Next-to-leading order corrections for scattering amplitudes in high-energy QCD

Yair Mulian (Under the supervision of M. Lublinsky)
Ben-Gurion University of the Negev

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How scattering amplitudes and cross sections depend on the collision energy $\sqrt{s}$?

At low energy hadrons consists of relatively small number of partons. As the collision energy (rapidity) increases, new partons are emitted (*Weizsacker Williams radiation*).

As long as the density of partons remains small, new particles are created *linearly* (BFKL): the number of new partons due to the increase in collision energy is proportional to the number of the emitting partons (so the density grows exponentially).

Unfortunately, BFKL solutions suffer from two problems:

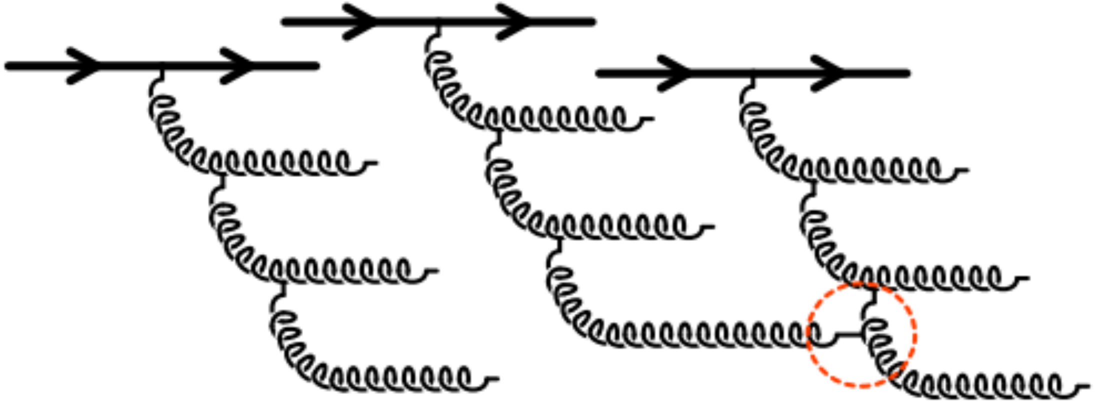
a. ***Infrared Diffusion***

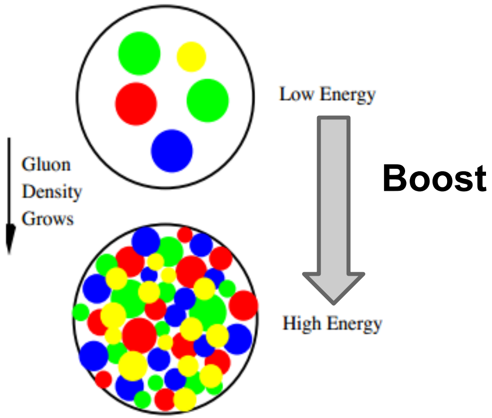
b. ***Unitarity Violation*** **- the cross sections computed by using BFKL grow as** $\sim s^{\alpha_P - 1}$ **and thus violate the *Froissart bound.***

What happens if we push to higher collision energy?

Eventually gluons start overlapping with each other and **new parton emission become a *collective process***.

Emission process becomes *non-linear* and leads to ***gluon saturation*** phenomena (also known as *Color Glass Condensate (CGC) / JIMWLK*). Gluon density grows logarithmically instead of exponentially (as in the case of BFKL).





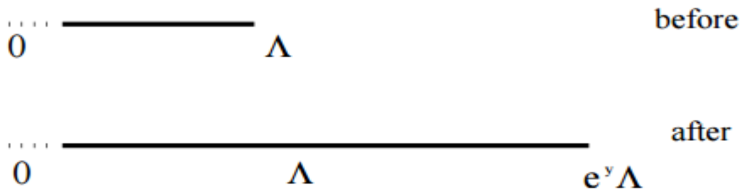
What is happening to a boosted state?

As we are dealing with fast particles, we will use the Eikonal approximation approach together with light-cone formalism. The advantage of this formalism is that boosting the LC hadronic wave function is simple:

$$k^+ \to e^y k^+$$

We separate the momenta in the following way: gluons with $k^+ \leq \Lambda_{UV}$ will be called soft, while those with $k^+ > \Lambda_{UV}$ will be called valence.

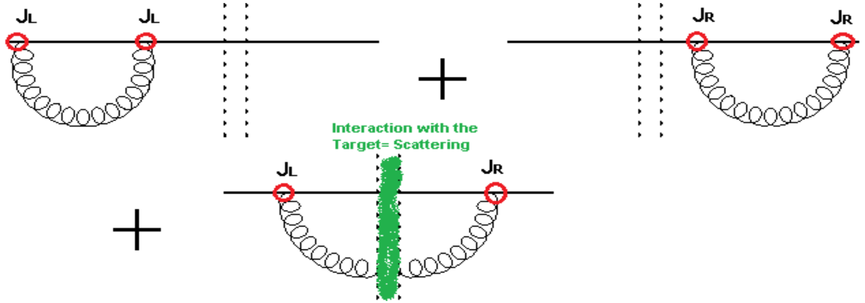
As the figure below suggest, by increasing the C.O.M energy (boosting) we are creating more valence particles.



Growth of the longitudinal phase space under boost.

What is JIMWLK Equation?

The JIMWLK equation, $\frac{d}{dY}\mathcal{O} = -H^{JIMWLK}\mathcal{O}$ describes the rapidity (denoted by Y) evolution of observables $\mathcal{O}$ in a dilute-dense scattering process. At the leading order, it consists of three terms - interaction of the probe with the target fields, and two virtual terms:



J.Jalilian-Marian,
E.Iancu,
L. McLerran,
H. Weigert,
A. Leonidov,
A. Kovner/ 97'-99'

The target is modeled by a fixed background field A. The S-matrix of a fast particle interacting with a gluonic field A is the Wilson line (*Eikonal approximation*):

$$S^{ab}(x) = \left[P \exp\left\{ig \int dx^+ T_c A_c^-(x^+, x)\right\}\right]^{ab}$$

x is a two dimensional transverse coordinate (LC gauge). As expected, JIMWLK reduces to BFKL for low probability of scattering.

The LO JIMWLK Equation

The JIMWLK Hamiltonian can be obtained by computing the expectation value of the S-matrix operator (expanded to first order in longitudinal phase space):

$$H^{JIMWLK} = \langle\psi|\,\hat{S} - 1\,|\psi\rangle$$

Where $|\psi\rangle$ is the soft part of the wave function. The general expression for the LO wave function in momenta space (before normalization) is:

$$|\psi\rangle = |0\rangle + |g_i^a(k)\rangle \frac{\langle g_i^a(k)\,|H_g|\,0\rangle}{E(g)} = |0\rangle + \int d^3k \frac{-igk_\perp^i \rho^a(k_\perp)}{2\pi^{3/2}\sqrt{k^+}k_\perp^2} |g_a^i(k)\rangle$$

Where H_g denotes the relevant part in the QCD Hamiltonian at leading order (in the Eikonal approximation) which reads:

$$H_g = \int \frac{dk^+}{2\pi} \frac{d^2k_\perp}{(2\pi)^2} \frac{gk_i}{|k^+|^{3/2}} \left[a_i^{\dagger a}(k^+, k_\perp) + a_i^a(k^+, k_\perp)\right] \rho^a(k_\perp) \qquad \rho^a(x_\perp) = \int_{k^+>\Lambda} a_i^{\dagger b}(k^+, x_\perp) T_{bc}^a a_i^c(k^+, x_\perp)$$

We can transform the wave function to coordinate space:

$$|\psi\rangle = |0\rangle + \frac{g}{2\pi^{3/2}} \int_{\Lambda}^{(1+\delta Y)\Lambda} \frac{dk^+}{\sqrt{k^+}} \int d^2x_\perp d^2z_\perp \, \rho^a(x_\perp) \frac{(x_\perp - z_\perp)^i}{(x_\perp - z_\perp)^2} |g_a^i(z)\rangle$$

After normalization, we can use the LO wave-function in order to write the LO JIMWLK Hamiltonian:

$$\mathcal{H}_{LO\,JIMWLK} = -\frac{\alpha_s}{2\pi} \int d^2x \, d^2y \, d^2z \, \frac{(x-z)(y-z)}{(x-z)^2(y-z)^2} \left[J_L^a(x)J_L^a(y) + J_R^a(x)J_R^a(y) - 2J_L^a(x)S^{ab}(z)J_R^b(y)\right]$$

J_L and J_R are operators acting on Wilson lines as rotations:

$$J_R^a(x)S^{ij}(w) = (S(w)t^a)^{ij}\delta(w-x)$$

$$J_L^a(x)S^{ij}(w) = (t^a S(w))^{ij}\delta(w-x)$$

This Hamiltonian describes the evolution of scattering amplitudes *for large values of rapidity*. It is a *non-linear functional* equation and it takes into account both linear growth as well as saturation effects.

Motivations for NLO JIMWLK Equation

The LO JIMWLK is only a first term in an infinite perturbative series:

$$H_{JIMWLK} = H_{LO}(\alpha_s) + \boxed{H_{NLO}(\alpha_s^2)} + H_{NNLO}(\alpha_s^3) + \ldots$$

The NLO term is necessary because:

a. NLO corrections are *known to be large*.

b. Built-in information on the *running coupling* - better phenomenology. The running is known to slow down the evolution.

c. To get the *region of applicability* of the leading order equation.

d. Important step towards *all order resummation*.

The NLO Wave Function

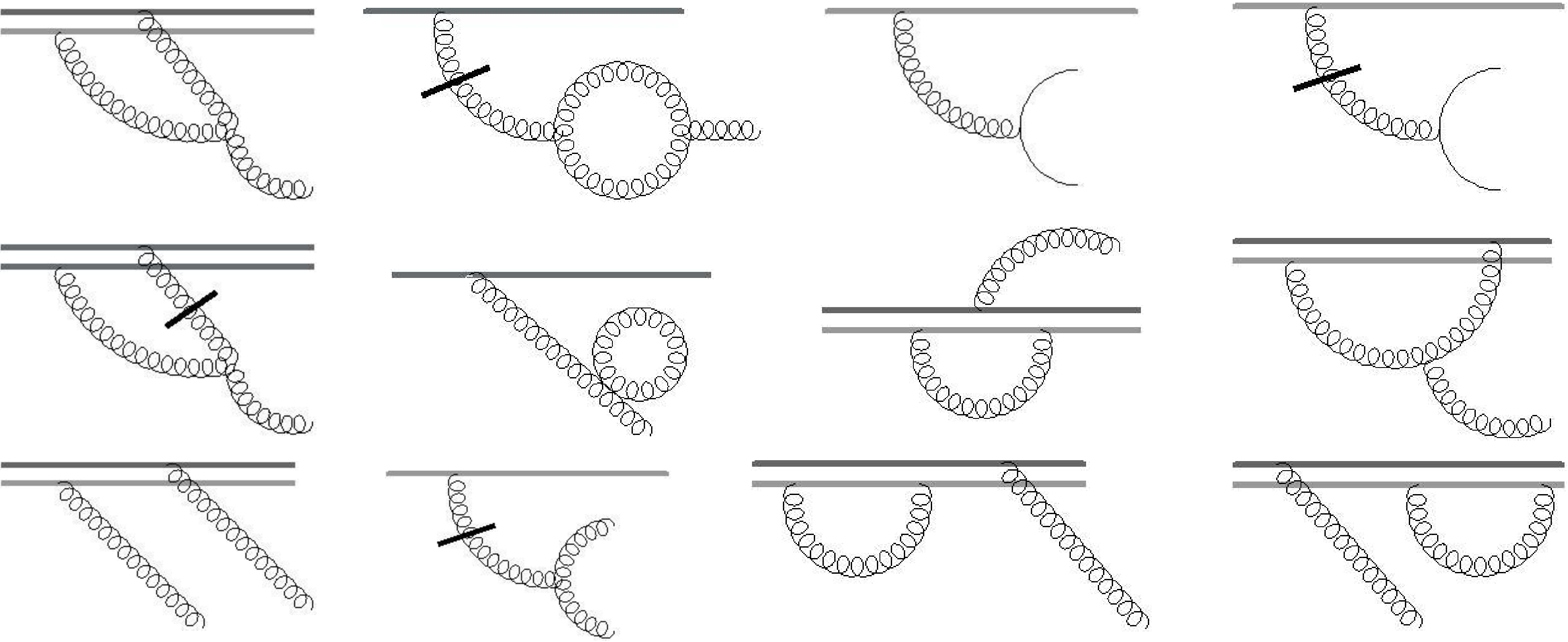
$$|\psi\rangle = |0\rangle\Bigg(1 - \frac{|\langle i\,|H_{int}|\,0\rangle|^2}{2E_i^2} - \frac{|\langle i\,|H_{int}|\,j\rangle\,\langle j\,|H_{int}|\,0\rangle|^2}{2E_i^2E_j^2} + \frac{\langle 0\,|H_{int}|\,i\rangle\,\langle i\,|H_{int}|\,j\rangle\,\langle j\,|H_{int}|\,0\rangle}{2E_iE_j}\left(\frac{1}{E_i}+\frac{1}{E_j}\right)$$
$$- \frac{\langle 0\,|H_{int}|\,i\rangle\,\langle i\,|H_{int}|\,k\rangle\,\langle k\,|H_{int}|\,j\rangle\,\langle j\,|H_{int}|\,0\rangle}{2E_iE_jE_k}\left(\frac{1}{E_i}+\frac{1}{E_j}\right) + \frac{3\,|\langle i\,|H_{int}|\,0\rangle|^4}{8E_i^4}\Bigg)$$
$$- |i\rangle\,\frac{\langle i\,|H_{int}|\,0\rangle}{E_i} + |i\rangle\,\frac{\langle i\,|H_{int}|\,j\rangle\,\langle j\,|H_{int}|\,0\rangle}{E_iE_j} - |i\rangle\,\frac{\langle i\,|H_{int}|\,k\rangle\,\langle k\,|H_{int}|\,j\rangle\,\langle j\,|H_{int}|\,0\rangle}{E_iE_jE_k}$$
$$+ |i\rangle\,\frac{\langle i\,|H_{int}|\,0\rangle\,|\langle j\,|H_{int}|\,0\rangle|^2\,(E_i+2E_j)}{2E_i^2E_j^2}$$

Where there is a summation over the different states, which can possibly be:
One soft gluon, two soft gluons or quark anti-quark state.
H_{int} is the sum of all non-kinetic terms in the light-cone Hamiltonian (notice that in light-cone gauge we have also instantaneous interactions). Based on the above expression we can write the general form of the NLO wave function:

$$|\psi\rangle \;=\; (1 - g_s^2\,\kappa_0\,JJ - g_s^4(\delta_1\,JJ + \delta_2\,JJJ + \delta_3\,JJJJ)\,|\,no\,soft\,gluons\rangle +$$
$$+\,(\,g_s\kappa_1\,J + g_s^3\epsilon_1\,J + g_s^3\,\epsilon_2\,J\,J)\,|\,one\,soft\,gluon\rangle + g_s^2(\epsilon_3\,J + \epsilon_4\,JJ)\,|\,two\,soft\,gluons\rangle + g_s^2\,\epsilon_5\,J\,|\,q\,\bar{q}\rangle$$

My interest was to determine the 10 unknown coefficients.

Diagrammatic Representation



In total around 30 diagrams, time ordering is important.

The General Form of NLO JIMWLK Hamiltonian

$$H^{NLO\ JIMWLK} = \int_{x,y,z} K_{JSJ}(x,y;z)\,\left[J_L^a(x)\,J_L^a(y) + J_R^a(x)\,J_R^a(y) - 2\,J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y)\right] +$$

$$+\int_{x\,y\,z\,z'} K_{JSSJ}(x,y;z,z')\left[f^{abc}\,f^{def}\,J_L^a(x)\,S_A^{be}(z)\,S_A^{cf}(z')\,J_R^d(y) - N_c\,J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y)\right] +$$

$$+\int_{x,y,z,z'} K_{q\bar{q}}(x,y;z,z')\left[2\,J_L^a(x)\,tr[S^\dagger(z)\,T^a\,S(z')T^b]\,J_R^b(y) - J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y)\right] +$$

$$+\int_{w,x,y,z,z'} K_{JJSSJ}(w;x,y;z,z')f^{acb}\left[J_L^d(x)\,J_L^e(y)\,S_A^{dc}(z)\,S_A^{eb}(z')\,J_R^a(w) - J_L^a(w)\,S_A^{cd}(z)\,S_A^{be}(z')\,J_R^d(x)\,J_R^e(y) +\right.$$

$$\left.+\frac{1}{3}[\,J_L^c(x)\,J_L^b(y)\,J_L^a(w) - J_R^c(x)\,J_R^b(y)\,J_R^a(w)]\right] +$$

$$+\int_{w,x,y,z} K_{JJSJ}(w;x,y;z)\,f^{bde}\left[J_L^d(x)\,J_L^e(y)\,S_A^{ba}(z)\,J_R^a(w) - J_L^a(w)\,S_A^{ab}(z)\,J_R^d(x)\,J_R^e(y) +\right.$$

$$\left.+\frac{1}{3}[\,J_L^d(x)\,J_L^e(y)\,J_L^b(w) - J_R^d(x)\,J_R^e(y)\,J_R^b(w)]\right]$$

The 5 kernels can be found based on dipole and baryon evolution.

UV and IR Divergences

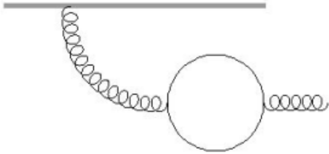
Many of the diagrams we are calculating diverge. In order to make them finite we are using the following regularizations:

1) ***Dimensional regularization*** on the transverse momenta.

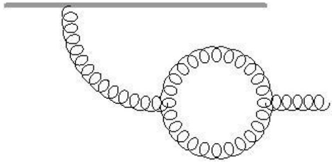
2) ***Sharp cutoff*** on the longitudinal momenta. The minimal possible longitudinal momenta will be denoted by Λ .
These divergences cancel between the different diagrams.

Explicit Example - JSJ Kernel

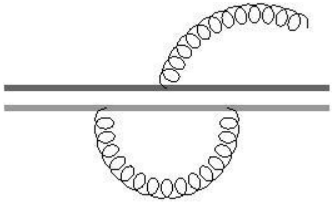
$$\int d^3k\, \frac{ig^3 N_f \rho^a(k_\perp) k_\perp^i}{16\pi^{7/2}\sqrt{k^+}k_\perp^2} \left(-\frac{1}{3}\log\left(\frac{k_\perp^2}{\mu_{\overline{MS}}^2}\right) + \frac{5}{9} \right) |g_a^i(k)\rangle$$



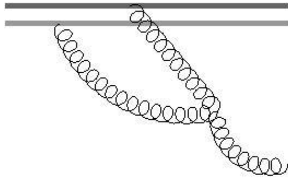
$$\int d^3k\, \frac{ig^3 N_c \rho^a(k_\perp) k_\perp^i}{16\pi^{7/2}k_\perp^2\sqrt{k^+}} \left(\left[\frac{11}{6} + 2\log\left(\frac{\Lambda}{k^+}\right)\right] \log\left(\frac{k_\perp^2}{\mu_{\overline{MS}}^2}\right) + \log^2\left(\frac{\Lambda}{k^+}\right) - \frac{67}{18} + \frac{\pi^2}{3} \right) |g_a^i(k)\rangle$$



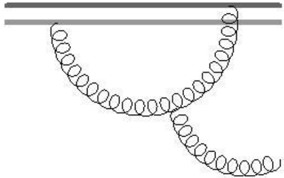
$$\int d^3k\, \frac{ig^3 N_c \rho^a(k_\perp) k_\perp^i}{32\pi^{7/2}\sqrt{k^+}k_\perp^2} \left(2\left[\log\left(\frac{\Lambda_{UV}}{k^+}\right) - \log\left(\frac{\Lambda}{k^+}\right)\right] \log\left(\frac{k_\perp^2}{\mu_{\overline{MS}}^2}\right) + \log^2\left(\frac{\Lambda_{UV}}{k^+}\right) - \log^2\left(\frac{\Lambda}{k^+}\right) \right) |g_a^i(k)\rangle$$



$$\int d^3k\, \frac{ig^3 N_c \rho^a(k_\perp) k_\perp^i}{32\pi^{7/2}k_\perp^2\sqrt{k^+}} \left(3\log\left(\frac{\Lambda}{k^+}\right) \log\left(\frac{k_\perp^2}{\mu_{\overline{MS}}^2}\right) + \frac{\pi^2}{3} + 2\log^2\left(\frac{\Lambda}{k^+}\right) \right) |g_a^i(k)\rangle$$



$$\int d^+k\, d^2p_\perp \int_0^1 d\xi \frac{g^3 f^{abc}\rho(p_\perp)\sqrt{\xi}p_\perp^j}{32\pi^{7/2}p_\perp^2\xi(1-\xi)\sqrt{k^+}} \left((2-\xi)\log(\xi) - (1+\xi)\log(1-\xi) + (2-\xi)\log\left(\frac{p_\perp^2}{\mu_{\overline{MS}}^2}\right) \right) |g_j^b(p)\rangle$$



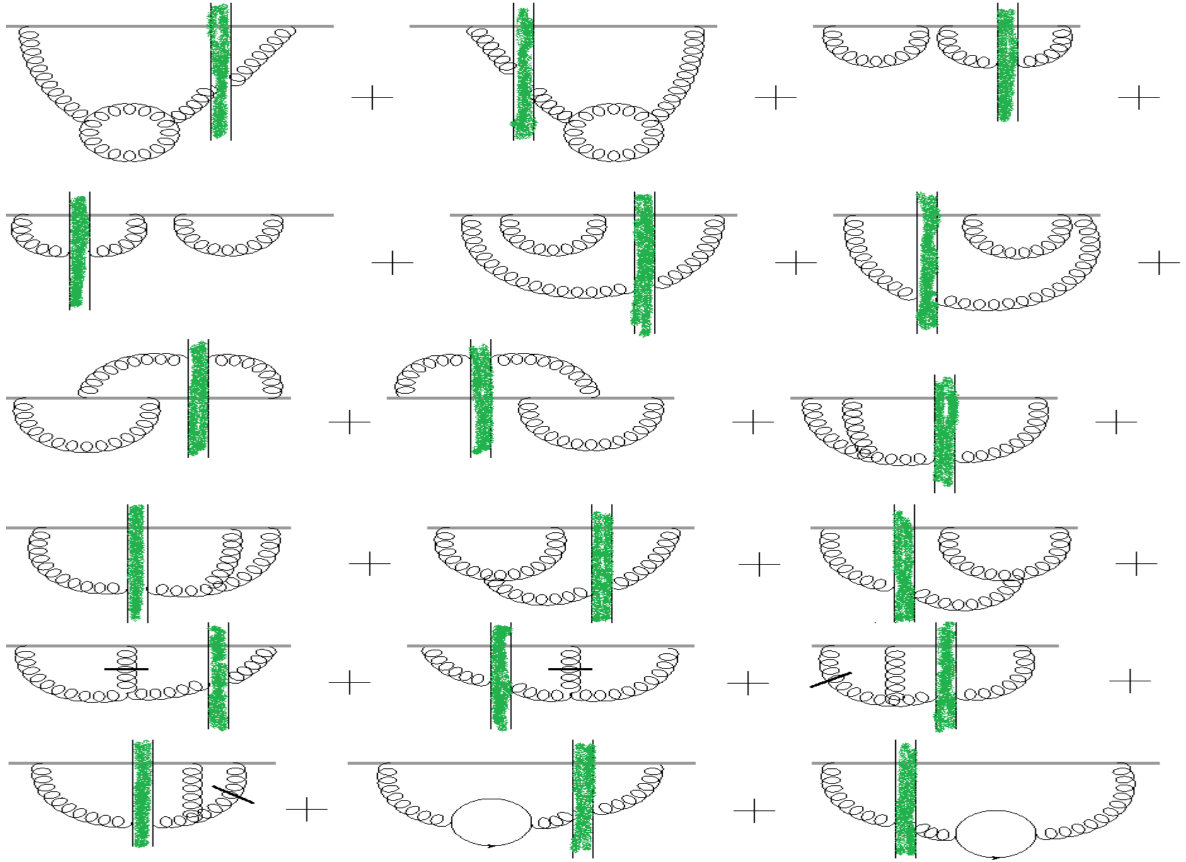
Kernel JSJ

By adding together all the different contributions, and after multiplying by the conjugate wave function, we find (after performing Fourier transform):

$$\int_{x_\perp, y_\perp, z_\perp} \frac{\alpha_s^2 J_L(x_\perp) S^{ab}(z_\perp) J_R(y_\perp) X^i Y^i}{8\pi^3 X^2 Y^2} \left(b \log\left(\frac{k_\perp^2}{\mu_{\overline{MS}}^2}\right) + \left(\frac{\pi^2}{3} - \frac{67}{9}\right) N_c + \frac{10}{9} N_f \right) \qquad b = \frac{11}{3} N_c - \frac{2}{3} N_f$$

Notice that as we expect, all the IR divergences (terms which involve $\log^2\left(\frac{\Lambda}{k^+}\right)$ and $\log\left(\frac{\Lambda}{k^+}\right)\log\left(\frac{k_\perp^2}{\mu_{\overline{MS}}^2}\right)$) were canceled at the final result.

From the Wave Function to the Kernel JSJ



Subtraction of the (LO)^2 Contributions

The contribution which corresponds to (LO)^2 can be written as:

$$(H^{LO\ JIMWLK})^2 = \frac{\alpha_s^2}{4\pi^4}\int_{x_\perp,y_\perp,u_\perp,v_\perp,z_\perp,z'_\perp}\frac{(X\cdot Y)(U\cdot V)}{X^2Y^2U^2V^2}\left[J_L^a(x)J_L^a(y)+J_R^a(x)J_R^a(y)-2J_L^a(x)S_A^{ab}(z)J_R^b(y)\right]$$
$$\times\left[J_L^c(u)J_L^c(v)+J_R^c(u)J_R^c(v)-2J_L^c(u)S_A^{cd}(z')J_R^d(v)\right]$$

This contribution does not correspond to the same power of logs of rapidity we are trying to resum, therefore this contribution has to be subtracted.

Current Status of the Calculation

Kernel	Calculated?	Comparison with previous papers (1405.0418, 1310.0378)
JSJ	✓	✓
JJSSJ	✓	✓
JSSJ	✓	✗
QQ	✓	✗
JJSJ	✓	✓

Conclusions

1) The light-cone was shown to be an efficient tool for calculating the NLO correction.

2) As expected the IR divergences disappear after adding all the relevant contributions together.

3) The coefficient of $\log\left(\frac{k^+}{\Lambda}\right)$ in the wave-function exactly corresponds to (LO)^2.

4) So far the results are consistent with our previous works based on Balitsky, Chirilly and Grabovsky.

The Kernels for Gauge Invariant Operators (color singlet amplitudes)

$$K_{JSJ}(x,y;z) = -\frac{\alpha_s^2}{16\pi^3}\frac{(x-y)^2}{X^2Y^2}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2} + (\frac{67}{9}-\frac{\pi^2}{3})N_c - \frac{10}{9}n_f\Big] - \frac{N_c}{2}\int_{z'}\tilde{K}(x,y,z,z')$$

Here μ is the normalization point in the $\overline{MS}$ scheme and $b = \frac{11}{3}N_c - \frac{2}{3}n_f$ is the first coefficient of the β-function.

$$K_{JSSJ}(x,y;z,z') = \frac{\alpha_s^2}{16\,\pi^4}\Bigg[-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]} + \frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Bigg] + \tilde{K}(x,y,z,z')$$

$$\tilde{K}(x,y,z,z') = \frac{i}{2}\,[K_{JJSSJ}(x;x,y;z,z') - K_{JJSSJ}(y;x,y;z,z') - K_{JJSSJ}(x;y,x;z,z') + K_{JJSSJ}(y;y,x;z,z')]$$

$$K_{q\bar{q}}(x,y;z,z') = -\frac{\alpha_s^2\,n_f}{8\,\pi^4}\Big\{\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}\ln\frac{X^2Y'^2}{X'^2Y^2} - \frac{2}{(z-z')^4}\Big\}$$

$$K_{JJSJ}(w;x,y;z) = -i\,\frac{\alpha_s^2}{4\,\pi^3}\Big[\frac{X\cdot W}{X^2\,W^2} - \frac{Y\cdot W}{Y^2\,W^2}\Big]\ln\frac{Y^2}{(x-y)^2}\ln\frac{X^2}{(x-y)^2}$$

$$K_{JJSSJ}(w;x,y;z,z') = -i\,\frac{\alpha_s^2}{2\,\pi^4}\Big(\frac{X_iY'_j}{X^2Y'^2}\Big)\Big(\frac{\delta_{ij}}{2(z-z')^2} + \frac{(z'-z)_iW'_j}{(z'-z)^2W'^2} + \frac{(z-z')_jW_i}{(z-z')^2W^2} - \frac{W_iW'_j}{W^2W'^2}\Big)\ln\frac{W^2}{W'^2}$$