Soft Theorems and Their Implementation

Hui Luo

University and INFN, Padua
ERC Advanced Grant DaMeSyFla

H.L., P. Mastrolia and W.J. Torres Bobadilla, Phys.Rev. D91(2015) 065018
H.L. & Y. Du, JHEP 1301(2013)129
Y. Huang., H. L., C. Wen, working in progress

@PADUA, Sep 3rd 2015

Overview:

- New methods and techniques of scattering amplitudes gained huge progress in past 20 years, i.e. classical-level has been well-studied and understood, more loops, more difficult;
- Interest in universal properties of low energy particle emissions was renewed;. Novel factorization results have been discovered down to the sub-(sub)-leading order in a soft momentum expansion
- Single/double/multiple soft structures are studied which might connect to some hidden symmetries, e.g. the hidden infinite dimensional bms4 symmetry of quantum gravity S-matrix, underlying patterns of symm. breaking

Outlines:

- On-shell method and spinor notation
- Single soft theorem in QCD @ tree-level
- Double soft Goldstone theorems @ tree-level
- Conclusion and Outlook

On-Shell Method and Boundary contribution

- *Scattering amplitudes are determined by their poles through complex deformation of external momenta*

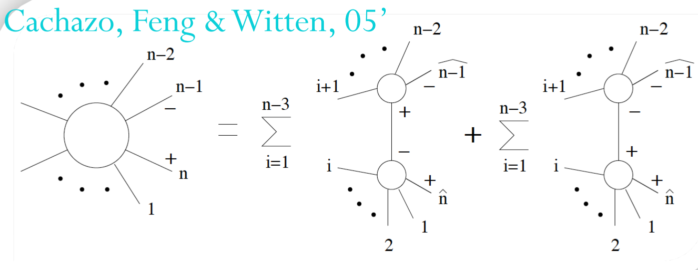
$$I=\oint\frac{dz}{z}A(z)=A(z=0)+\sum_{z_\alpha}\text{Res}\left(\frac{A(z)}{z}\right)_{z_\alpha}\qquad \frac{1}{(p+p_i(z))^2}=\frac{1}{(p+p_i)^2+z(2q\cdot(p+p_i))}$$

- *If no boundary contribution in the contour integration (BCFW) :*

$z\to\infty \quad A(z)\to 0$

Britto, Cachazo & Feng, 05';Britto, Cachazo, Feng & Witten, 05'

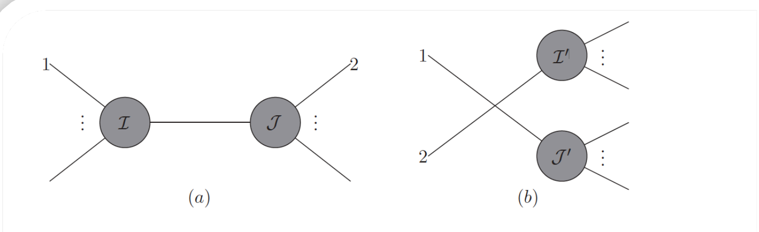
$$\left(\frac{A(z)}{z}\right)_{z_\alpha}=-\sum_{h=\pm}A_L(p_i(z_\alpha),p^h(z_\alpha))\frac{1}{P_\alpha^2}A_R(-p^{-h}(z_\alpha),p_j(z_\alpha))$$



Higher-point amplitude constructed from sub-amplitudes

- *Boundary contribution of on-shell recurrence relation*

B.Feng,Y. Jia, H.L.& M. Luo, 11'
R. H. Boels, 10'; Benincasa & Conde, 12'; Feng & Jin, 14'



(a) Diagrams not giving boundary contributions (b) Diagrams giving boundary contributions

- Introduce auxiliary field to enlarge the theory;
- Analyze Feynman diagrams and isolate boundary contribution, which can be evaluated;
- Express boundary in terms of roots of Amp's;
- Analyze pole structures of boundary

$$A_n=\sum_{z_\alpha,h=\pm}A_L(p_i(z_\alpha),p^h(z_\alpha))\frac{1}{p_\alpha^2}A_R(-p^{-h}(z_\alpha),p_j(z_\alpha))+B$$

Open Question!
Choose a good momentum-deformation!

A Quick Review of the Spinoral Notation

- *Given a null momentum in 4dim space-time, define a 2-dim Weyl spinor* λ *and an anti-spinor* $\widetilde{\lambda}$ *by Dirac equations*

$$k_{\dot{a}a}\lambda^a(k) = 0, \quad \widetilde{\lambda}^{\dot{a}}(k)k_{\dot{a}a} = 0$$

See review: B.Feng, & M. Luo, 11'

the null momentum can be decomposed as $k_{\dot{a}a} = \widetilde{\lambda}_{\dot{a}}\lambda_a$

- *Lorentz invariant inner products of 2 spinors or anti-spinors*

$$\langle i|j\rangle \equiv \lambda_i^a\lambda_{ja}, \qquad [i|j] \equiv \widetilde{\lambda}_{i\dot{a}}\widetilde{\lambda}_j^{\dot{a}}$$

- *For massless fermions, definite helicity can be identified as*

$$u_\pm(k) = \frac{1\pm\gamma_5}{2}u(k), \quad v_\mp(k) = \frac{1\pm\gamma_5}{2}u(k),$$

$$\overline{u_\pm(k)} = \overline{u(k)}\frac{1\mp\gamma_5}{2}, \quad \overline{v_\mp(k)} = \overline{v(k)}\frac{1\mp\gamma_5}{2}$$

or write in terms of spinorial notations (angel/square brackets)

$$|i\rangle \equiv |k_i^+\rangle = u_+(k_i) = v_-(k_i), \qquad |i] \equiv |k_i^-\rangle = u_-(k_i) = v_+(k_i)$$

$$\langle i| \equiv \langle k_i^-| = \overline{u_-}(k_i) = \overline{v_+}(k_i), \qquad [i| \equiv \langle k_i^+| = \overline{u_+}(k_i) = \overline{v_-}(k_i)$$

- The polarization vector

$$\epsilon_\nu^+(k|\mu) = \frac{\langle\mu|\gamma_\nu|k]}{\sqrt{2}\,\langle\mu|k\rangle} \qquad \epsilon_\nu^-(k|\mu) = \frac{[\mu|\gamma_\nu|k\rangle}{\sqrt{2}\,[\mu|k]}$$

Single Soft：
Subleading Soft Behavior of QCD Amplitudes

H.L., P. Mastrolia and W.J. Torres Bobadilla, Phys.Rev. D91(2015) 065018

Soft-Limit Behaviors at Tree-Level

- ***Graviton Amplitudes obeying a soft identity*** *Cachazo & Strominger, 14'*

$$\mathcal{M}_{n+1}(k_1, k_2, ...k_n, q) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right) \mathcal{M}_n(k_1, k_2, ...k_n) + \mathcal{O}(q^2)$$

- Soft Operators derived from BCFW
- Gauge invariance property requires the vanishing of these pole terms under the gauge transformation
- Invariant under gauge transformation according to momentum conservation, angular momentum conservation and anti-symmetry of Lorentz generator

$$S^{(0)} \equiv \sum_{a=1}^{n} \frac{E_{\mu\nu} k_a^\mu k_a^\nu}{q \cdot k_a} \qquad S^{(1)} \equiv -i \sum_{a=1}^{n} \frac{E_{\mu\nu} k_a^\mu (q_\rho J_a^{\rho\nu})}{q \cdot k_a}$$

$$S^{(2)} \equiv -\frac{1}{2} \sum_{a=1}^{n} \frac{E_{\mu\nu} (q_\rho J_a^{\rho\mu})(q_\sigma J_a^{\sigma\nu})}{q \cdot k_a}$$

- ***Extensions of the soft-limit topic***

Bern, Davies, Vecchia & Nohle；
Geyer, Lipstein, Mason;
Schwab & Volovich；
Larkoski； E. Casali；
Broedel, Leeuw, Plafka & Rosso；
He, Huang & Wen;
Bonocore, Laenen, Magnean, Vernazza &White；
Afkhami-Jeddi
…

➢ In Different Theoretical Frames:

Pure-YM, String theory, with/without SUSY…

➢ With Different Methods:

BCFW, Scattering Equation; Conformal Invariance; Gauge Invariance…

➢ In Different Dimensions:

4−dim; D-dim

➢ Involve Quantum Contributions：

Loop-correction; Soft-Collinear ET;…

A Complementary Missing Piece:
Soft-limit property in quark-gluon amplitudes with a soft gauge boson emitted from fermions

Soft−Photon Limit in QCD Amplitude:

- The soft gauge boson emitted from a bosonic leg has been studied, one can consider the radiation from a fermion line
- To isolate the fermionic emitter behavior, we would first study a soft photon case from BCFW and gauge invariance approaches

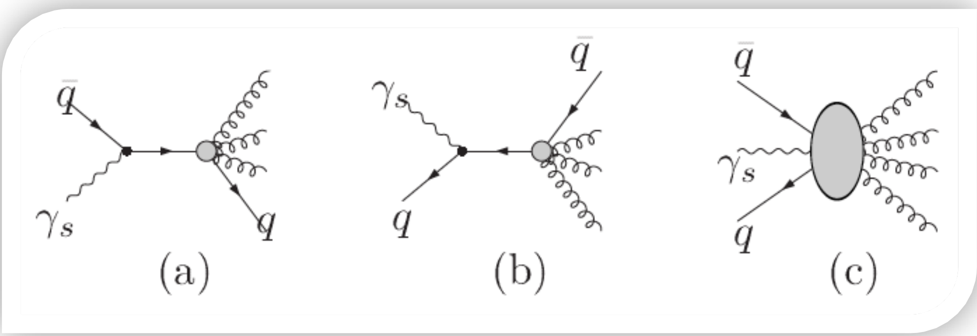
From On-Shell Recursion Relation Derivation

$$A_{n+3}(\Lambda_{\bar{q}}, \gamma_s^+, \Lambda_q, g_1, \cdots, g_n) = \left(\frac{1}{\epsilon^2} S^{(0)\lambda} + \frac{1}{\epsilon} S^{(1)\lambda}\right) A_{n+2}(\Lambda_{\bar{q}}, \Lambda_q, g_1, \cdots, g_n) + \mathcal{O}(1)$$

- Holomorphic spinor to the soft limit $|s\rangle \to \epsilon|s\rangle$
- Leading soft singularity is the well-known universal factor;
- Sub-leading soft operator contains derivatives of quark/anti-quark spinors

$$S^{(0)\lambda} = \frac{\langle nq\rangle}{\langle ns\rangle\langle sq\rangle} - \frac{\langle n\bar{q}\rangle}{\langle ns\rangle\langle s\bar{q}\rangle} = \frac{\langle \bar{q}q\rangle}{\langle \bar{q}s\rangle\langle sq\rangle}$$

$$S^{(1)\lambda} = \frac{1}{\langle sq\rangle}\tilde{\lambda}_s^{\dot{a}}\frac{\partial}{\partial\tilde{\lambda}_q^{\dot{a}}} - \frac{1}{\langle s\bar{q}\rangle}\tilde{\lambda}_s^{\dot{a}}\frac{\partial}{\partial\tilde{\lambda}_{\bar{q}}^{\dot{a}}}$$



Soft–Photon Limit in QCD Amplitude, cont'd:

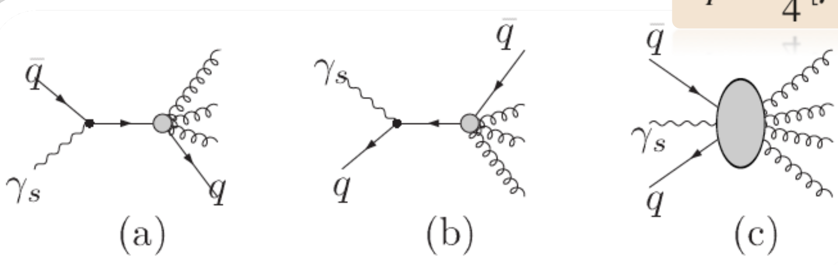
■ From Gauge Invariance Approach

$$A_{n+3}(k_s; k_{\bar{q}}, k_q, k_1, \ldots, k_n)$$
$$= \left(\frac{\varepsilon^+(k_s; r_s)\cdot k_{\bar{q}}}{\sqrt{2}k_{\bar{q}}\cdot k_s} - \frac{\varepsilon^+(k_s; r_s)\cdot k_q}{\sqrt{2}k_q\cdot k_s}\right) \times \left(\overline{u}(k_q)\tilde{A}(k_{\bar{q}}, k_q, k_1, \ldots, k_n)v(k_{\bar{q}})\right)$$
$$+ A^{(1)}_{n+3}\left(\varepsilon^+_\mu(k_s; r_s)k^\nu_s, \frac{\overline{u}(k_q)\,\Sigma^{\mu\nu}_F}{k_q\cdot k_s}, \frac{\Sigma^{\mu\nu}_F v(k_{\bar{q}})}{k_{\bar{q}}\cdot k_s}, \left(\frac{L^{\mu\nu}_{\bar{q}}}{k_{\bar{q}}\cdot k_s} - \frac{L^{\mu\nu}_q}{k_q\cdot k_s}\right)\widetilde{A}(k_{\bar{q}}, k_q, k_1, \ldots, k_n)\right) + \mathcal{O}(k_s)$$

- Use soft-momentum k_s to denote the singularity
- Non-radiative amplitudes ingredients: Dirac states $\overline{u}(k_q)$ $v(k_{\bar{q}})$; $\widetilde{A}(k_{\bar{q}}, k_q, k_1, \ldots, k_n)$ as a function of explicit momenta
- Leading soft singularity comes from diagram (a) and (b)
- Sub-leading soft behavior from (a), (b) and (c) consists of soft-particle kinematic information, spin angular-momentum actions and obital angular-momentum actions

$$\Sigma^{\mu\nu}_F \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$L^{\mu\nu}_{f_i} = i\left(k^\mu_i \frac{\partial}{\partial k_{i\nu}} - k^\nu_i \frac{\partial}{\partial k_{i\mu}}\right)$$



Spin operators of fermionic emitter cannot be naively disentangled as bosonic emitter…
Careful treatment to connect two derivations!

Equivalence of Two Derivations :

Leading Singularities

- Derived from direct calculation

$$S^{(0)} = \frac{\varepsilon^{+}(k_s;r_s)\cdot k_{\bar{q}}}{\sqrt{2}k_{\bar{q}}\cdot k_s} - \frac{\varepsilon^{+}(k_s;r_s)\cdot k_q}{\sqrt{2}k_q\cdot k_s} = \frac{\langle q\bar{q}\rangle}{\langle \bar{q}s\rangle\langle sq\rangle} = -S^{(0)\lambda},$$

$$L^{\mu\nu}_{f_i} u_\pm(k_i) = L^{\mu\nu}_{f_i} v_\pm(k_i) = 0$$
$$\overline{u}_\pm(k_i) L^{\mu\nu}_{f_i} = \overline{v}_\pm(k_i) L^{\mu\nu}_{f_i} = 0$$

Sub-Leading Singularities:

Strategy: BCFW Derivation ⟺ (Spinorial Notation) Gauge-Invariance Derivation

Sub-leading soft operators acting on Dirac field states, only spin operator contributes(R.H.S.), obital angular momenta do not contribute

Proposition 1: $S^{(1)\lambda}\, v(k_{\bar{q}}) = -\left[\frac{i\,\varepsilon^{+}_{\mu}(k_s;r_s)\,k_{s\nu}}{\sqrt{2}k_{\bar{q}}\cdot k_s}\,\Sigma^{\mu\nu}_{F}\, v(k_{\bar{q}})\right]$

Proof: Consider an outgoing antiquark: $h_{\bar{q}} = +\frac{1}{2}, \qquad v_+(k_{\bar{q}}) = \widetilde{\lambda}^{\dot{\alpha}}_{\bar{q}} = |\bar{q}]$

BCFW Derivation: $S^{(1)\lambda}\, v_+(k_{\bar{q}}) = \left(\frac{1}{\langle s\, q\rangle}\widetilde{\lambda}^{\dot{a}}_{s}\frac{\partial}{\partial\widetilde{\lambda}^{\dot{a}}_{q}} - \frac{1}{\langle s\,\bar{q}\rangle}\widetilde{\lambda}^{\dot{a}}_{s}\frac{\partial}{\partial\widetilde{\lambda}^{\dot{a}}_{\bar{q}}}\right)\widetilde{\lambda}^{\dot{b}}_{\bar{q}} = -\frac{1}{\langle s\,\bar{q}\rangle}|s]$

Gauge-Invariance Derivation: $\frac{i\,\varepsilon^{+}_{\mu}(k_s;r_s)\,k_{s\nu}}{\sqrt{2}k_{\bar{q}}\cdot k_s}\,\Sigma^{\mu\nu}_{F}\, v_+(k_{\bar{q}}) = +\frac{1}{\langle s\,\bar{q}\rangle}|s]$

Proposition 2: $S^{(1)\lambda}\,\overline{u}(k_q) = -\left[\overline{u}(k_q)\frac{i\,\varepsilon^{+}_{\mu}(k_s;r_s)\,k_{s\nu}}{\sqrt{2}k_q\cdot k_s}\,\Sigma^{\mu\nu}_{F}\right]$

Outgoing quark, proof is similar as proposition 1.

Equivalence Proof of Next-leading Soft Singularity :

- Next-to-Leading Soft Singularity, cont'd

Proposition 3:

Sub-leading soft operators acting on functions of momenta, only obital angular momenta operators contribute (R.H.S.), spin angular momenta do not contribute

$$S^{(1)\lambda}\tilde{A}(k_{\bar{q}},k_q,k_1,\ldots,k_n)$$
$$= i\frac{\varepsilon^+_\mu(k_s;r_s)k_{s\nu}}{\sqrt{2}}\left[\left(\frac{L^{\mu\nu}_{\bar{q}}}{k_{\bar{q}}\cdot k_s}-\frac{L^{\mu\nu}_{q}}{k_q\cdot k_s}\right)\tilde{A}(k_{\bar{q}},k_q,k_1,\ldots,k_n)\right]$$

Proof: Consider an outgoing antiquark

$\tilde{A}(k_{\bar{q}},k_q,k_1,\ldots,k_n)$

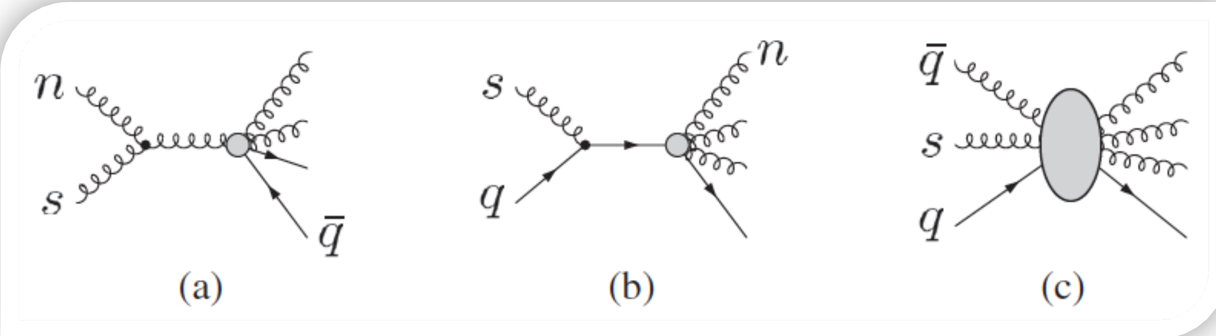
- A rational function of polarizations and momenta of gluons, momenta of quark/anti-quark & gamma matrices: expressed with spinor chains

BCFW Derivation:	Gauge-Invariance Derivation:
$S^{(1)\lambda}\,[\bullet\,\bar{q}] = -\frac{1}{\langle s\,\bar{q}\rangle}[\bullet\,s]$	$-\frac{i\,\varepsilon^+_\mu(k_s;r_s)\,k_{s\,\nu}}{\sqrt{2}}\left(\frac{L^{\mu\nu}_{\bar{q}}}{k_{\bar{q}}\cdot k_s}-\frac{L^{\mu\nu}_{q}}{k_q\cdot k_s}\right)k^\rho_{\bar{q}} = +\frac{1}{\langle s\,\bar{q}\rangle}\frac{\langle\bar{q}\vert\gamma^\rho\vert s]}{2}$
$S^{(1)\lambda}\frac{1}{[p\,\bar{q}]} = +\frac{1}{[p\,\bar{q}]}\frac{[p\,s]}{\langle s\,\bar{q}\rangle\,[p\,\bar{q}]}$	$-\frac{i\,\varepsilon^+_\mu(k_s;r_s)\,k_{s\,\nu}}{\sqrt{2}}\left(\frac{L^{\mu\nu}_{\bar{q}}}{k_{\bar{q}}\cdot k_s}-\frac{L^{\mu\nu}_{q}}{k_q\cdot k_s}\right)\frac{1}{p\cdot k_{\bar{q}}} = -\frac{1}{p\cdot k_{\bar{q}}}\frac{[p\,s]}{\langle s\,\bar{q}\rangle\,[p\,\bar{q}]}$

Connections Up to An Overall Minus Sign, Equivalence Proof Done!

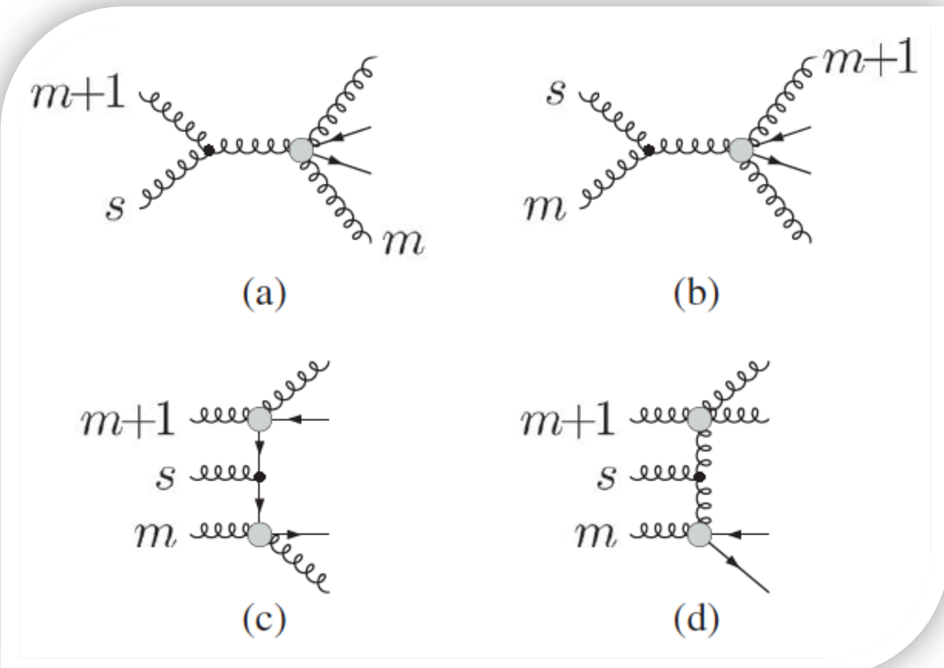
- *Soft-Gluon Limit in QCD Amplitude:*

➢ Case 1: Soft gluon adjacent to one quark and one gluon



Combine results of pure-YM and soft-photon from fermionic emitter

➢ Case 2: Soft gluon adjacent to two gluons



Although, results are similar to pure-YM, However, physics insight is different according to diagram (c) and (d).

Further1: Can we inverse the soft limit procedure to derive the n+1-pt amplitude from n-pt? Open question!

Further2: IR Divergence structure involving both soft and collinear effects, and quantum corrections?

Two-loop, talk by Zhu, Amplitude 2015

Soft Emissions of Off-shell Currents & Their On-Shell Limits In NLSM

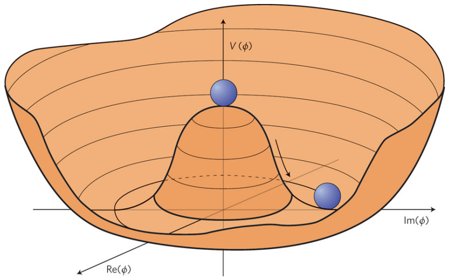
Yi-jian Du & H.L., JHEP 1508(2015) 058

Soft Behaviors and Symmetries

- *Soft behaviors of S-matrix connected to symmetries*

  Potential for discovery of hidden symmetries of quantum gravity or YM S-matrix

- *Soft limits for massless Goldstone bosons of spontaneously broken symmetry can be studied via Amplitude*



- Single soft emission: Adler zero [Kampf, Novotny & Trnka, 2013; Du & H.L.,2015]
- Double soft emission:

$$\lim_{\delta\to 0}\mathcal{A}_{n+2}(\phi^i(\delta q_1),\phi^j(\delta q_2),3,\ldots n+2)=\sum_{a=3}^{n+2}\frac{p_a\cdot(q_1-q_2)}{p_a\cdot(q_1+q_2)}\,f^{ijk}\hat{T}_k\mathcal{A}_n(3,\ldots n+2)$$

[Plefka, Amplitude 2015]

One can read out symmetry algebra from double soft limit (rotation in the vaccum)!

Examples: Soft pions, Hidden E7(7) symmetry in N=8 SUGRA [Arkani-Hamed, Cachazo, Kaplan, 08']

- *Related works:*

  - Soft limits of Scalars & Fermions in N<8 SUGRAs [Chen, Huang & Wen, 14']
  - Soft limits of Scalars & Photons in DBI, Galileon, Einstein-Maxwell-Scalar and NLSM [Cachazo, He & Ye, 15'; Du & Luo, 15']
  - Double/Triple soft gluons from string theory [Klose, McLoughlin, Nandan, Plefka & Travaglini; Volovich, Wen & Zlotnikov; Di Vecchia, Marotta & Mojaza15']

Theoretic Framework & Method in $SU(N)\times SU(N)\to SU(N)$

- *Lagrangian for NLSM with Cayley parameterization*

$$\mathcal{L}=\frac{F^2}{4}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger) \qquad U=1+2\sum_{n=1}^{\infty}\left(\frac{1}{2F}\phi\right)^n$$

➢ Vertices: $V_{2n+1}=0$ → **Odd-point amplitude vanishes in NLSM**

$$V_{2n+2}=\left(-\frac{1}{2F^2}\right)^n\left(\sum_{i=0}^{n}p_{2i+1}\right)^2=\left(-\frac{1}{2F^2}\right)^n\left(\sum_{i=0}^{n}p_{2i+2}\right)^2$$

- *Color-like (Flavor) Decomposition:* [Kampf, Novotny & Trnka, 2015]

$$M(1^{a_1},\ldots,n^{a_n})=\sum_{\sigma\in S_{n-1}}\mathrm{Tr}(T^{a_1}T^{a_{\sigma_2}}\ldots T^{a_{\sigma_n}})A(1,\sigma)$$

- *Berends-Giele recursion for NLSM with*

$$J(2,\ldots,2n)$$

$$=\frac{i}{P_{2,2n}^2}\sum_{m=2}^{n}\sum_{\text{Divisions}}iV_{2m}(p_1=-P_{2,2n},P_{A_1},\cdots,P_{A_{2m-1}})\times\prod_{k=1}^{2m-1}J(A_k)$$

➢ Divisions: all possible divisions of on-shell particles

$$\{2,\ldots,2n\}\to\{A_1\},\ldots,\{A_{2m-1}\}$$

Soft Behaviors of the off-shell currents & on-shell limits:

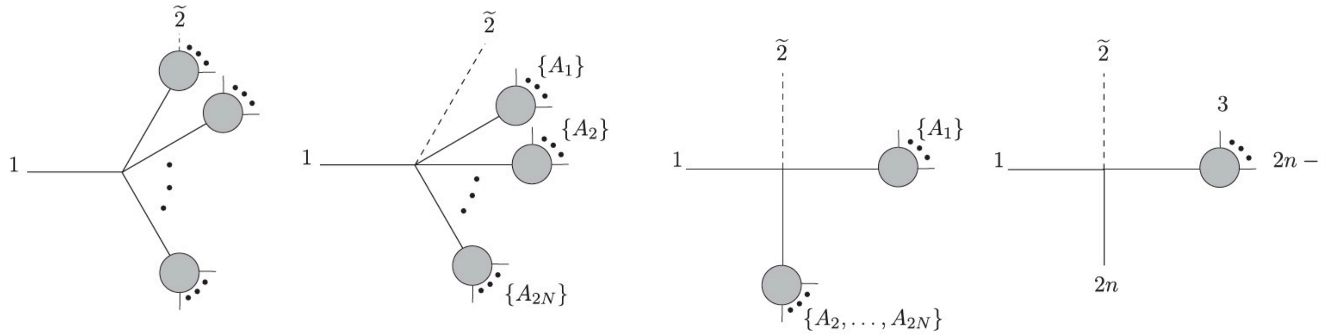
- *Single soft behaviors* (τ parameterizes the soft momentum)

$$J(2,\ldots,i-1,\tilde{i},i+1,\ldots,2n) = \left\{ \begin{array}{cc} 0 & (i \text{ is even}) \\ \left(\frac{1}{2F^2}\right) J(2,\ldots,i-1)J(i+2,\ldots,2n) & (i \text{ is odd}) \end{array} \right\} + \mathcal{O}(\tau),$$

Taking on-shell limit $P^2_{2,2n} \to 0$ → " *Adler Zero* "

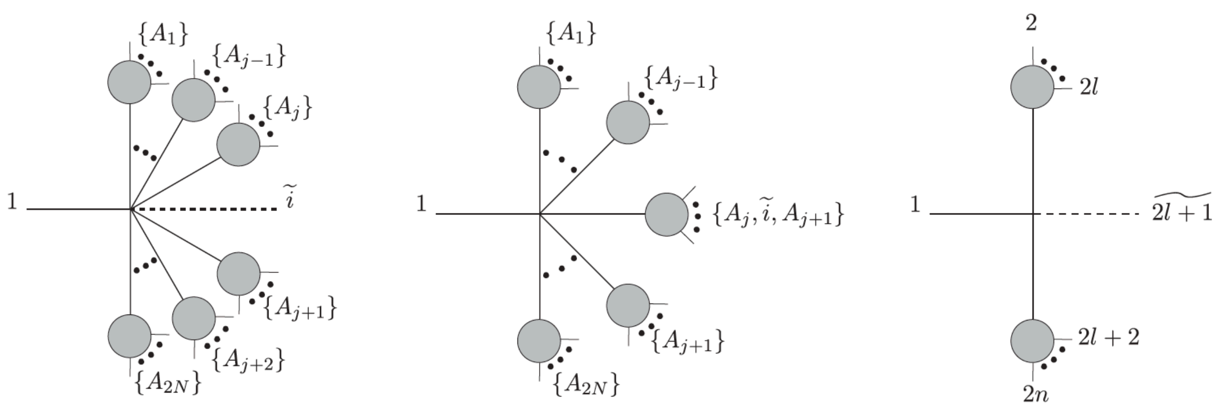
Soft Limit and on-shell limit can be exchanged

- <u>*Boundary case:*</u> Soft particle adjacent to the off-shell line $J^{(0)}(\tilde{2},3,\ldots,2n) = 0$



#Matches the "even" case;
#Inductive proof from Berends-Giele recursion

- <u>*Other cases:*</u> The even/odd soft particle non-adjacent to the off-shell line



Inductive assumption for lower-point off-shell currents
Results of boundary case should be used during the inductive proof
In "i is even" condition, the first 2 diagrams cancel each other
In "i is odd" condition, the first 2 cancel out, the last gives non-zero term

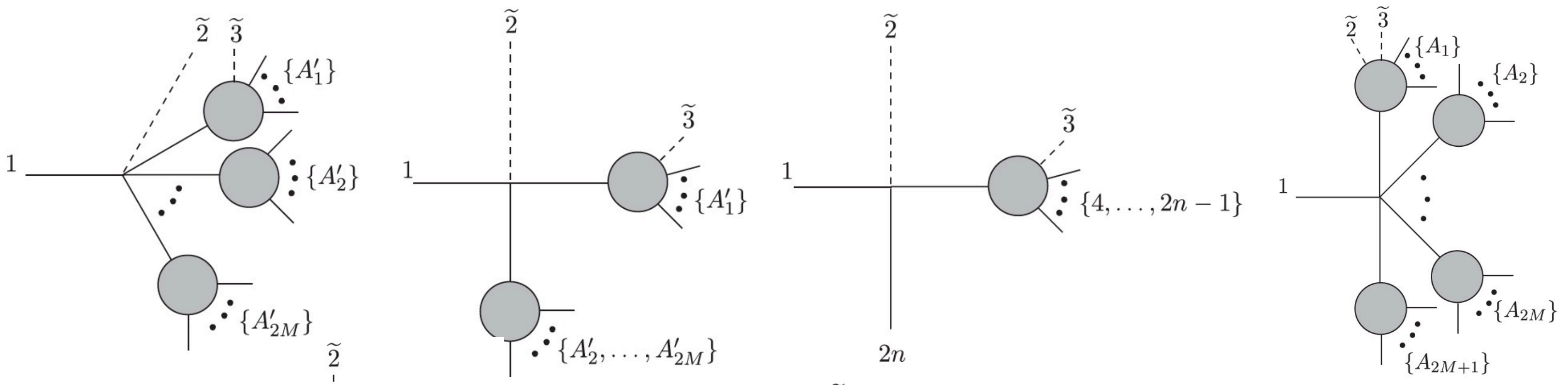
Adjacent Double Soft Behaviors of the off-shell currents-1

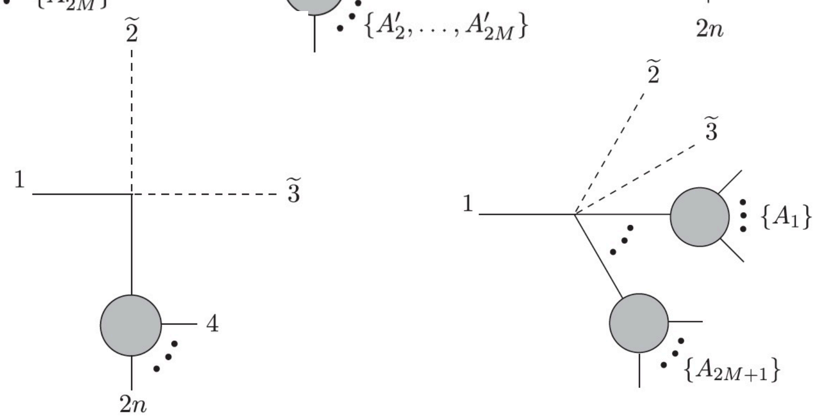
- *Consider two soft Goldstone bosons which are near each other*
- *Boundary case:* One of the soft particles adjacent to the off-shell line

i.e. $J(\widetilde{2},\widetilde{3},4,\ldots,2n)$ *with 2 and 3 as soft particles*

$$J(\widetilde{2},\widetilde{3},4,\ldots,2n)=\tau^0 S^{(0)}_{2,3}J(4,\ldots,2n)+\tau^1 S^{(1)}_{2,3}J(4,\ldots,2n)+\mathcal{O}(\tau)$$

$$S^{(0)}_{2,3}=\left(\frac{1}{2F^2}\right)\frac{k_4\cdot p}{k_4\cdot(q+p)}\qquad S^{(1)}_{2,3}=\left(-\frac{1}{2F^2}\right)\left[(p\cdot q)\frac{k_4\cdot p}{(k_4\cdot(p+q))^2}+\frac{q_\mu p_\nu\mathcal{J}_4^{\mu\nu}}{k_4\cdot(p+q)}\right]$$





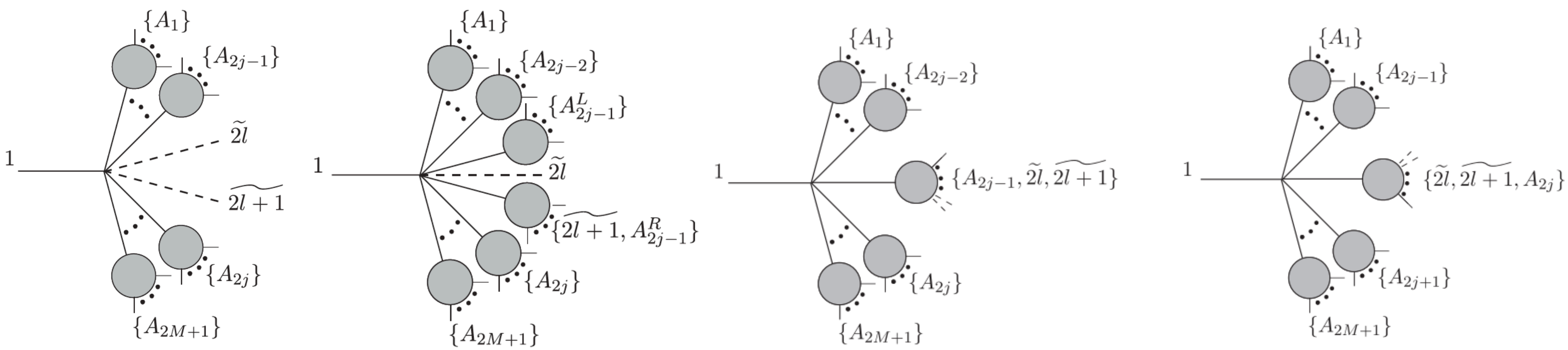
Inductive proof via Berends-Giele recursion while considering different divisions (6 types)
Single soft limit results list on the previous page should be used

Adjacent Double Soft Behaviors of the off-shell currents-2

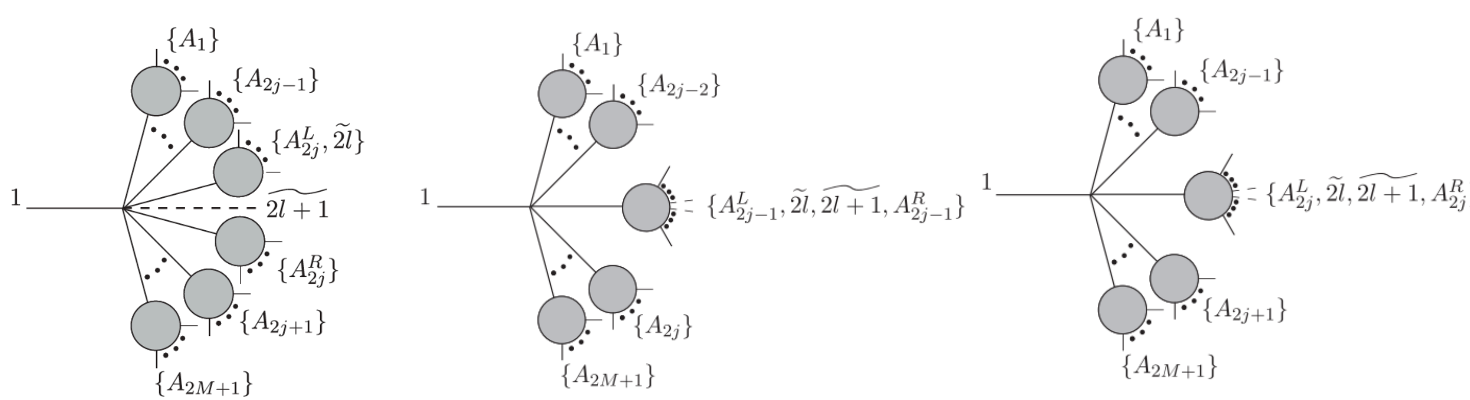
- *Other cases:* the soft particles non-adjacent to the off-shell line

Expressions of these two operators, see next page

$$
\begin{aligned}
&J(2,\ldots,i-1,\widetilde{i},\widetilde{i+1},i+2,\ldots,2n) \\
&= \tau^0 S^{(0)}_{i,i+1} J(2,\ldots,i-1,i+2,\ldots,2n) + \tau^1 \Bigg[S^{(1)}_{i,i+1} J(2,\ldots,i-1,i+2,\ldots,2n) \\
&\quad + \left\{ \begin{array}{ll} \left(\frac{1}{2F^2}\right) J^{(1)}(2,\ldots,i-1,\widetilde{i}) J(i+2,\ldots,2n) & (i \text{ is even }) \\ \left(\frac{1}{2F^2}\right) J(2,\ldots,i-1) J^{(1)}(\widetilde{i+1},i+2,\ldots,2n) & (i \text{ is odd}) \end{array} \right\} \Bigg] + \mathcal{O}(\tau^2)
\end{aligned}
$$



Inductive proof via Berends-Giele recursion
Single soft results and boundary case results should be applied during the proof



On-shell Limits of the Adjacent Double Soft Behaviors

- *Boundary case: Notice the orders in taking soft and on-shell limits:*

➢ While taking the on-shell limit of the off-shell leg $P^2_{2,2n} \to 0$ after deriving the soft limits, there is a 0/0 illed form

➢ In the boundary case, the on-shell limit should be imposed first, then the soft limits

- *Other cases: soft and on-shell limits can be exchanged*

- *With a careful treatment, the double soft behaviors of the amplitudes in the NLSM can be achieved as*

$$A(1,\ldots,\widetilde{i},\widetilde{i+1},\ldots,2n) = \left(\tau^0 \mathbb{S}^{(0)}_{i,i+1} + \tau^1 \mathbb{S}^{(1)}_{i,i+1}\right) A(1,\ldots,i-1,i+2,\ldots,2n) + \mathcal{O}(\tau^2)$$

$$\mathbb{S}^{(0)}_{i,i+1} = \left(-\frac{1}{2F^2}\right)\frac{1}{2}\left[\frac{k_{i-1}\cdot(p-q)}{k_{i-1}\cdot(p+q)} + \frac{k_{i+2}\cdot(q-p)}{k_{i+2}\cdot(q+p)}\right]$$

$$S^{(0)}_{i,i+1} = \mathbb{S}^{(0)}_{i,i+1}$$

$$\mathbb{S}^{(1)}_{i,i+1} = \left(-\frac{1}{2F^2}\right)(p\cdot q)\left[\frac{k_{i-1}\cdot q}{(k_{i-1}\cdot(p+q))^2} + \frac{k_{i+2}\cdot p}{(k_{i+2}\cdot(p+q))^2}\right] + \left(-\frac{1}{2F^2}\right)\left[\frac{p_\mu q_\nu}{k_{i-1}\cdot(p+q)}\mathcal{J}^{\mu\nu}_{i-1} + \frac{q_\mu p_\nu}{k_{i+2}\cdot(p+q)}\mathcal{J}^{\mu\nu}_{i+2}\right]$$

$$S^{(1)}_{i,i+1} = \mathbb{S}^{(1)}_{i,i+1}$$

$$\mathcal{J}^{\mu\nu}_a \equiv k^\mu_a \frac{\partial}{\partial k_{a,\nu}} - k^\nu_a \frac{\partial}{\partial k_{a,\mu}}$$

Non-Adjacent Double Soft Behaviors

- *Two soft particles Do Not Share any common adjacent particle:*

$$A(\tilde{1},2,\ldots,i-1,\tilde{i},i+1,\ldots,2n)=0+\mathcal{O}(\tau^2),\qquad (3<i<2n-1)$$

- *Two soft particles Share One common adjacent particle:*

$$A(\tilde{1},2,\tilde{3},4,\ldots,2n)=\begin{cases}0+\mathcal{O}(\tau^2) & (n=2)\\ \tau^1\left(\frac{1}{2F^2}\right)\frac{p\cdot q}{k_2\cdot(p+q)}A(2,4,\ldots,2n)+\mathcal{O}(\tau^2) & (n>2)\end{cases}$$

- *The above Proof by Kleiss-Kuijf (KK) relation in NLSM*

$$A(1,2,\ldots,i-1,i,i+1,\ldots,2n)=\sum_{\alpha\in OP(\{2,\ldots,i-1\}\bigcup\{i+1,\ldots,2n\}^T)}(-1)^{2n-i}A(1,\{\alpha\},i)$$

Those results can also be derived as before by Berends-Giele recursions in off-shell currents and then take the on-shell limits

We have checked the results from two ways are identical

Discussion and possible implementations

- Natural Question: can we derive the same sub-leading double soft operators by PCAC? What kind of symmetry/physics insight does the sub-leading operators indicate?
- Soft limits of Goldstone-boson amplitudes encode underlying patterns of symm. breaking, which can also be implemented in N=8 SUGRA, where the classical theory has global continuous E7(7) symm. broken to SU(8)
- We can use the scalar limits to test the candidate counter terms for high-loop orders in N=8 SUGRA, in principle they should be E7(7)–compatible and match the scalar soft limits factorization
- Only one 7-loop counter term D^8R^4 pass the test of single and double scalar limits up to 6-point [Beisert, Elvang, Freedman, Kiermaier, Morales & Stieberger, 2010]
- Further tests are required for D^8R^4 with more constraints: we are working on the constraints from multi-scalar limits to test its E7(7) compatibility [Huang, H.L. & Wen, working in progress]

Conclusions and Outlook

- We study the single gluon soft limit in QCD amplitude and the double soft Goldstone bosons structures in NLSM all up to the sub-leading order
- It's quite interesting to discover the hidden (if exists) symmetry which makes the sub-(sub-)leading soft behaviors universal
- It's natural to ask: Multi-soft particles? Especially, to probe the coset of the broken symm. in the NLSM and hidden E7(7) symmetry of N=8 SUGRA
- The soft behaviors constrain (partially) the candidate counterterms of the N=8 SUGRA theory, to test the UV finite (if true) conjecture

Thanks!

Back-Up:

Bondi-van der Burg-Metzner-Sachs(BMS) symm.:

- Study of classical gravitational waves: Expected Poincaré symmetry enlarged by BMS_4 group
- Acts at null infinity ($\mathcal{I}^{\pm}$) for asympt. flat space-times
- Coordinates: u (retarded time), r (radius), $x^A = \{\Theta, \phi\} \in S^2$ at $\mathcal{I}^{\pm}$

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}\,du\,dr + g_{AB}(dx^A + U^A du)(dx^B + U^B du)$$

Metric functions β, V, U^A, g_{AB} have fall-off conditions in r:

$$g_{AB} = r^2(d\Theta^2 + \sin^2\Theta\,d\phi^2) + \mathcal{O}(r),\;\; \beta = \mathcal{O}(r^{-2}),\;\; \frac{V}{r} = \mathcal{O}(r),\;\; U^A = \mathcal{O}(r^{-2})$$

- BMS_4 group: Maps asymptotically flat space-times onto themselves

$$\Theta' = \Theta'(\Theta,\phi) \qquad \phi' = \phi'(\Theta,\phi) \qquad u' = K(\Theta,\phi)\,(u - \alpha(\Theta,\phi))$$

Where $(\Theta,\phi) \to (\Theta',\phi')$ is conformal transformation on S^2:

$$d\Theta'^2 + \sin^2\Theta' d\phi'^2 = K(\Theta,\phi)^2(d\Theta^2 + \sin^2\Theta\,d\phi^2)$$

- For $\Theta' = \Theta$ & $\phi' = \phi$ one has "supertranslations": $u' = u - \alpha(\Theta,\phi)$ with a general function $\alpha(\Theta,\phi)$.

● BMS_4 *Algebra:*

In standard complex coordinates $z = e^{i\phi}\cot(\Theta/2)$ conformal symmetry generated by Virasoro generators ("superrotations")

$$l_n = -z^{n+1}\,\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\,\partial_{\bar{z}}$$

Supertranslations generated by $T_{m,n} = z^m\,\bar{z}^n\,\partial_u$

Extended $\mathfrak{bms}_4$ algebra [Barnich, Troessart]

$$[l_n, l_m] = (m-n)\,l_{m+n} \qquad [\bar{l}_n, \bar{l}_m] = (m-n)\,\bar{l}_{m+n}$$

$$[l_l, T_{m,n}] = -m\,T_{m+l,n} \qquad [\bar{l}_l, T_{m,n}] = -n\,\bar{T}_{m,n+l}$$

Poincaré subalgebra spanned by $\underbrace{l_{-1}, l_0, l_1; \bar{l}_{-1}, \bar{l}_0, \bar{l}_1}_{\text{Lorentz}}$ $\underbrace{T_{0,0}, T_{0,1}, T_{1,0}, T_{1,1}}_{\text{Translation}}$

BMS_4 group maps gravitational wave solutions onto each other.

Claim: | Supertranslations $\hat{=}\ S_{\mathsf{G}}^{(0)}$ | Superrotations $\hat{=}\ S_{\mathsf{G}}^{(1)}$ | [Cachazo, Strominger]

Soft-Limit in Pure−YM

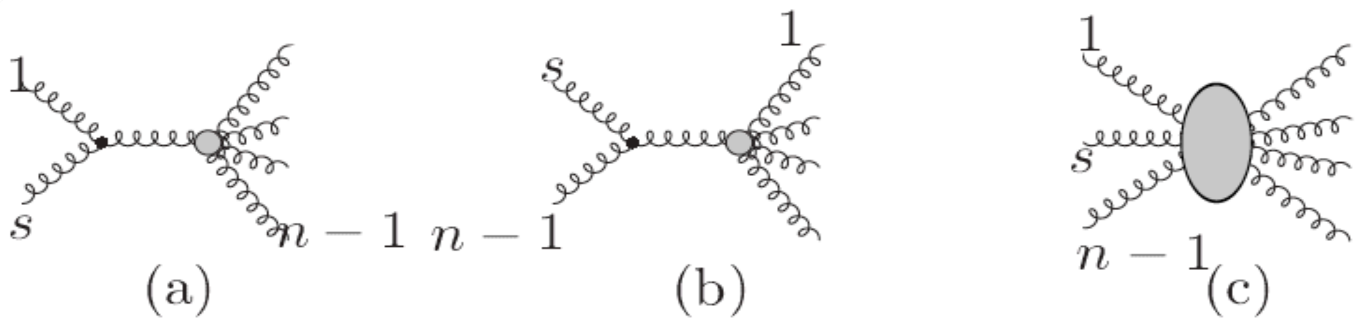


FIG. 1: Soft-gluon behaviour of pure-gluon amplitudes

$$A_n(k_s;k_1,\dots,k_{n-1}) = \left[S_{\rm G}^{(0)}+S_{\rm G}^{(1)}\right]A_{n-1}(k_1,\dots,k_{n-1})+\mathcal{O}(k_s)$$

"s" : soft particle
"i" : hard particles

On-Shell Gauge Invariance

$$S_{\rm G}^{(0)} \equiv \frac{k_1\cdot\varepsilon(k_s;r_s)}{\sqrt{2}(k_1\cdot k_s)}-\frac{k_{n-1}\cdot\varepsilon(k_s;r_s)}{\sqrt{2}(k_{n-1}\cdot k_s)}\,,$$

$$S_{\rm G}^{(1)} \equiv -i\varepsilon_\mu(k_s;r_s)k_{s\sigma}\left(\frac{J^{\mu\sigma}_{{\rm G}1}}{\sqrt{2}(k_1\cdot k_s)}-\frac{J^{\mu\sigma}_{{\rm G}n-1}}{\sqrt{2}(k_{n-1}\cdot k_s)}\right)$$

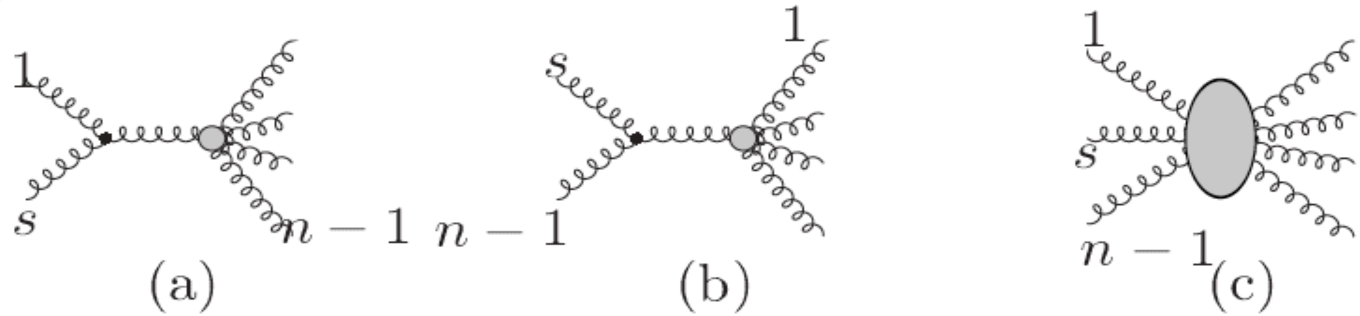
$$J^{\mu\sigma}_{{\rm G}i} \equiv L^{\mu\sigma}_{{\rm G}i}+\Sigma^{\mu\sigma}_{{\rm G}i}$$

$$L^{\mu\sigma}_{{\rm G}i}\varepsilon^\nu_i=0.$$

$$L^{\mu\sigma}_{{\rm G}i} \equiv i\left(k_i^\mu\frac{\partial}{\partial k_{i\sigma}}-k_i^\sigma\frac{\partial}{\partial k_{i\mu}}\right)$$

$$\Sigma^{\mu\sigma}_{{\rm G}i} \equiv i\left(\varepsilon_i^\mu\frac{\partial}{\partial \varepsilon_{i\sigma}}-\varepsilon_i^\sigma\frac{\partial}{\partial \varepsilon_{i\mu}}\right)$$

Soft-Limit in Pure−YM



$$S_{\rm G}^{(0)} \equiv \frac{k_1\cdot\varepsilon(k_s;r_s)}{\sqrt{2}(k_1\cdot k_s)} - \frac{k_{n-1}\cdot\varepsilon(k_s;r_s)}{\sqrt{2}(k_{n-1}\cdot k_s)}\,,$$

$$S_{\rm G}^{(1)} \equiv -i\varepsilon_\mu(k_s;r_s)k_{s\sigma}\left(\frac{J^{\mu\sigma}_{{\rm G}1}}{\sqrt{2}(k_1\cdot k_s)} - \frac{J^{\mu\sigma}_{{\rm G}n-1}}{\sqrt{2}(k_{n-1}\cdot k_s)}\right)$$

$$S_{\rm G}^{(1)}\varepsilon^{+\rho}(k_1;r_1) = -\frac{\langle r_1,\, s\rangle}{\langle r_1,\, 1\rangle\,\langle 1,\, s\rangle}\varepsilon^{+\rho}(k_s;r_1)$$

$$S_{\rm G}^{(1)}\varepsilon^{-\rho}(k_1;r_1) = +\frac{[r_1,\, s]}{[r_1,\, 1]\,[1,\, s]}\varepsilon^{+\rho}(k_s;r_1)$$

$$S_{\rm G}^{(0)\lambda} = \frac{\langle n-1,\, 1\rangle}{\langle s,\, 1\rangle\,\langle n-1,\, s\rangle}\,,$$

$$S_{\rm G}^{(1)\lambda} = \frac{1}{\langle s,\, 1\rangle}\widetilde{\lambda}_s^{\dot\alpha}\frac{\partial}{\partial\widetilde{\lambda}_1^{\dot\alpha}} + \frac{1}{\langle n-1,\, s\rangle}\widetilde{\lambda}_s^{\dot\alpha}\frac{\partial}{\partial\widetilde{\lambda}_{n-1}^{\dot\alpha}}$$

$$S_{\rm G}^{(1)\lambda}\varepsilon^{+\rho}(k_1;r_1) = -\frac{\langle r_1,\, s\rangle}{\langle r_1,\, 1\rangle\,\langle 1,\, s\rangle}\varepsilon^{+\rho}(k_s;r_1),$$

$$S_{\rm G}^{(1)\lambda}\varepsilon^{-\rho}(k_1;r_1) = +\frac{[r_1,\, s]}{[r_1,\, 1]\,[1,\, s]} \times\left[\varepsilon^{+\rho}(k_s;r_1) - \frac{\sqrt{2}\,[r_1,\, s]}{[r_1,\, 1]\,\langle 1,\, s\rangle}k_1^\rho\right]$$