Amplitudes in High-Energy-factorisation via BCFW recursion

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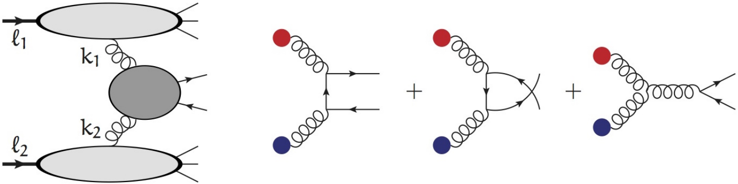
High-Energy-factorisation

High-Energy-factorisation *(Catani,Ciafaloni,Hautmann, 1991 / Collins,Ellis, 1991)*



$$\sigma_{h_1,h_2\to q\bar{q}} = \int d^2k_{1\perp}\, d^2k_{2\perp}\, \frac{dx_1}{x_1}\, \frac{dx_2}{x_2}\, f_g(x_1,k_{1\perp})\, f_g(x_2,k_{2\perp})\, \hat{\sigma}_{gg}\left(\frac{m^2}{x_1x_2s}, \frac{k_{1\perp}}{m}, \frac{k_{2\perp}}{m}\right)$$

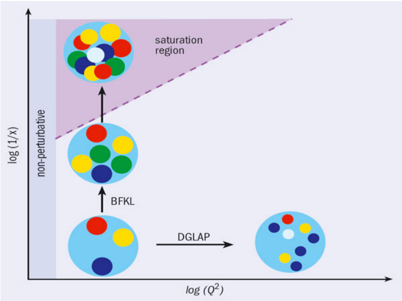
where the f_g's are the gluon densities, obeying BFKL, BK, CCFM evolution equations.

Non neglectable transverse momentum is associated to small-x physics.

Momentum parameterization:

$$k_1^\mu = x_1\, p_1^\mu + k_{1\perp}^\mu \quad , \quad k_2^\mu = x_2\, p_2^\mu + k_{2\perp}^\mu \quad \text{for} \quad p_i \cdot k_i = 0 \quad k_i^2 = -k_{i\perp}^2 \quad i = 1,2$$

To be applied in the regime: $s >> M^2 \sim k_\perp^2$



Problem: general partonic processes must be described by **gauge invariant amplitudes**

$\Rightarrow$ ordinary Feynman rules are not enough !

Applications:
- production of forward dijets initiated with gluons : $gg^* \to gg$
- production of forward dijets initiated with quarks : $q\bar{q}^* \to gg$
- Dilute-dense hadronic collisions in TMD factorization (see Kotko's talk)
- Test of TMDs in multi-jet production : $p\,p \to n\,$jets

Is there a general method to compute such gauge-invariant amplitudes ?

Just for the formalism: Weyl spinors

High energy limit ⇒ **massless particles** ⇒ Weyl basis for spinors.
If $p^2 = 0$, it can be cast in the Pauli matrices language,

$$|p] = \begin{pmatrix} L(p) \\ \mathbf{0} \end{pmatrix} \qquad L(p) = \frac{1}{\sqrt{|p^0 + p^3|}} \begin{pmatrix} -p^1 + i\,p^2 \\ p^0 + p^3 \end{pmatrix}$$

$$|p\rangle = \begin{pmatrix} \mathbf{0} \\ R(p) \end{pmatrix} \qquad R(p) = \frac{\sqrt{|p^0 + p^3|}}{p^0 + p^3} \begin{pmatrix} p^0 + p^3 \\ p^1 + i\,p^2 \end{pmatrix}$$

and the charge-conjugated spinors

$$[p| = \left((\mathcal{E} L(p))^T \,,\, \mathbf{0} \right) \qquad \langle p| = \left(\mathbf{0} \; (\mathcal{E}^T R(p))^T \right) \qquad \text{where} \quad \mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

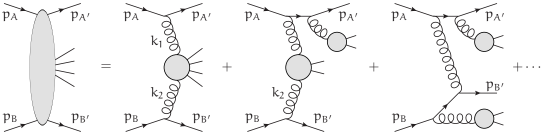
$$p \cong p^\mu \, \sigma_\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + i\,p^2 \\ -p^1 - i\,p^2 & p^0 + p^3 \end{pmatrix} = |p]\langle p|$$

Prescription for off-shell gluons

ONE IDEA:

on-shell amplitudes are gauge invariant, so off-shell gauge-invariant amplitudes could be got by embedding them into on-shell processes...

...first result...: 1) For off-shell gluons: represent g^* as coming from a $\bar{q}qg$ vertex, with the quarks taken to be on-shell



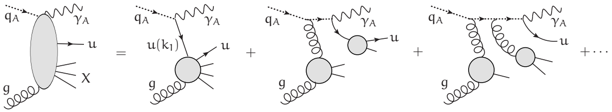
- embed the scattering of the off-shell gluons in the scattering of two quark pairs carrying momenta $p_A^\mu = k_1^\mu$, $p_B^\mu = k_2^\mu$, $p_{A'}^\mu = 0$, $p_{B'}^\mu = 0$
- Assign the spinors $|p_1\rangle, |p_1]$ to the A-quark and the propagator $\frac{i\,\not{p}_1}{p_1 \cdot k}$ instead of $\frac{i\not{k}}{k^2}$ to the propagators of the A-quark carrying momentum k; the same goes for the B-quark line.
- multiply the amplitude by $g_s^{-1} x_1 \sqrt{-2\,k_{1\perp}^2} \times g_s^{-1} x_2 \sqrt{-2\,k_{2\perp}^2}$.
- ordinary Feynman rules must be used everywhere else and the procedure holds for any number of off-shell gluons.

K. Kutak, P. Kotko, A. van Hameren, JHEP 1301 (2013) 078

Prescription for off-shell quarks

... and second result:

2) for off-shell quarks: represent q^* as coming from a $\gamma\bar{q}q$ vertex, with a 0 momentum and $\bar{q}$ on shell (and vice-versa)



- embed the scattering of the quark with whatever set of particles in the scattering of an auxiliary quark-photon pair, q_A and γ_A carrying momenta $p^\mu_{q_A} = k^\mu_1$, $p^\mu_{\gamma_A} = 0$
- Let q_A-propagators of momentum k be $\frac{i\,\not{p}_1}{p_1 \cdot k}$ and assign the spinors $|p_1\rangle, |p_1]$ to the A-quark.
- Assign the polarization vectors $\epsilon^\mu_+ = \frac{\langle q|\gamma^\mu|p_1]}{\sqrt{2}\langle p_1 q\rangle}$, $\epsilon^\mu_- = \frac{\langle p_1|\gamma^\mu|q]}{\sqrt{2}[p_1 q]}$ to the auxiliary photon, with q a light-like auxiliary momentum.
- Multiply the amplitude by $x_1\sqrt{-k^2_{1\perp}/2}$ and use ordinary Feynman rules everywhere else.

K. Kutak, T. Salwa, A. van Hameren, Phys.Lett. B727 (2013) 226-233

Prescription for off-shell gluons: derivation 1

Auxiliary vectors (complex in general):
$$\begin{cases} p_3^\mu = \frac{1}{2}\langle p_2|\gamma^\mu|p_1] \\ p_4^\mu = \frac{1}{2}\langle p_1|\gamma^\mu|p_2] \\ p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0 \\ p_{1,2}\cdot p_{3,4} = 0\,, \quad p_1\cdot p_2 = -p_3\cdot p_4 \end{cases}$$

Auxiliary momenta:
$$\begin{cases} p_A^\mu = (\Lambda + x_1)p_1^\mu - \frac{p_4\cdot k_{1\perp}}{p1\cdot p_2}p_3^\mu\,, \quad p_{A'}^\mu = \Lambda p_1^\mu + \frac{p_3\cdot k_{1\perp}}{p1\cdot p_2}p_4^\mu \\ p_B^\mu = (\Lambda + x_2)p_2^\mu - \frac{p_3\cdot k_{2\perp}}{p1\cdot p_2}p_4^\mu\,, \quad p_{B'}^\mu = \Lambda p_2^\mu + \frac{p_4\cdot k_{2\perp}}{p1\cdot p_2}p_3^\mu \end{cases}$$

For any Λ:
$$\begin{cases} p_A^\mu - p_{A'}^\mu = x_1\,p_1^\mu + k_{1\perp}^\mu \\ p_B^\mu - p_{B'}^\mu = x_2\,p_2^\mu + k_{2\perp}^\mu \\ p_A^2 = p_{A'}^2 = p_B^2 = p_{B'}^2 = 0 \end{cases}$$

Prescription for off-shell gluons: derivation 2

Momentum flowing through a propagator of an auxiliary quark line:

$$k^\mu = (\Lambda + x_k)p_1^\mu + y_k\, p_2^\mu + k_\perp$$

Final step: remove complex components taking the $\Lambda \to \infty$ limit.

$$\frac{\not{k}}{k^2} = \frac{(\Lambda + x_k)\not{p}_1 + y_k\, \not{p}_2 + \not{k}}{2(\Lambda + x_k) y_k p_1 \cdot p_2 + k_\perp^2} \xrightarrow{\Lambda \to \infty} \frac{\not{p}_1}{2\, y_k p_1 \cdot p_2} = \frac{\not{p}_1}{2 p_1 \cdot k}$$

...and the factor $x_1\sqrt{-k_\perp^2/2}$ is to match the collinear limit.

Also another approach based on Wilson lines (for gluons only) presented :
P. Kotko, JHEP 1407 (2014) 128

In agreement with other approaches (e.g. Lipatov's effective action)

Reminds us of BCFW...

The analytical results derived with the mentioned trick are strikingly similar to the ones obtained in the on-shell case via, for example, the BCFW recursion relation (which does not require auxiliary particles).

Computing scattering amplitudes in Yang-Mills theories via ordinary Feynman diagrams: soon overwhelming !

Number of Feynman diagrams at tree level on-shell:

# of gluons	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

And there are even more with the proposed method for amplitudes with off-shell particles due to the gauge-restoring terms.

A method to efficiently compute helicity amplitudes: **BCFW recursion relation**

Britto, Cachazo, Feng, Nucl.Phys. B715 (2005) 499-522
Britto, Cachazo, Feng, Witten, Phys.Rev.Lett. 94 (2005) 181602

BCFW recursion relation

Two very simple ideas for tree level amplitudes:

1. Cauchy's residue theorem: if the amplitude is formally treated as a function of a complex variable z and if it is rational and vanishes for $z \to \infty$, then the integral extended to an infinite contour enclosing all poles vanishes

$$\lim_{z\to\infty} \mathcal{A}(z) = 0 \Rightarrow \frac{1}{2\pi i}\oint dz\, \frac{\mathcal{A}(z)}{z} = 0$$

   implying that the value at $z = 0$ (physical amplitude) can be determined as a sum of the residues at the poles:

$$\mathcal{A}(0) = -\sum_i \frac{\lim_{z\to z_i}[(z - z_i)\, f(z)]}{z_i}$$

   where z_i is the location of the i-th pole

2. Unitarity: Poles in Yang-Mills tree level amplitudes can only be due to gluon propagators dividing the n-point amplitude into two on-shell sub-amplitudes with $k+1$ and $n-k+1$ gluons $\Rightarrow$ it is all about finding the proper way to "complexify" an amplitude.

To properly "complexify" $\mathcal{A}$: for helicities $(h_1, h_n) = (-,+)$ (no loss of generality...)

$$
\begin{aligned}
|1] \quad &\to \quad |\hat{1}] \equiv |1] - z\,|n] \Rightarrow p_1 \to \hat{p}_1 = |1]\langle 1| - z|1]\langle n| \\
|n\rangle \quad &\to \quad |\hat{n}\rangle \equiv |n\rangle + z|1\rangle \Rightarrow p_n \to \hat{p}_n = |n]\langle n| + z|1]\langle n|
\end{aligned}
$$

With such a choice

- On-shellness, gauge invariance and momentum conservation preserved throughout.
- the most serious issue is the behaviour for $z \to \infty$, but either a result derived with twistor methods (*Cachazo,Svrcek and Witten JHEP 0409 (2004) 006*) or a smart choice of reference lines always allow to overcome the problem, so that $\lim_{z\to\infty} \mathcal{A}(z) = 0$ holds

BCFW applies to* color-ordered partial amplitudes*, for which the kinematics and gauge structure are factorised like

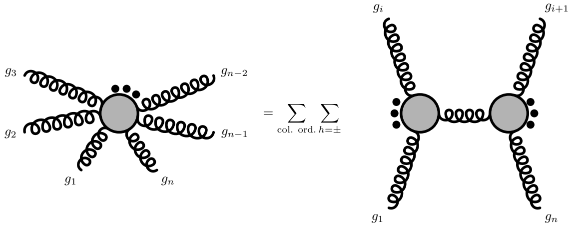
$$
\mathcal{M}_n = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T_{\sigma(1)} \dots T_{\sigma(n)})\,\mathcal{A}(g_{\sigma(1)}, \dots, g_{\sigma(n)})
$$

The result is an amazingly simple recursive relation:

any tree-level color-ordered amplitude is the sum of residues of the poles it develops when it is made dependent on a complex variable as above.
Such residues are simply products of color-ordered lower-point amplitudes evaluated at the pole times an intermediate propagator. Shifted particles are always on opposite sides of the propagator.

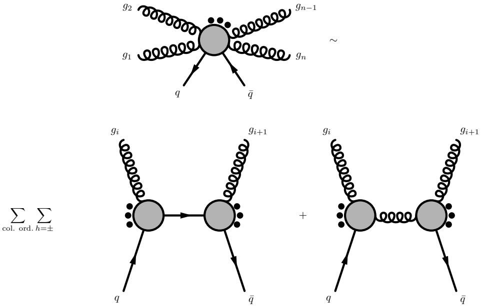
$$\mathcal{A}(g_1,\ldots,g_n) = \sum_{i=2}^{n-2}\sum_{h=+,-}\mathcal{A}(g_1,\ldots,g_i,\hat{P}^h)\,\frac{1}{(p_1+\cdots+p_i)^2}\,\mathcal{A}(-\hat{P}^{-h},g_{i+1},\ldots,g_n)$$

$$z_i = \frac{(p_1+\cdots+p_i)^2}{[1|p_1+\cdots+p_i|n\rangle}$$ location of the pole corresponding for the "i-th" partition



The inclusion of fermions and MHV amplitudes

The BCFW recursion was promptly extended to Yang-Mills theories with fermions:
M. Luo, C. Wen, JHEP 0503 (2005) 004



A couple of MHV amplitudes:

$$\mathcal{A}(g_1^+, g_2^+, \ldots, g_i^-, \ldots, g_j^-, \ldots, g_n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n-1\, n \rangle \langle n1 \rangle}$$

$$\mathcal{A}(q^-, g_1^-, g_2^+, \ldots, g_n^+, \bar{q}^+) = \frac{\langle q1 \rangle^3 \langle \bar{q}1 \rangle}{\langle \bar{q}q \rangle \langle q1 \rangle \langle 12 \rangle \ldots \langle n\bar{q} \rangle}$$

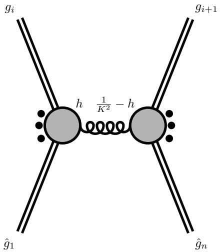
It is natural to ask whether something like a BCFW recursion relation exists with off-shell particles. For off shell, gluons, the answer was first found in *A. van Hameren, JHEP 1407 (2014) 138*

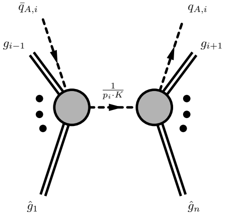
$$\mathcal{A}(0) = \sum_{s=g,f} \left(\sum_{p} \sum_{h=+,-} \mathrm{A}^{s}_{p,h} + \sum_{i} \mathrm{B}^{s}_{i} + \mathrm{C}^{s} + \mathrm{D}^{s} \right) ,$$

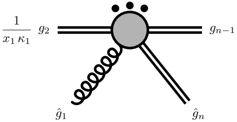
- $\mathrm{A}^{g/f}_{p,h}$ are due to the poles which appear in the original BCFW recursion for on-shell amplitudes. The pole appears because one of the intermediate virtual gluon, whose shifted momentum squared $K^2(z)$ goes on-shell.
- $\mathrm{B}^{g/f}_{i}$ are due to the poles appearing in the propagator of auxiliary eikonal quarks. This means $p_i \cdot \hat{K}(z) = 0$ for $z = -\frac{2\,p_i \cdot K}{2\,p_i \cdot e}$. $\hat{K}$ is the momentum flowing through the eikonal propagator.
- $\mathrm{C}^{g/f}$ and $\mathrm{D}^{g/f}$ show up us the first/last shifted particle is off-shell and their external propagator develops a pole.
  The external propagator for off-shell particles is necessary to ensure

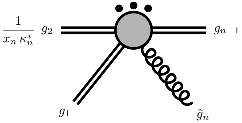
$$\lim_{z \to \infty} \mathcal{A}(z) = 0$$

Classification of poles in the gluon case

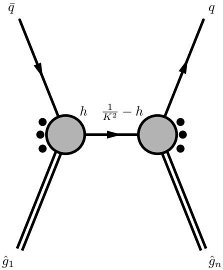


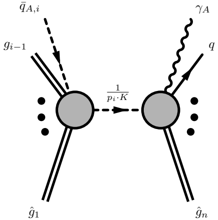


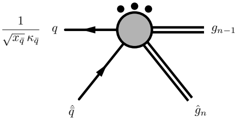


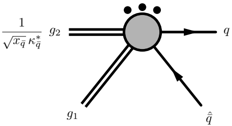


Classification of poles in the fermion case









Some simple amplitudes

Transverse momentum parameterization:
$$\begin{cases} k^{\mu}_{T\,i} = -\frac{\kappa_i}{2}\frac{\langle p_i|\gamma^\mu|q]}{[p_i q]} - \frac{\kappa_i^*}{2}\frac{\langle q|\gamma^\mu|p_i]}{\langle q p_i\rangle} \\ \kappa_i \equiv \frac{\langle q|\not{k}_i|p_i]}{\langle q p_i\rangle} \qquad \kappa_i^* \equiv \frac{\langle p_i|\not{k}_i|q]}{[p_i q]} \\ q^2 = 0 \quad \text{auxiliary momentum} \end{cases}$$

Subleading contribution: this is zero in the on-shell case !

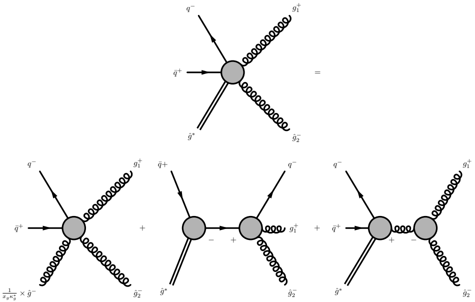
$$\mathcal{A}(g_1^+, g_2^+, \ldots, g_{n-1}^+, \bar{q}, q, g_n^+) = \frac{\langle \bar{q} q\rangle^3}{\langle 12\rangle\langle 23\rangle \ldots \langle \bar{q} q\rangle\langle q n\rangle\langle n1\rangle}$$

Structure of MHV amplitudes

$$\mathcal{A}(g_1^+, g_2^+, \ldots, g_{n-1}^+, \bar{q}^*, q^+, g_n^-) = \frac{1}{\kappa_{\bar{q}}^*}\frac{\langle \bar{q} n\rangle^3\langle q n\rangle}{\langle 12\rangle\langle 23\rangle \ldots \langle \bar{q} q\rangle\langle q n\rangle\langle n1\rangle}$$

$$\mathcal{A}(g^*, \bar{q}^+, q^-, g_1^+, g_2^+, \ldots, g_n^+) = \frac{1}{\kappa_g^*}\frac{\langle g q\rangle^3\langle g \bar{q}\rangle}{\langle g \bar{q}\rangle\langle \bar{q} q\rangle \ldots \langle n-1|n\rangle\langle n g\rangle}$$

But not everything is so smooth...



$$\mathcal{A}(g^*, \bar{q}^+, q^-, g_1^+, g_2^-) = \frac{1}{\kappa_g^*} \frac{[\bar{q}1]^3 \langle 2g\rangle^4}{[\bar{q}q]\langle g|\not{p}_2 + \not{k}_g|1]\langle 2|\not{k}_g\ (\not{k}_g + \not{p}_2)\,|g]\langle 2|\not{k}_g|\bar{q}]}$$

$$+ \frac{1}{\kappa_g} \frac{1}{(k_g + p_{\bar{q}})^2} \frac{[g\bar{q}]^2 \langle 2q\rangle^3 \langle 2|\not{k}_g + \not{p}_{\bar{q}}|g]}{\langle 1q\rangle\langle 12\rangle \left\{(k_g + p_{\bar{q}})^2[\bar{q}g]\langle 2q\rangle - \langle 2|\not{k}_g + \not{p}_{\bar{q}}|g]\langle q|\not{k}_g|\bar{q}]\right\}}$$

$$+ \frac{\langle gq\rangle^3 [g1]^4}{\langle \bar{q}q\rangle [12][g2]\langle q|\not{p}_1 + \not{p}_2|g]\langle g|\not{p}_1 + \not{p}_2|g]\langle g|\not{k}_g + \not{p}_2|1]}$$

General outline of the results

- It is necessary to understand which shifts are legitimate in the off-shell case, i.e. for which choices $\lim_{z\to\infty} \mathcal{A}(z) = 0$. **Full classification of the possible shifts**.
- It turns out that amplitudes which are MHV in the on-shell case (2 of the partons have different helicity sign w.r.t. all the others) preserve a similar structure in the off-shell case.
- So 4-point amplitudes are always MHV , juts as in the on-shell case.
- **First calculation of 5-point amplitudes, which are not always MHV**
- **Sub-leading amplitudes absent in the on-shell case do not vanish here**
- **Numerical cross-checks** are always successful. They were performed cross checked with a program implementing **Berends-Giele** recursion relation, *A. van Hameren, M. Bury, arXiv:1503.08612*

Thorough discussion is in

A. van Hameren, M.S. JHEP 1507 (2015) 010 .

Conclusions and perspectives

- High-energy factorisation requires gauge invariant scattering amplitudes with off-shell partons.
- "Embedding tricks" to provide such amplitudes have been devised they work very well numerically. We wanted an efficient framework for the analytic recursive generation of these amplitudes,
- BCFW construction was extended to Yang Mills with fermions with off-shell particles. This implies identifying a new set of poles in the auxiliary complex variable. A complete set of 5 point amplitudes with 1 off-shell parton has been obtained...available on a Mathematica notebook with arXiv:1504.00315.
- It is possible, in the same way, to obtain the scattering amplitudes with more off-shel partons...just some more work (in progress !).
- Next natural step in phenomenology: applications of these results to multi-jet production in HEF factorisation....
- Connection between Berends-Giele recursion relation and Wilson line approach via complex shifts of momenta (with P. Kotko and A. Stasto)

The end

THANK YOU FOR YOUR ATTENTION