Deformations of special geometry and the holomorphic anomaly equation

Gabriel Lopes Cardoso

CERN

with Bernard de Wit and Swapna Mahapatra, arXiv:1406.5478

Introduction

In supersymmetric field theories and string theories:

class of terms in effective action that play a special role:

F-terms.

Example: Type II superstring theory on $M_4 \times CY_3$

N=2 supersymmetry in four dimensions.

For every integer $g \geq 0$: $\exists$ F-term at precisely g-loop order in string perturbation theory, schematically

$$\int d^4x\, d^4\theta \sum_{g \geq 0} F^{(g)}(Y^I)\,(W^2)^g$$

Y^I: conical affine special Kähler manifold $\mathcal{M}$, $Y^I \rightarrow \lambda\, Y^I\,,\ \lambda \in \mathbb{C}^*$, $I = 0, \ldots, n = h^{1,1}$ (or $h^{2,1}$)

Special Kähler geometry. de Wit, van Proeyen

$F^{(0)}(Y)$ determines Kähler metric $N_{IJ} = 2\,\mathrm{Im}\left(\partial^2 F^{(0)}/\partial Y^I \partial Y^J\right)$.

Projective special Kähler manifold $\bar{\mathcal{M}} = \mathcal{M}/\mathbb{C}^*$ $\quad (t^I = Y^I/Y^0)$.

Assemble: $g \geq 1$

$$w(Y) = \sum_{g=1}^{\infty} F^{(g)}(Y)$$

Special significance of F-terms:

- determine entropy of half-BPS black holes OSV, 2004
- $F^{(g)}$ topological string amplitudes.

Actually, the $F^{(g)}$ are not quite holomorphic ($g \geq 1$):

holomorphic anomaly equation (recursive relation, $g \geq 2$)

Bershadsky, Cecotti, Ooguri, Vafa, 1993

Deformed special geometry

Q1: what is the precise relation between TST and the LEEA?

Q2: Consistent extension of special geometry for incorporating non-holomorphic corrections encoded in $F^{(g)}, g \geq 1$?

- Q2: deformed special geometry, based on

$$F(Y, \bar{Y}) = F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y}) \;, \quad \Omega \quad \text{real}$$

Symplectic $Sp(2(n+1), \mathbb{R})$-transformations act on

$$(Y^I, F_I) \qquad , \qquad F_I = \frac{\partial F(Y, \bar{Y})}{\partial Y^I} \quad , \quad I = 0, \ldots, n$$

LEEA. When Ω harmonic: Wilsonian limit

$$\Omega(Y, \bar{Y}) = w(Y) + \bar{w}(\bar{Y})$$

Natural deformation leads to perturbative topological string.

Deformed special geometry

- Q1: Generalized Hesse potential $H(\phi, \chi)$ `Freed 1997`

$$\phi^I = Y^I + \bar{Y}^I \quad , \quad \chi_I = F_I + \bar{F}_I$$

Obtained by Legendre transform with respect to $Y^I - \bar{Y}^I$:

$$H(\phi, \chi) = 4\left[\mathrm{Im}F^{(0)}(Y) + \Omega(Y, \bar{Y})\right] + i\,\chi_I\,(Y - \bar{Y})^I$$

Significance:

- H 'Hamiltonian' (Legendre transform of LEEA), symplectic function $\longrightarrow \Omega$ transforms in prescribed, non-trivially way.
- Want to understand the structure of the Hesse potential:

a unique subsector captures perturbative topological string.

Deformation of special geometry

Classical mechanics system: n dof $i = 1, \ldots n$

coordinates ϕ^i, velocities $\dot\phi^i$, Lagrangian $L(\phi, \dot\phi)$

Hamiltonian $H(\phi, \pi) = \dot\phi^i \, \pi_i - L(\phi, \dot\phi)$. Patch of phase space (ϕ^i, π_i).

Complex coordinates $z^k = \frac{1}{2}\left(\phi^k + i\,\dot\phi^k\right)$.

Theorem: $\exists$ function

$$F(z, \bar z) = F^{(0)}(z) + 2i\,\Omega(z, \bar z) \quad , \quad \Omega \;\; \text{real}$$

$$\begin{pmatrix} \phi^i \\ \pi_i \end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix} z^i \\ F_i(z, \bar z) \end{pmatrix} \quad , \quad F_i = \frac{\partial F(z, \bar z)}{\partial z^i}$$

Equivalence relation: $F(z, \bar z) \rightarrow F(z, \bar z) + \bar g(\bar z)$

$$F^{(0)}(z) \rightarrow F^{(0)}(z) + g(z) \quad , \quad \Omega \rightarrow \Omega - \mathrm{Im}\, g(z)$$

(z^i, F_i) complexification of phase space coordinates (ϕ^i, π_i):

canonical transformations = $\mathrm{Sp}(2n, \mathbb{R})$ transformations

$$\begin{pmatrix} z^i \\ F_i(z,\bar z) \end{pmatrix} \rightarrow \begin{pmatrix} \tilde z^i \\ \tilde F_i(\tilde z, \bar{\tilde z}) \end{pmatrix} = \begin{pmatrix} U^i_j & Z^{ij} \\ W_{ij} & V_i{}^j \end{pmatrix} \begin{pmatrix} z^j \\ F_j(z,\bar z) \end{pmatrix}$$

Transformation is integrable: $F(z,\bar z) \rightarrow \tilde F(\tilde z, \bar{\tilde z})$.

$$\begin{aligned} L &= 4\left[\mathrm{Im}\, F - \Omega\right] \\ H &= -i\left(z^i\, \bar F_{\bar\imath} - \bar z^{\bar\imath}\, F_i\right) - 2\left(2\Omega - z^i \Omega_i - \bar z^{\bar\imath}\, \Omega_{\bar\imath}\right) \\ &\quad -4\mathrm{Im}\left[F^{(0)} - \tfrac{1}{2} z^i F^{(0)}_i\right] \end{aligned}$$

H is a symplectic function: $\tilde H(\tilde\phi, \tilde\pi) = H(\phi, \pi)$.

When $\Omega(z,\bar z) = w(z) + \bar w(\bar z)$: $F(z) = F^{(0)}(z) + 2i\, w(z)$

Evaluating the Hesse potential

In $N = 2$ supergravity, $-H$ becomes the Hesse potential $H(\phi, \chi)$, the Legendre transform of $\mathrm{Im}\, F - \Omega$,

$$
\begin{aligned}
F(Y, \bar{Y}) &= F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y}) \\
H(\phi, \chi) &= -i\left(\bar{Y}^I F_I - Y^I \bar{F}_{\bar{I}}\right) + 2\left(2\Omega - Y^I \Omega_I - \bar{Y}^{\bar{I}} \Omega_{\bar{I}}\right)
\end{aligned}
$$

Q1: New variables:

$$
\begin{pmatrix} \phi^I \\ \chi_I \end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix} Y^I \\ F_I(Y, \bar{Y}) \end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix} \mathcal{Y}^I \\ F_I^{(0)}(\mathcal{Y}) \end{pmatrix}
$$

Relation between $\mathcal{Y}^I$ and (ϕ, χ) only involves $F^{(0)}$.

Y^I: sugra variables $\qquad$ $\mathcal{Y}^I$: new variables

$$
\begin{aligned}
\mathcal{Y}^I &= Y^I + \Delta Y^I(\Omega) \quad , \quad \Omega \neq 0 \\
\mathcal{Y}^I &= Y^I \quad , \quad \Omega = 0\,.
\end{aligned}
$$

Evaluating the Hesse potential

Evaluate H in terms of $\mathcal{Y}^I \Rightarrow$ expansion of ΔY^I in powers of $\Omega(\mathcal{Y}, \bar{\mathcal{Y}})$.

The Hesse potential transforms as a function under symplectic transformations: $\tilde{H}(\tilde{\phi}, \tilde{\pi}) = H(\phi, \pi)$.

- H as series of symplectic functions, $H = \sum_{k=0}^{\infty} H^{(k)}(\mathcal{Y}, \bar{\mathcal{Y}})$
- $H^{(0)} = -i\left[\bar{\mathcal{Y}}^I F_I^{(0)}(\mathcal{Y}) - \text{c.c.}\right]$
- $H^{(1)}$ is the only one that contains $\Omega(\mathcal{Y}, \bar{\mathcal{Y}})$ (the other $H^{(k)}$ contain derivatives thereof)

$$H^{(1)} = 4\Omega - 4N^{IJ}\left(\Omega_I \Omega_J + \Omega_{\bar{I}} \Omega_{\bar{J}}\right) + \mathcal{O}(\Omega^3)$$

$$N_{IJ} = -i\left(F_{IJ}^{(0)} - \bar{F}_{\bar{I}\bar{J}}^{(0)}\right)$$

- $N^{IJ} \rightarrow \ldots (N - \mathrm{i}\mathcal{Z})^{IJ}$

$$\tilde{\Omega}(\tilde{\mathcal{Y}}, \bar{\tilde{\mathcal{Y}}}) = \Omega - \mathrm{i}\left(\mathcal{Z}^{IJ}\,\Omega_I \Omega_J - \bar{\mathcal{Z}}^{\bar{I}\bar{J}}\,\Omega_{\bar{I}} \Omega_{\bar{J}}\right) + \mathcal{O}(\Omega^3)$$

The holomorphic anomaly equation

Wilsonian set-up:

$$\Omega(\mathcal{Y},\bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}})$$

Preserved by symplectic transformations:

$$\tilde{\Omega}(\tilde{\mathcal{Y}},\bar{\tilde{\mathcal{Y}}}) = \tilde{w}(\tilde{\mathcal{Y}}) + \bar{\tilde{w}}(\bar{\tilde{\mathcal{Y}}})$$

Now add non-holomorphic term whose variation is harmonic:

$$\Omega(\mathcal{Y},\bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}}) + \alpha \ln\det N_{IJ} + \mathcal{O}(\alpha^2) \quad , \quad \alpha \in \mathbb{R}$$

$$\ln\det\tilde{N} = \ln\det N - \ln\det\mathcal{S}(\mathcal{Y}) - \ln\det\bar{\mathcal{S}}(\bar{\mathcal{Y}})$$

Transformation law of Ω requires further modifications:

The holomorphic anomaly equation

$$\Omega(\mathcal{Y},\bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}}) + \alpha \ln\det N_{IJ} + \Big[2\alpha\, N^{IJ}\, w_{IJ} - \alpha^2 \left(iF^{(0)}_{IJKL} - \tfrac{2}{3} F^{(0)}_{IKM} F^{(0)}_{JLN} N^{MN}\right) N^{IJ} N^{KL} + \text{h.c.}\Big] + \dots$$

Double expansion: $w(\mathcal{Y}) = \sum_{n=1}^{\infty} \beta^n\, w^{(n)}(\mathcal{Y})$, $\beta = 1$

Expanding $H^{(1)}$ in powers of (w,α):

$$H^{(1)} = 4\Big[F^{(1)}(\mathcal{Y},\bar{\mathcal{Y}}) + \left(F^{(2)}(\mathcal{Y},\bar{\mathcal{Y}}) + \text{h.c.}\right) + \left(\left(F^{(3)}(\mathcal{Y},\bar{\mathcal{Y}}) + \text{h.c.}\right) + 4\,\alpha\, D_I F^{(1)}_J\, N^{IK}\, N^{JL}\, \bar{D}_{\bar{K}} \bar{F}^{(1)}_{\bar{L}}\right) + \mathcal{O}(\alpha^4)\Big]$$

Expansion in terms of symplectic functions $F^{(g)}$.

The holomorphic anomaly equation

$$
\begin{aligned}
F^{(1)}(\mathcal{Y},\bar{\mathcal{Y}}) &= w^{(1)}(\mathcal{Y}) + \bar{w}^{(1)}(\bar{\mathcal{Y}}) + \alpha \ln\det N_{IJ} \quad , \quad \text{real} \\
F^{(2)}(\mathcal{Y},\bar{\mathcal{Y}}) &= w^{(2)}(\mathcal{Y}) + 2\alpha\, N^{IJ}\, w^{(1)}_{IJ} \\
&\quad -i\alpha^2\, N^{IJ} N^{KL}\, F^{(0)}_{IJKL} + \tfrac{2}{3}\alpha^2 N^{IJ} N^{KP} N^{LQ} F^{(0)}_{IKL} F^{(0)}_{JPQ} \\
&\quad -N^{IJ} \left(w^{(1)}_I - i\alpha N^{KL} F^{(0)}_{IKL} \right) \left(w^{(1)}_J - i\alpha N^{PQ} F^{(0)}_{JPQ} \right) \\
F^{(3)}(\mathcal{Y},\bar{\mathcal{Y}}) &= 41 \quad \text{terms}
\end{aligned}
$$

$F^{(g)}(\mathcal{Y},\bar{\mathcal{Y}}),\ g \geq 2$: polynomials in N^{IJ}, degree $3g-3$.

Non-holomorphicity entirely through $N_{IJ} = -i\left(F^{(0)}_{IJ} - \bar{F}^{(0)}_{\bar{I}\bar{J}}\right)$.

$F^{(2)}$: `Grimm+Klemm+Mariño+Weiss, hep-th/0702187`

$F^{(3)}$: new.

The holomorphic anomaly equation

By explicit calculation, verify that the $F^{(g)}(\mathcal{Y},\bar{\mathcal{Y}})$ $(g \geq 2)$ satisfy the holomorphic anomaly equation of topological string theory in big moduli space:

$$\partial_{\bar{I}} F^{(g)} = i\, \bar{F}^{(0)}_{\bar{I}\bar{P}\bar{Q}}\, N^{PJ} N^{QK} \left(2\alpha\, D_J \partial_K F^{(g-1)} + \sum_{r=1}^{g-1} \partial_J F^{(r)} \partial_K F^{(g-r)} \right)$$

where $\partial_{\bar{I}} F^{(g)} = \frac{\partial F^{(g)}(\mathcal{Y},\bar{\mathcal{Y}})}{\partial \bar{\mathcal{Y}}^I}$, etc.

`Bershadsky, Cecotti, Ooguri, Vafa, hep-th/9309140`

Consequence of symplectic covariance.

Conclusions

- Described a consistent deformation of special geometry:
  - framework for incorporating non-holomorphic corrections into LEEA
  - $F(Y, \bar{Y}) = F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y})$
- Discussed structure of the associated Hesse potential H, by expanding it in new variables $\mathcal{Y}^I$.
- Identified a unique subsector of H that captures the topological string free energies $F^{(g)}(\mathcal{Y}, \bar{\mathcal{Y}})$: $\quad H^{(1)}$
- Taking as non-holomorphic deformation

$$\Omega(\mathcal{Y}, \bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}}) + \alpha \ln\det N_{IJ} + \dots$$

  and expanding $H^{(1)}$ in powers of (w, α), obtained expansion functions $F^{(g)}(\mathcal{Y}, \bar{\mathcal{Y}})$ that satisfy the holomorphic anomaly equation of TST. Diagrammatic expansion.

Thanks!