pi}\Bigg(\sum_i \mathcal{D}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n-1}\right] + \sum_i \mathcal{E}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n}\right] \\
&\quad + \sum_i \mathcal{F}_i^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n+1}\right] + \mathcal{G}^n \Bigg) + \mathcal{O}(\alpha_s^2) \Bigg\}
\end{aligned}
$$

NNLO ($\alpha\alpha_s$)

Moment Sum Rule for F_2^γ

LO :

$$\mathcal{L}_i^n = \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{1}{d_i^n + 1}$$

$$d_i^n = \frac{\lambda_i^n}{2\beta_0} \qquad i = +, -, NS$$

NLO :

$$\mathcal{A}_i^n = -\boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{d_i^n} - \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1 - d_i^n}{d_i^n}$$
$$+\boldsymbol{K}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{1}{d_i^n} - 2\beta_0 \widetilde{\boldsymbol{A}}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)}$$

$$\mathcal{B}_i^n = \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1 + d_i^n}$$
$$+ \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(1)} \frac{1}{1 + d_i^n} - \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{d_i^n}{1 + d_i^n}$$

$$\mathcal{C}^n = 2\beta_0 (C_{2,n}^{\gamma(1)} + \widetilde{\boldsymbol{A}}_n^{(1)} \cdot \boldsymbol{C}_{2,n}^{(0)})$$

Moment Sum Rule for F_2^γ

NNLO :

$$
\begin{aligned}
\mathcal{D}_i^n \;=\; & -\boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \frac{1}{1-d_i^n} \right) \left(1 - \frac{d_i^n}{2} \right) \\
& -\boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1-d_j^n}{1-d_i^n} \\
& -\boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 4\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \left(\frac{1 - d_i^n + d_j^n}{1-d_i^n} \right) \\
& +\boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(2)} P_i^n}{\lambda_j^n - \lambda_i^n + 4\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} \\
& -\boldsymbol{K}_n^{(0)} \sum_{j,k} \frac{P_k^n \hat{\gamma}_n^{(1)} P_j^n \hat{\gamma}_n^{(1)} P_i^n}{(\lambda_j^n - \lambda_i^n + 2\beta_0)(\lambda_k^n - \lambda_i^n + 4\beta_0)} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} \\
& +\boldsymbol{K}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} + \boldsymbol{K}_n^{(1)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} \\
& -\boldsymbol{K}_n^{(2)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} + 2\beta_0 \widetilde{\boldsymbol{A}}_n^{(1)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \\
& -2\beta_0 \widetilde{\boldsymbol{A}}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} d_i^n - 2\beta_0 \widetilde{\boldsymbol{A}}_n^{(2)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \; ,
\end{aligned}
$$

Moment Sum Rule for F_2^γ

NNLO :

$$
\begin{aligned}
\mathcal{E}_i^n \;=\; & -\boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(1)} \frac{\beta_1}{\beta_0}\frac{1-d_i^n}{d_i^n} - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(1)} \frac{1}{d_i^n} \\
& + \boldsymbol{K}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(1)} \frac{1}{d_i^n} + \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1^2}{\beta_0^2}(1-d_i^n) \\
& - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0}\frac{1-d_i^n}{d_i^n} + \boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \\
& - \boldsymbol{K}_n^{(0)} \sum_{j,k} \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n \hat{\gamma}_n^{(1)} P_k^n}{(\lambda_i^n - \lambda_k^n + 2\beta_0)(\lambda_j^n - \lambda_i^n + 2\beta_0)} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{d_i^n} \\
& - \boldsymbol{K}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} + \boldsymbol{K}_n^{(1)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{d_i^n} \\
& - 2\beta_0 \widetilde{\boldsymbol{A}}_n^{(1)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} + 2\beta_0 \widetilde{\boldsymbol{A}}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} d_i^n - 2\beta_0 \widetilde{\boldsymbol{A}}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(1)}
\end{aligned}
$$

Moment Sum Rule for F_2^γ

NNLO：

$$\begin{aligned}
\mathcal{F}_i^n \;=\; & \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(2)} \frac{1}{1+d_i^n} - \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(1)} \frac{\beta_1}{\beta_0} \frac{d_i^n}{1+d_i^n} \\
& + \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(1)} \frac{1}{1+d_i^n} \\
& + \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \frac{1}{1+d_i^n} \right) \frac{d_i^n}{2} \\
& - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{d_j^n}{1+d_i^n} \\
& - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 4\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1+d_i^n-d_j^n}{1+d_i^n} \\
& + \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(2)} P_j^n}{\lambda_i^n - \lambda_j^n + 4\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1+d_i^n} \\
& + \boldsymbol{K}_n^{(0)} \sum_{j,k} \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n \hat{\gamma}_n^{(1)} P_k^n}{(\lambda_j^n - \lambda_k^n + 2\beta_0)} \boldsymbol{C}_{2,n}^{(0)} \left(\frac{1}{\lambda_i^n - \lambda_j^n + 2\beta_0} - \frac{1}{\lambda_i^n - \lambda_k^n + 4\beta_0} \right) \frac{1}{1+d_i^n} \;,
\end{aligned}$$

$$\mathcal{G}^n \;=\; 2\beta_0 \left(C_{2,n}^{\gamma(2)} + \widetilde{\boldsymbol{A}}_n^{(1)} \cdot \boldsymbol{C}_{2,n}^{(1)} + \widetilde{\boldsymbol{A}}_n^{(2)} \cdot \boldsymbol{C}_{2,n}^{(0)} \right)$$

Needed parameters in $\overline{\mathrm{MS}}$

- QCD β-function

Tarasov-Vladimirov-Zarkov (1980)
Larin-Vermaseren (1993)

$$\beta(g) = -\frac{g^3}{16\pi^2}\beta_0 - \frac{g^5}{(16\pi^2)^2}\beta_1 - \frac{g^7}{(16\pi^2)^3}\beta_2 + \mathcal{O}(g^9)$$

1-loop 2-loop 3-loop

- Anomalous dimensions of hadronic operators

Moch-Vermaseren-Vogt (2004)

$$\hat{\gamma}^n(g) = \frac{g^2}{16\pi^2}\hat{\gamma}^{(0),n} + \frac{g^4}{(16\pi^2)^2}\hat{\gamma}^{(1),n} + \frac{g^6}{(16\pi^2)^3}\hat{\gamma}^{(2),n} + \mathcal{O}(g^8)$$

1-loop 2-loop 3-loop

- Hadronic coefficient functions

van Neerven-Zijlstra (1991,1992)

$$\boldsymbol{C}_2^n(g) = \boldsymbol{C}_2^{(0),n} + \frac{g^2}{16\pi^2}\boldsymbol{C}_2^{(1),n} + \frac{g^4}{(16\pi^2)^2}\boldsymbol{C}_2^{(2),n} + \mathcal{O}(g^6)$$

tree 1-loop 2-loop

Needed parameters in $\overline{\mathrm{MS}}$

- Mixing anomalous dimensions

$$\boldsymbol{K}^n(g,\alpha) = -\frac{\alpha}{4\pi}\left[\boldsymbol{K}^{(0),n} + \frac{g^2}{16\pi^2}\boldsymbol{K}^{(1),n} + \frac{g^4}{(16\pi^2)^2}\boldsymbol{K}^{(2),n} + \mathcal{O}(g^6)\right]$$

1-loop   2-loop

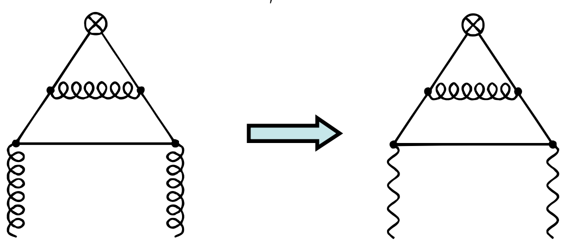
- Photon coefficient function

$$C^n_{2,\gamma}(g,\alpha) = \frac{\alpha}{4\pi}\left[C^{(1),n}_{2,\gamma} + \frac{g^2}{16\pi^2}C^{(2),n}_{2,\gamma} + \mathcal{O}(g^4)\right]$$

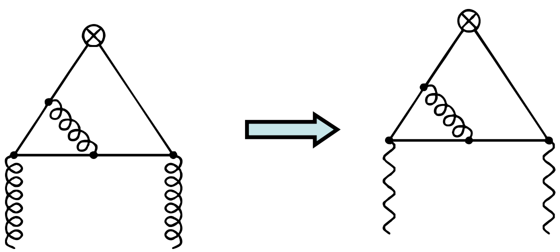
1-loop   2-loop

$\boldsymbol{K}^{(1),n}$, $C^{(2),n}_{2,\gamma}$ are obtained by the replacement of group factors

[example] $\gamma^{(1),n}_{\psi G} \Longrightarrow K^{(1),n}_{\psi}$



$n_f T_F C_F$ $\qquad$ $-3\langle e^2\rangle n_f C_F$



$n_f T_F\left(C_F - \frac{C_A}{2}\right)$ $\qquad$ $-3\langle e^2\rangle n_f C_F$

$n_f T_F \to -3\langle e^2\rangle n_f$ $\qquad$ $C_A \to 0$

Needed parameters in $\overline{\mathrm{MS}}$

- Mixing anomalous dimensions

$$\boldsymbol{K}^n(g,\alpha) = -\frac{\alpha}{4\pi}\left[\boldsymbol{K}^{(0),n} + \frac{g^2}{16\pi^2}\boldsymbol{K}^{(1),n} + \frac{g^4}{(16\pi^2)^2}\boldsymbol{K}^{(2),n} + \mathcal{O}(g^6)\right]$$

3-loop

No exact expressions for 3-loop $K_\psi^{(2),n}$, $K_{NS}^{(2),n}$, $K_G^{(2),n}$ yet

But the compact parametrization of

3-loop splitting functions $P_{ns\gamma}^{(2),\mathrm{approx}}(x)$, $P_{G\gamma}^{(2),\mathrm{approx}}(x)$

(and exact $P_{ps\gamma}^{(2)}(x)$) exist

(the deviation < 0.1 %)

A.Vogt, S.Moch and J.Vermaseren
Acta Phys.Polon. B37,683(2006)

- We use

$$\begin{aligned}
K_{\psi,\mathrm{approx}}^{(2),n} &= \int dx x^{n-1}\left[P_{ns\gamma}^{(2),\mathrm{approx}}(x) + P_{ps\gamma}^{(2)}(x)\right] \\
K_{NS,\mathrm{approx}}^{(2),n} &= \int dx x^{n-1}\left[P_{ns\gamma}^{(2),\mathrm{approx}}(x)\right] \\
K_{G,\mathrm{approx}}^{(2),n} &= \int dx x^{n-1}\left[P_{G\gamma}^{(2),\mathrm{approx}}(x)\right]
\end{aligned}$$

Needed parameters in $\overline{\mathrm{MS}}$

- Photon matrix elements Perturbatively calculable when $\Lambda^2 \ll P^2$

$$\boldsymbol{A}^n(g,\alpha) = \frac{\alpha}{4\pi}\left[\boldsymbol{A}^{(1),n} + \frac{g^2}{16\pi^2}\boldsymbol{A}^{(2),n} + \mathcal{O}(g^4)\right]$$

1-loop 2-loop

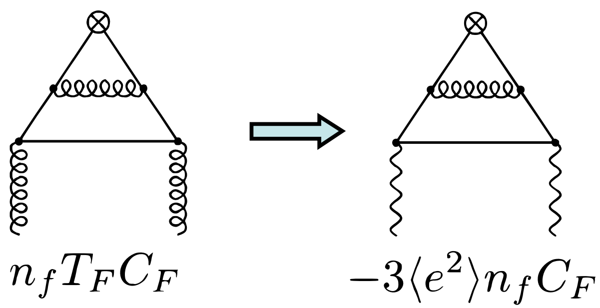
- Unrenormalized gluon matrix elements of hadronic operators were calculated in x-space

Matiounine-Smith-van Neerven (1998)

- We obtain $A_\psi^{(2),n}$, $A_{NS}^{(2),n}$, $A_G^{(2),n}$, after taking moments,

$$\hat{A}^k_{qg}\left(n, \frac{-p^2}{\mu^2}, \frac{1}{\epsilon}\right) = \int_0^1 dx\, x^{n-1}\hat{A}^k_{qg}\left(x, \frac{-p^2}{\mu^2}, \frac{1}{\epsilon}\right)$$

renormalization and the replacement of group factors



Numerical Inversion

- Inverse Mellin transformation

over complex n -plane

$$F_2^\gamma(x, Q^2, P^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{1-n} M_2^\gamma(n, Q^2, P^2)$$

numerically inverted

  - Analytic continuation fron even- n
  - Harmonic sums appear in the expressions of coefficient functions, anomalous dimensions and photon matrix elements

$$S_k(n) = \sum_{j=1}^{n} \frac{1}{j^k} \qquad S_{-k}(n) = \sum_{j=1}^{n} \frac{(-1)^j}{j^k} \qquad k = 1, 2, \cdots$$

$$S_{k,\vec{m}}(n) = \sum_{j=1}^{n} \frac{1}{j^k} S_{\vec{m}}(j) \qquad S_{-k,\vec{m}}(n) = \sum_{j=1}^{n} \frac{(-1)^j}{j^k} S_{\vec{m}}(j)$$

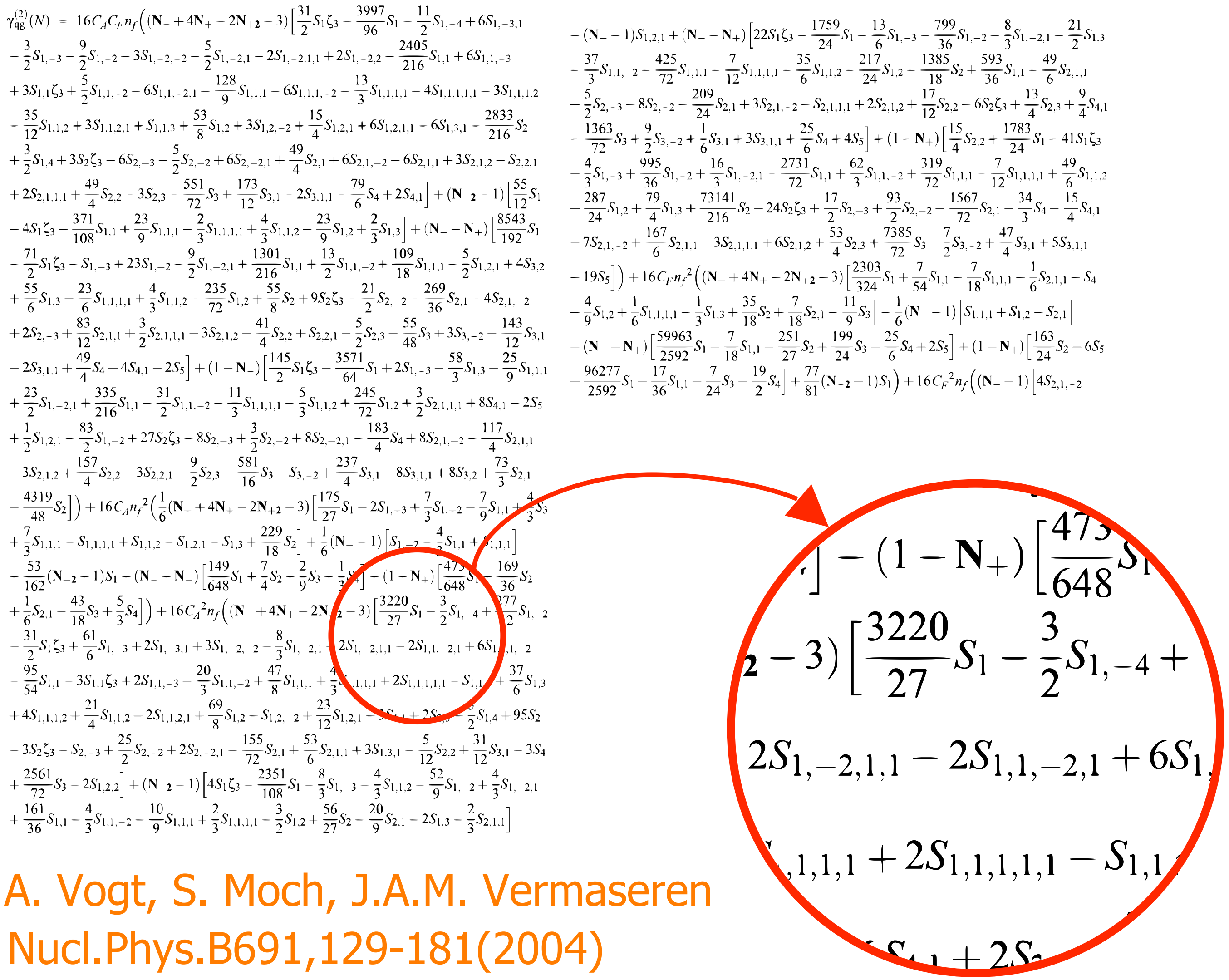
[example] $S_{1,-2,2}(n) = \sum_{j=1}^{n} \frac{1}{j} \sum_{k=1}^{j} \frac{(-1)^k}{k^2} \sum_{l=1}^{k} \frac{1}{l^2}$ (up to weight-5)

  - We need to evaluate harmonic sums numerically on the complex n-plane

up to NLO: we only need $S_1, S_2, S_{-2}, S_3, S_{-3}, S_{-2,1}$

Evaluation of the Harmonic Sums

- [Example] $\gamma_{\rm qg}^{(2)}(n)$



A. Vogt, S. Moch, J.A.M. Vermaseren
Nucl.Phys.B691,129-181(2004)

Evaluation of the Harmonic Sums

- Our strategy
  - For large $|n| > n_0$, we use asymptotic expansions of harmonic sums
  - For small $|n| < n_0$, we use translation relations and shift the argument n, as $n \to n+1 \to \cdots \to \tilde{n}$, till $|\tilde{n}| > n_0$ so that the asymptotic expansion can be used $(n_0 = 16)$

[Example] $$S_1(n) = \sum_{j=1}^{n} \frac{1}{j} = \psi(n+1) + \gamma_E$$

- Asymptotic expansion:

Euler-Maclaurin formula

$$S_1(n) = \ln(n) + \gamma_E + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \mathcal{O}\left(\frac{1}{n^6}\right)$$

- Translation relation: shift $n \to n+1$

$$S_1(n) = S_1(n+1) - \frac{1}{n+1}$$

Evaluation of the Harmonic Sums

- [Another example] $S_{1,1,-2,1}(n) = \sum_{i=1}^{n} \frac{1}{i} \sum_{j=1}^{i} \frac{1}{j} \sum_{k=1}^{j} \frac{(-1)^k}{k^2} \sum_{l=1}^{k} \frac{1}{l}$
  - Asymptotic expansion:

$$S_{1,1,-2,1}^{\text{even}}(n) = c_{0,2}\ln^2(n) + c_{0,1}\ln(n) + c_{0,0} + \frac{c_{1,1}\ln(n)}{n} + \frac{c_{1,0}}{n} + \frac{c_{2,1}\ln(n)}{n^2} + \frac{c_{2,0}}{n^2} + \frac{c_{3,0}}{n^3} + \frac{c_{4,1}\ln(n)}{n^4} + \frac{c_{4,0}}{n^4} + \frac{c_{5,1}\ln(n)}{n^5} + \frac{c_{5,0}}{n^5} + \mathcal{O}\left(\frac{1}{n^6}\right)$$

Euler-Maclaurin formula

(straightforward but tedious)

$$c_{0,2} = c_{1,1} = -\frac{5}{16}\zeta(3) \qquad c_{0,1} = -\frac{5}{8}\gamma_E\zeta(3) - \frac{3}{40}\zeta^2(2)$$

$$c_{0,0} = -\frac{3}{40}\gamma_E\zeta^2(2) - \frac{5}{16}\gamma_E^2\zeta(3) - \ln(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{7}{16}\ln^2(2)\zeta(3) + \frac{1}{6}\ln^3(2)\zeta(2) - \frac{1}{30}\ln^5(2) + \frac{1}{8}\zeta(2)\zeta(3) + \frac{1}{8}\zeta(5) - \mathrm{Li}_5\left(\frac{1}{2}\right)$$

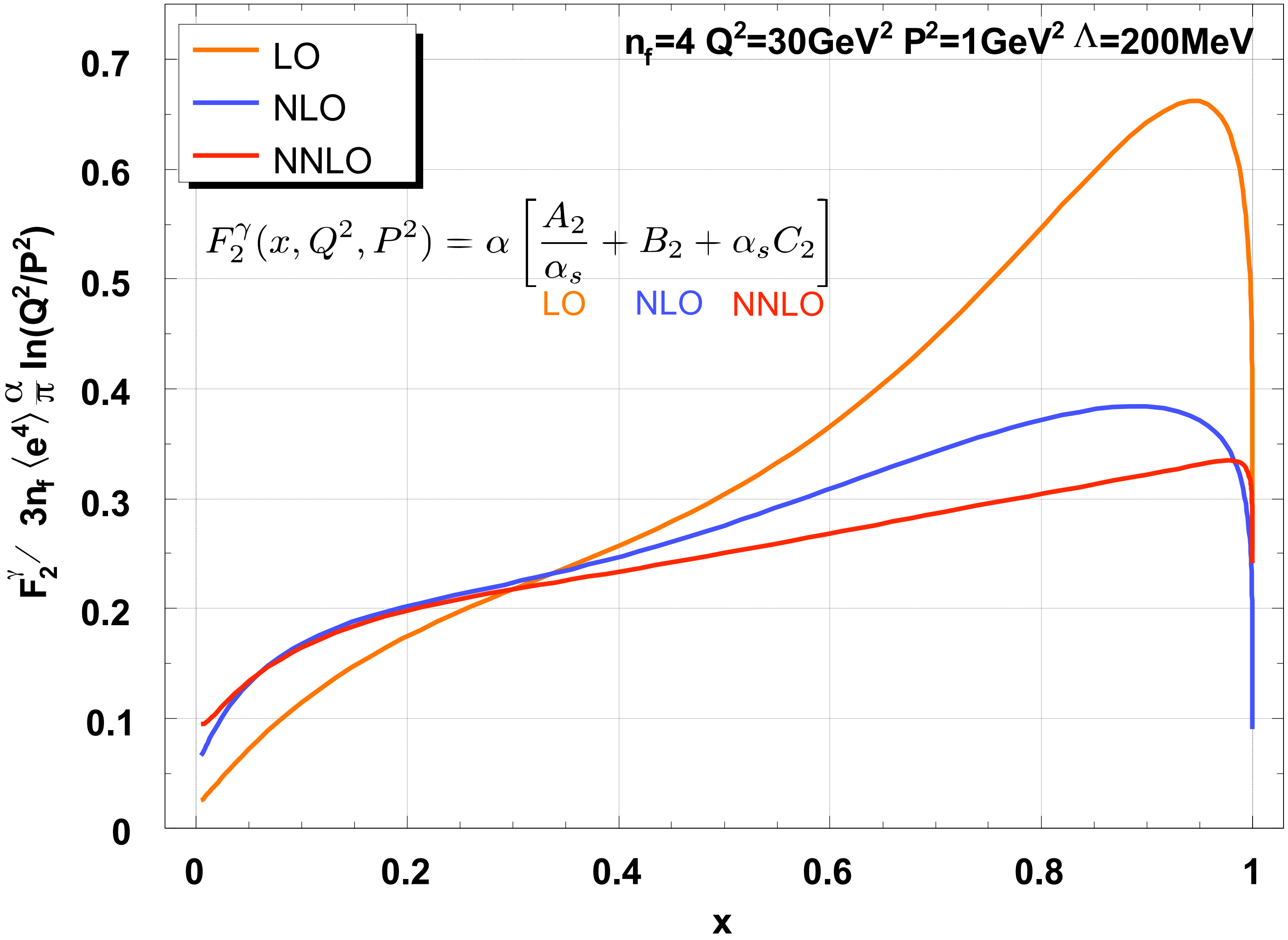
$$c_{1,0} = -\frac{5}{16}\gamma_E\zeta(3) - \frac{3}{80}\zeta^2(2) + \frac{5}{16}\zeta(3)$$

$$c_{2,1} = \frac{5}{96}\zeta(3) \qquad c_{2,0} = \frac{5}{96}\gamma_E\zeta(3) + \frac{1}{160}\zeta^2(2) - \frac{15}{64}\zeta(3) \qquad c_{3,0} = \frac{5}{64}\zeta(3) \qquad c_{4,1} = \frac{1}{8} - \frac{1}{192}\zeta(3)$$

  - Translation relation: shift $n \to n+2$

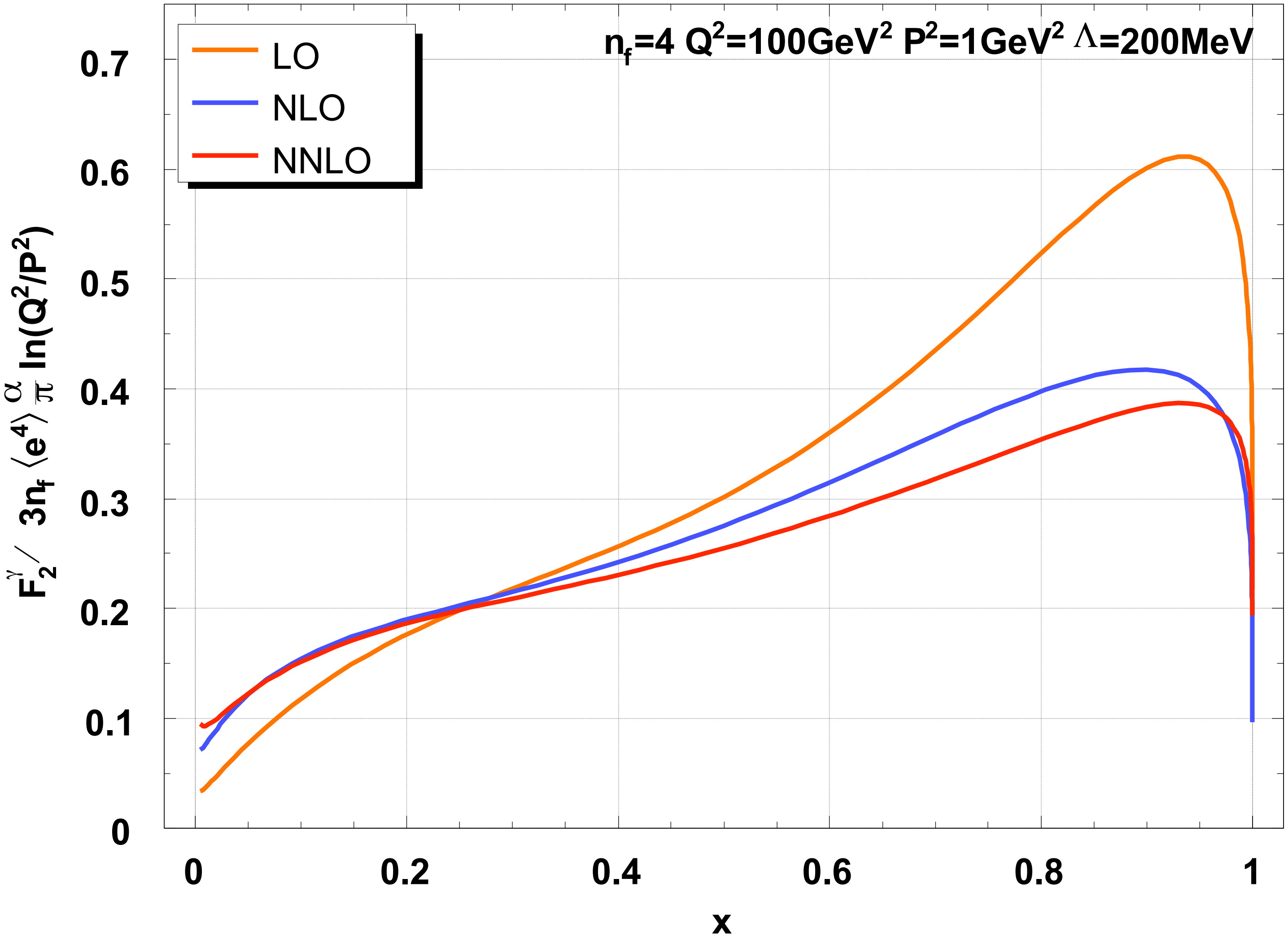
$$S_{1,1,-2,1}^{\text{even}}(n) = S_{1,1,-2,1}^{\text{even}}(n+2) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right) S_{1,-2,1}^{\text{even}}(n+2) + \frac{1}{(n+1)(n+2)} S_{-2,1}^{\text{even}}(n+2)$$

Numerical Plot $F_2^\gamma(x, Q^2, P^2)$

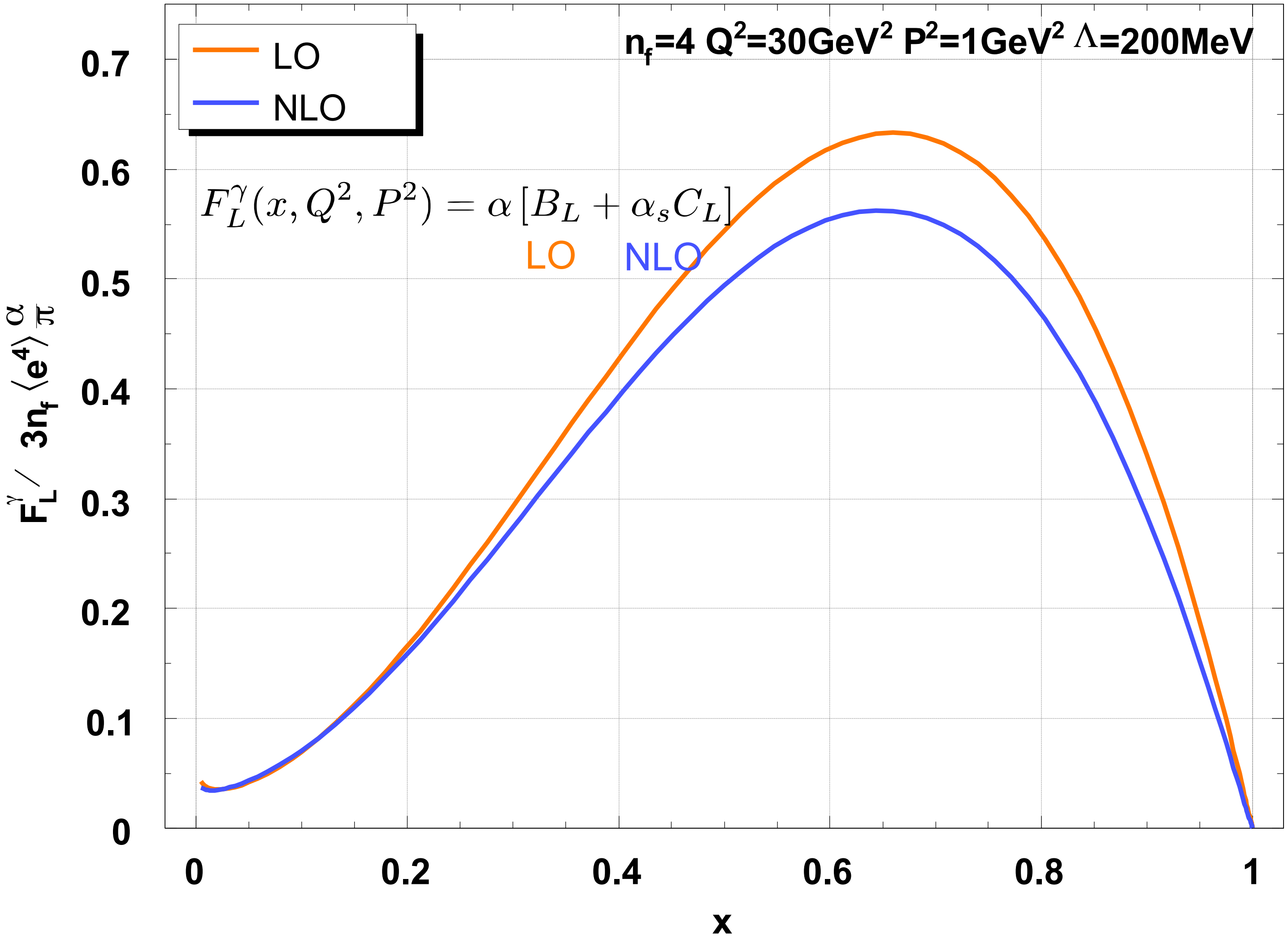


Result

Numerical Plot $F_2^\gamma(x, Q^2, P^2)$

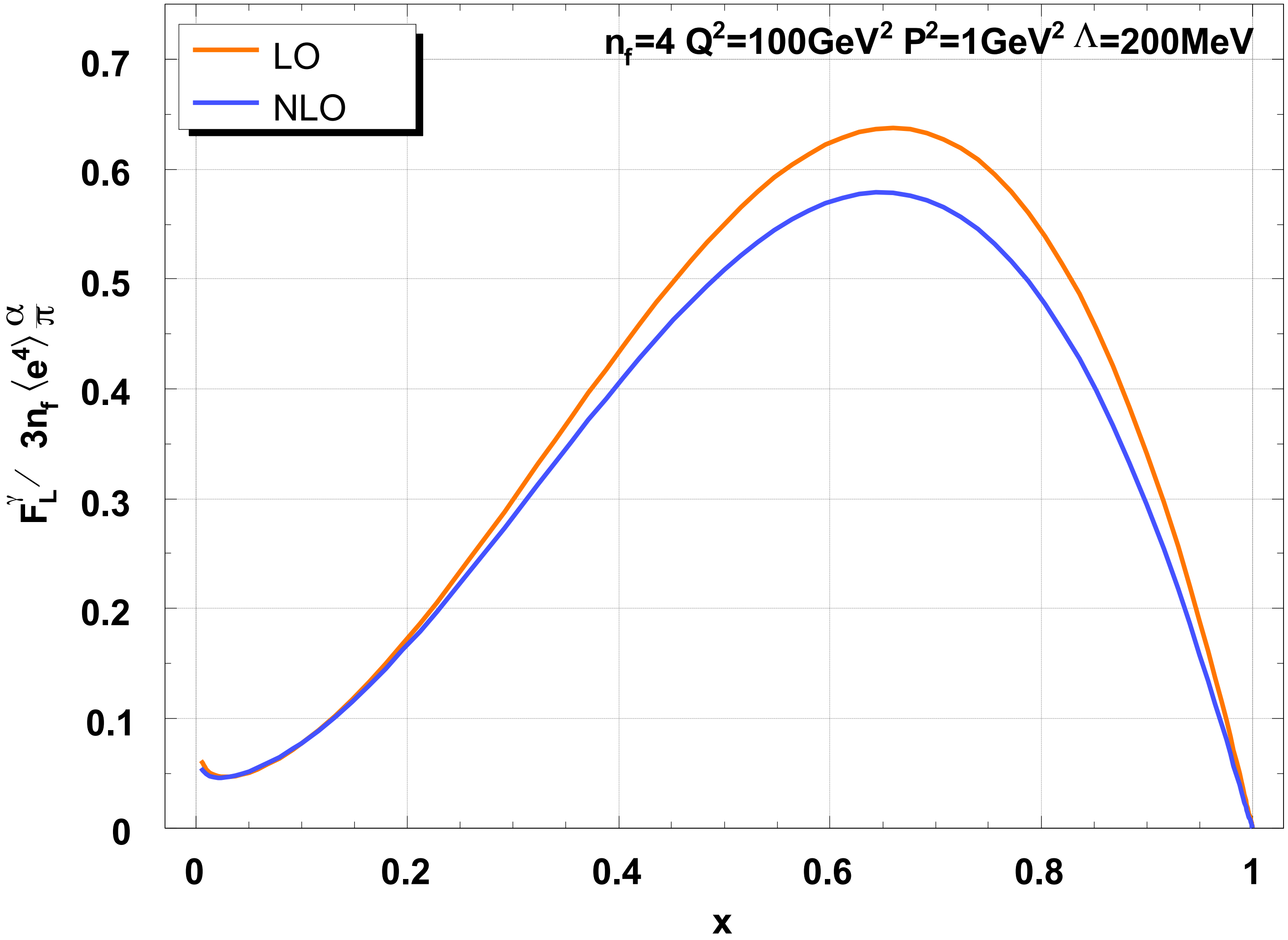


Numerical Plot $F_L^\gamma(x, Q^2, P^2)$



Result

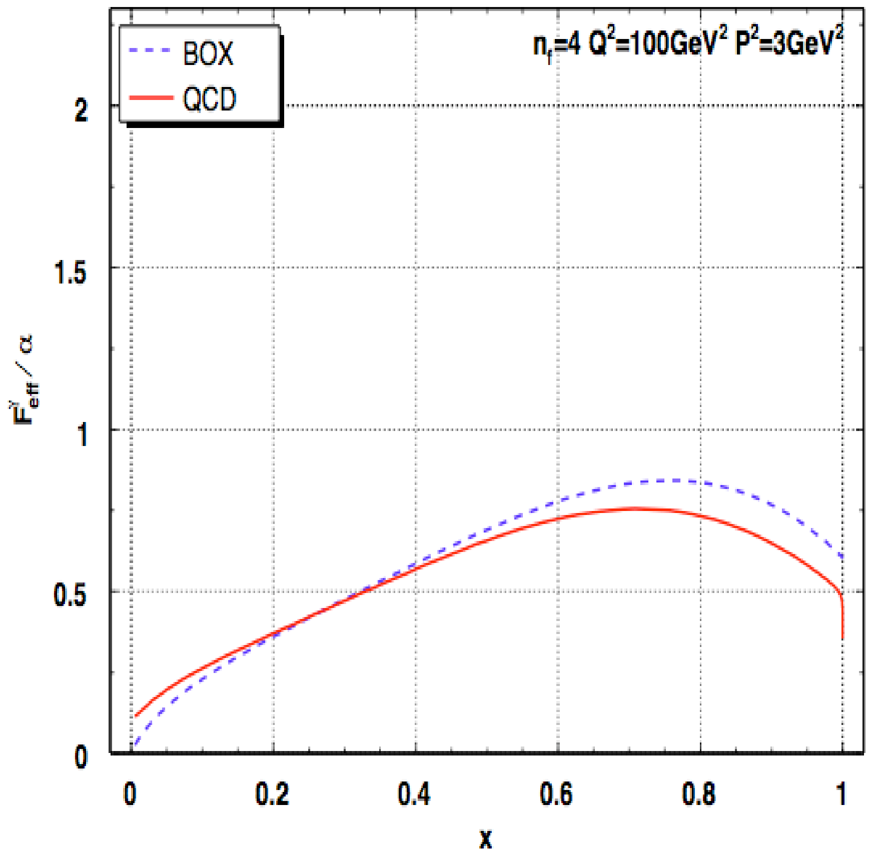
Numerical Plot $F_L^\gamma(x, Q^2, P^2)$



Numerical Plot $F_{\rm eff}^{\gamma}$

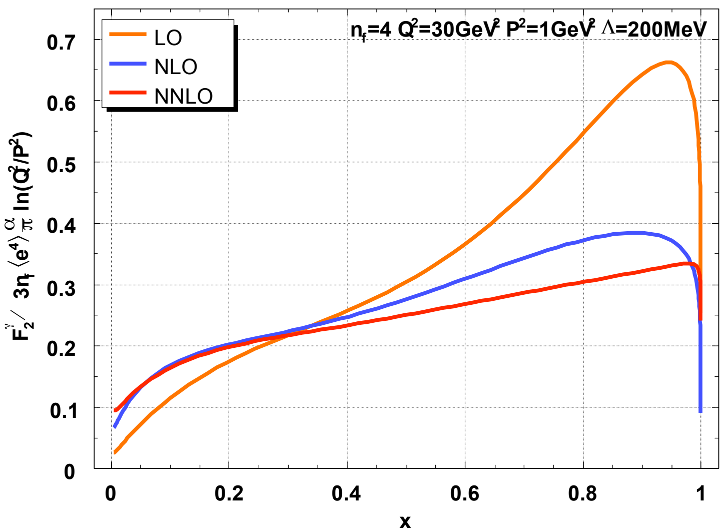
- $$F_{\rm eff}^{\gamma} \approx F_2^{\gamma} + \frac{3}{2} F_L^{\gamma}$$

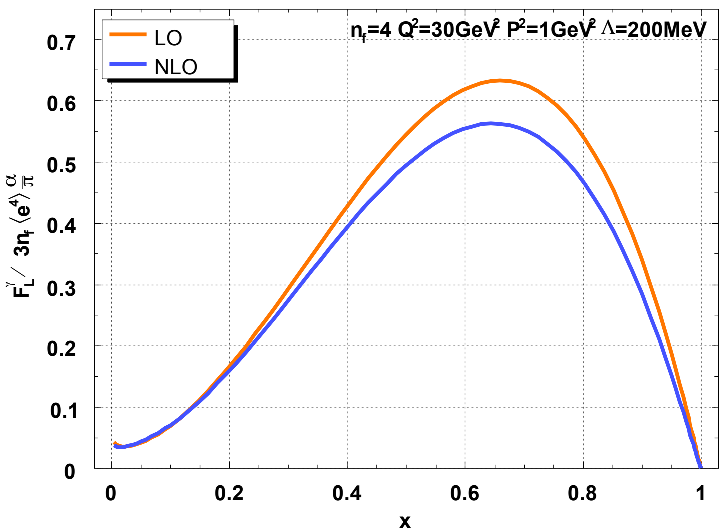
  directly accessible in the experiment



Summary

- The virtual photon structure function $F_2^\gamma(x, Q^2, P^2)$ was investigated in the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$
- Definite predictions were made perturbatively up to NNLO $(\alpha\alpha_s)$
- NNLO $(\alpha\alpha_s)$ corrections appear at large x
- We also analyzed the virtual photon structure function $F_L^\gamma(x, Q^2, P^2)$





Future works

- Quark and gluon distribution functions in the virtual photon up to NNLO

  T. Ueda, T. Uematsu, K.S. in preparation

- Phenomenological analysis of the virtual photon structure functions including target mass effects up to NNLO

  Y. Kitazono, T. Ueda, T. Uematsu, K.S. in preparation