

# Two-Loop Off-Shell QCD Amplitudes in FDR

Ben Page

Universidad de Granada/CERN

June 1, 2015 - LHCtheory

in collaboration with R. Pittau

# Problem Sketch

- Want to develop UV transition rules between  $DR/\overline{MS}$  and FDR for QCD @ 2-loops.

# Problem Sketch

- Want to develop UV transition rules between  $DR/\overline{MS}$  and FDR for QCD @ 2-loops.
- Different regularizations  $\Rightarrow$  different renormalization schemes.

- Want to develop UV transition rules between DR/ $\overline{MS}$  and FDR for QCD @ 2-loops.
- Different regularizations  $\Rightarrow$  different renormalization schemes.
- Different renormalization schemes  $\Rightarrow$  different renormalization constants.

$$Z_A \neq Z_B$$

- Want to develop UV transition rules between  $\overline{\text{DR}}/\overline{\text{MS}}$  and FDR for QCD @ 2-loops.
- Different regularizations  $\Rightarrow$  different renormalization schemes.
- Different renormalization schemes  $\Rightarrow$  different renormalization constants.
- Related by coupling shift.

$$Z_A \neq Z_B \Rightarrow$$

$$\alpha_B = \alpha_A [1 + c_1 \alpha_A + c_2 \alpha_A^2]$$

- Renormalization performed at the level of definition of UV-finite integral.

$$\int [d^4 q_1][d^4 q_2] J$$


- Renormalization performed at the level of definition of UV-finite integral.
- Achieved by subtraction of FDR Vacuum.

$$\int [d^4 q_1][d^4 q_2] J = \int_{\epsilon} J - \int_{\epsilon} J_V$$

# Renormalization in FDR

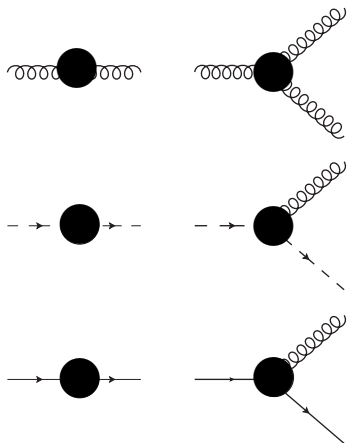
- Renormalization performed at the level of definition of UV-finite integral.
- Achieved by subtraction of FDR Vacuum.
- Two natural categories, global- and sub-vacuum.

$$\int [d^4 q_1][d^4 q_2] J = \int_{\epsilon} J - \int_{\epsilon} J_V$$

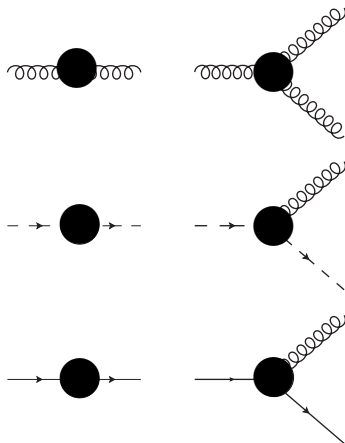
$$\int_{\epsilon} J_V \sim \text{[Diagram 1]} + \text{[Diagram 2]}$$




# What to Renormalize?

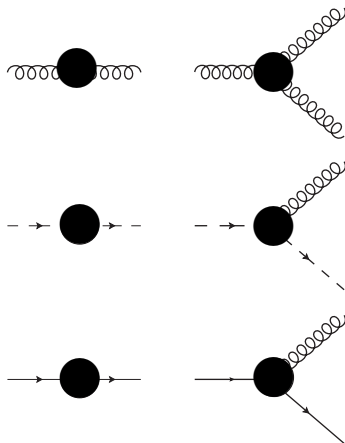


# What to Renormalize?



- $Z_G$
- $Z_c$
- $Z_\psi$
- $Z_\alpha \times 3$

# What to Renormalize?



- $Z_G$
- $Z_c$
- $Z_\psi$
- $Z_\alpha \times 3$

Universality check in  
fermionic and bosonic sectors.

# How to Renormalize? - *à la* FDR

Describe FDR in the same framework as  $\overline{MS}$ .

# How to Renormalize? - *à la* FDR

Describe FDR in the same framework as  $\overline{MS}$ .

- Take off-shell renormalized DR amplitude,

$$A_{\text{Ren}} = \left( \int d^n q_1 d^n q_2 J \right) + \text{CTs.}$$

# How to Renormalize? - *à la* FDR

Describe FDR in the same framework as  $\overline{MS}$ .

- Take off-shell renormalized DR amplitude,

$$A_{\text{Ren}} = \left( \int d^n q_1 d^n q_2 J \right) + \text{CTs.}$$

- Let FDR integral define the renormalization scheme,

$$A_{\text{Ren}} = \int [d^4 q_1][d^4 q_2] J$$

# How to Renormalize? - *à la* FDR

Describe FDR in the same framework as  $\overline{MS}$ .

- Take off-shell renormalized DR amplitude,

$$A_{\text{Ren}} = \left( \int d^n q_1 d^n q_2 J \right) + \text{CTs.}$$

- Let FDR integral define the renormalization scheme,

$$A_{\text{Ren}} = \int [d^4 q_1][d^4 q_2] J$$

- Corresponds to a choice of renormalization constants/CTs:

$$\Rightarrow \text{CTs} = ?$$

- Use the defining expansion to relate DR integral to FDR integral:

$$\int d^n q_1 d^n q_2 J = \left( \int [d^4 q_1][d^4 q_2] J \right) + \left( \int d^n q_1 d^n q_2 V[J] \right)$$



- Use the defining expansion to relate DR integral to FDR integral:

$$\int d^n q_1 d^n q_2 J = \left( \int [d^4 q_1][d^4 q_2] J \right) + \left( \int d^n q_1 d^n q_2 V[J] \right)$$

FDR Integral      +      FDR Vacuum  
(in DR)

- Use the defining expansion to relate DR integral to FDR integral:

$$\int d^n q_1 d^n q_2 J = \underbrace{\left( \int [d^4 q_1][d^4 q_2] J \right)}_{\text{FDR Integral}} + \underbrace{\left( \int d^n q_1 d^n q_2 V[J] \right)}_{\substack{\text{FDR Vacuum} \\ \text{(in DR)}}$$

$$\Rightarrow \text{CTs} = - \left( \int d^n q_1 d^n q_2 V[J] \right)$$

- Use the defining expansion to relate DR integral to FDR integral:

$$\int d^n q_1 d^n q_2 J = \underbrace{\left( \int [d^4 q_1][d^4 q_2] J \right)}_{\text{FDR Integral}} + \underbrace{\left( \int d^n q_1 d^n q_2 V[J] \right)}_{\substack{\text{FDR Vacuum} \\ \text{(in DR)}}$$

$$\Rightarrow \text{CTs} = - \left( \int d^n q_1 d^n q_2 V[J] \right)$$

- We use this to calculate the CTs, and then extract the renormalization constants.

# One Loop Warmup

Start with one loop as needed for order by order renormalization.

# One Loop Warmup

Start with one loop as needed for order by order renormalization.  
Consider:

The diagram shows a Feynman diagram on the left consisting of two wavy lines connected by a central circle with an 'X' inside. This is equated to a large square bracket containing a minus sign followed by a Feynman diagram of a wavy line connected to a loop of wavy lines, which is then connected to another wavy line. This is followed by a plus sign and an ellipsis, all within the square bracket.

$$\text{tadpole with } \times = -V \left[ \text{loop diagram} + \dots \right]$$

# One Loop Warmup

Start with one loop as needed for order by order renormalization.  
Consider:

The diagram shows a tree-level vertex correction on the left, represented by two wavy lines meeting at a central vertex marked with an 'X'. This is equated to a one-loop diagram on the right, enclosed in large square brackets. The one-loop diagram consists of two wavy lines meeting at a vertex, with a circular loop of wavy lines attached to that vertex. To the right of the bracketed diagram is a plus sign followed by an ellipsis, indicating higher-order terms.

$$\text{Tree-level vertex correction} = -V \left[ \text{One-loop diagram} + \dots \right]$$

## One-loop Renormalization constants

$$Z_i = \frac{A}{\epsilon} + \text{universal constant}$$

# Structure of 2-Loop FDR Renormalization

$$0 = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4}$$

The equation shows four Feynman diagrams representing 2-loop corrections. The first diagram is a tree-level exchange with a counterterm. The second is a self-energy correction. The third is a vertex correction. The fourth is a more complex 2-loop diagram.

# Structure of 2-Loop FDR Renormalization

$$0 = \text{[Diagram 1]} + \text{[Diagram 2]} +$$

Diagram 1: A horizontal chain of two gluon lines (represented by wavy lines) connected by a circular loop. The loop contains two 'X' marks, representing a ghost loop. Diagram 2: A circle divided into three sectors by three lines meeting at the center, representing a ghost vacuum diagram.

$$\text{[Diagram 3]} + \text{[Diagram 4]} = 0$$

Diagram 3: A circular loop of gluons with a ghost loop (a circle with an 'X') attached to it. Diagram 4: A ghost vacuum diagram (a circle divided into three sectors) with two gluon lines and one ghost line (a straight line) attached to it.

- Sub vacuum and loop counterterms *exactly* cancel.



# Structure of 2-Loop FDR Renormalization

$$0 = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]}$$
  
$$\text{[Diagram 4]} + \text{[Diagram 5]} = 0$$

The diagrams are:

- Diagram 1: A tree-level diagram with two external wavy lines and a central circle containing two 'x' marks.
- Diagram 2: A tree-level diagram with three external wavy lines meeting at a central vertex.
- Diagram 3: A tree-level diagram with two external wavy lines and a central circle containing one 'x' mark.
- Diagram 4: A one-loop diagram with two external wavy lines and a central circle containing one 'x' mark.
- Diagram 5: A one-loop diagram with three external wavy lines meeting at a central vertex, with a shaded sector removed.

- Sub vacuum and loop counterterms *exactly* cancel.
- Suggests way to calculate renormalization constants directly.

# Structure of 2-Loop FDR Renormalization

$$0 = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]}$$
  
$$\text{[Diagram 4]} + \text{[Diagram 5]} = 0$$

The diagrams are:  
1. A tree-level diagram with two external wavy lines and a central circle containing two 'x' marks.  
2. A tree-level diagram with three external wavy lines meeting at a central vertex.  
3. A tree-level diagram with two external wavy lines and a central circle containing one 'x' mark.  
4. A loop diagram with two external wavy lines and a central circle containing one 'x' mark.  
5. A loop diagram with three external wavy lines meeting at a central vertex, with a shaded sector removed.

- Sub vacuum and loop counterterms *exactly* cancel.
- Suggests way to calculate renormalization constants directly.
- Some extra terms needed in fermionic case.


## 2 Loop FDR Renormalization Constants



## 2 Loop FDR Renormalization Constants

$$\text{Diagram} \sim \int d^n q_1 d^n q_2 \left( a \frac{1}{\overline{q_1^4} \overline{q_2^2} \overline{q_{12}^2}} + b \frac{1}{\overline{q_1^4} \overline{q_2^4}} \right)$$

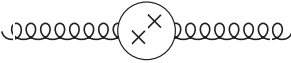
## 2 Loop FDR Renormalization Constants


$$\sim \int d^n q_1 d^n q_2 \left( a \frac{1}{\overline{q_1^4} \overline{q_2^2} \overline{q_{12}^2}} + b \frac{1}{\overline{q_1^4} \overline{q_2^4}} \right)$$

Two-loop renormalization constants

$$Z_i = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + c + d f_{11}$$

## 2 Loop FDR Renormalization Constants



$$\sim \int d^n q_1 d^n q_2 \left( a \frac{1}{\overline{q_1^4} \overline{q_2^2} \overline{q_{12}^2}} + b \frac{1}{\overline{q_1^4} \overline{q_2^4}} \right)$$

### Two-loop renormalization constants

$$Z_i = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + c + d f_{11}$$

- Single pole differs from  $\overline{MS}$ .

## 2 Loop FDR Renormalization Constants



$$\sim \int d^n q_1 d^n q_2 \left( a \frac{1}{\overline{q_1^4} \overline{q_2^2} \overline{q_{12}^2}} + b \frac{1}{\overline{q_1^4} \overline{q_2^4}} \right)$$

### Two-loop renormalization constants

$$Z_i = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + c + d f_{11}$$

- Single pole differs from  $\overline{MS}$ .
- $Z_i$  are universal.

## 2 Loop FDR Renormalization Constants


$$\sim \int d^n q_1 d^n q_2 \left( a \frac{1}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2} + b \frac{1}{\bar{q}_1^4 \bar{q}_2^4} \right)$$

### Two-loop renormalization constants

$$Z_i = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + c + d f_{11}$$

- Single pole differs from  $\overline{MS}$ .
- $Z_i$  are universal.
- Finite part in terms of Clausen function:

$$f_{11} = \frac{i}{\sqrt{3}} \left( \text{Li}_2(e^{i\frac{\pi}{3}}) - \text{Li}_2(e^{-i\frac{\pi}{3}}) \right) = -1.17195361 \dots$$



# Two-loop Coupling Constant Shift

Similar procedure to extract the  $\overline{MS}$  renormalization constants, allowing us to find the coupling shift:

$$\alpha_{FDR} = \alpha_{\overline{MS}} \left[ 1 + c_1 \alpha_{\overline{MS}} + (c_2 + c_3 f_{11}) \alpha_{\overline{MS}}^2 \right].$$

# Two-loop Coupling Constant Shift

Similar procedure to extract the  $\overline{MS}$  renormalization constants, allowing us to find the coupling shift:

$$\alpha_{FDR} = \alpha_{\overline{MS}} \left[ 1 + c_1 \alpha_{\overline{MS}} + (c_2 + c_3 f_{11}) \alpha_{\overline{MS}}^2 \right].$$

- Constant  $c_1$  is same as FDH/DRed  $\Rightarrow$  UV-equivalent @ one-loop.

# Two-loop Coupling Constant Shift

Similar procedure to extract the  $\overline{MS}$  renormalization constants, allowing us to find the coupling shift:

$$\alpha_{FDR} = \alpha_{\overline{MS}} \left[ 1 + c_1 \alpha_{\overline{MS}} + (c_2 + c_3 f_{11}) \alpha_{\overline{MS}}^2 \right].$$

- Constant  $c_1$  is same as FDH/DRed  $\Rightarrow$  UV-equivalent @ one-loop.
- Clausen function enters here @ two-loop.

# Two-loop Coupling Constant Shift

Similar procedure to extract the  $\overline{MS}$  renormalization constants, allowing us to find the coupling shift:

$$\alpha_{FDR} = \alpha_{\overline{MS}} \left[ 1 + c_1 \alpha_{\overline{MS}} + (c_2 + c_3 f_{11}) \alpha_{\overline{MS}}^2 \right].$$

- Constant  $c_1$  is same as FDH/DRed  $\Rightarrow$  UV-equivalent @ one-loop.
- Clausen function enters here @ two-loop.
- We have used this to calculate three-loop beta function in FDR.

# The Extra Terms and FDH

- Without extra terms, the calculated fermionic  $Z$  is *non-local*.

# The Extra Terms and FDH

- Without extra terms, the calculated fermionic  $Z$  is *non-local*.
- Interestingly, if we repeat the calculation in FDH, then the extra terms are not required.

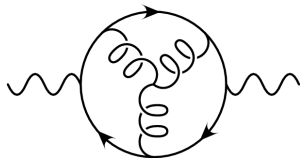
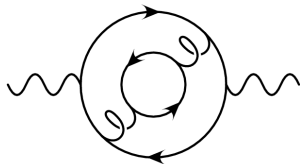
# The Extra Terms and FDH

- Without extra terms, the calculated fermionic  $Z$  is *non-local*.
- Interestingly, if we repeat the calculation in FDH, then the extra terms are not required.
- However, Kilgore\* showed that with external fermions FDH is not unitary.

\* [arXiv:1102.5353]

# The Extra Terms and FDH

- Without extra terms, the calculated fermionic  $Z$  is *non-local*.
- Interestingly, if we repeat the calculation in FDH, then the extra terms are not required.
- However, Kilgore\* showed that with external fermions FDH is not unitary.

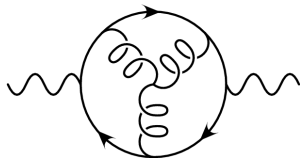
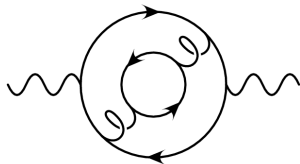


\* [arXiv:1102.5353]



# The Extra Terms and FDH

- Without extra terms, the calculated fermionic  $Z$  is *non-local*.
- Interestingly, if we repeat the calculation in FDH, then the extra terms are not required.
- However, Kilgore\* showed that with external fermions FDH is not unitary.
- Suggestive that analogous extra terms could make FDH consistent.



\*[arXiv:1102.5353]

- FDR at two loops is a different scheme to  $\overline{DR}/\overline{MS}$ .

# Conclusions

- FDR at two loops is a different scheme to  $\overline{DR}/\overline{MS}$ .
- We calculate the coupling constant shift between FDR and  $\overline{MS}$ .

- FDR at two loops is a different scheme to  $\overline{DR}/\overline{MS}$ .
- We calculate the coupling constant shift between FDR and  $\overline{MS}$ .
- We renormalize all two and three point vertices as a consistency check of universality.

- FDR at two loops is a different scheme to  $DR/\overline{MS}$ .
- We calculate the coupling constant shift between FDR and  $\overline{MS}$ .
- We renormalize all two and three point vertices as a consistency check of universality.
- We introduce some extra integrals to deal with the quark sector, which may have interesting consequences for FDH.

- FDR at two loops is a different scheme to  $DR/\overline{MS}$ .
- We calculate the coupling constant shift between FDR and  $\overline{MS}$ .
- We renormalize all two and three point vertices as a consistency check of universality.
- We introduce some extra integrals to deal with the quark sector, which may have interesting consequences for FDH.
- Next step is massive fermions.