Lattice computation of m_b

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Traditionally, by zero-temperature Lattice QCD we mean

ab initio or model independent MC simulation of Euclidean QCD

$$\mathcal{L} = F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i(\not{D} + m_i)\psi_i$$

- Assume we can "rotate back" to Minkowski space-time
- Take the infinite volume limit, continuum limit, infinite statistics limit

Lattice QCD $\Leftrightarrow$ *Euclidiean QCD*

- LQCD is not a model

Therefore, in principle, for a given number of flavour N_f

Input: bare parameters of QCD Lagrangian (g, $m_i = m_u, m_d, m_s, \ldots$)

$\downarrow$

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Output: Hadronic quantities, Masses, Decay constants, ...

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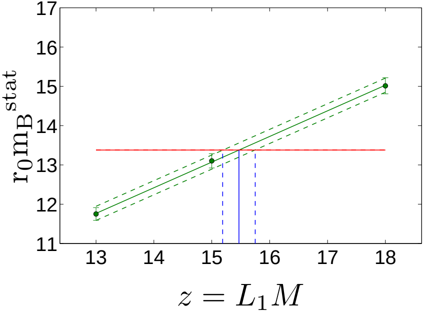
Simple way to extract a quark mass

- Compute a hadron mass m_H for several values of the quark mass m_q

$$\langle O^\dagger(t)O(0)\rangle \to |\langle 0|O|H\rangle|^2 e^{-m_H t}$$

- Use the experimental value for the hadron mass
- Interpolate, ie find the value of the bare quark mass m_q^{phys} such that $m_H(m_q^{phys}) = m_H^{exp}$
- Renormalise, match to $\overline{\mathrm{MS}}$ or your favourite scheme



Unfortunately this does not work for m_b

However

- Computer power not infinite
- Lattice QCD very demanding
- See quarks are expensive, Physical quark masses are expensive, ...
- Critical slowing down when $a \to 0$. Standard MC fails toward the continuum limit
- Very different scales: a few MeV for m_{light} vs $m_b \sim 5\,GeV$
- Cut some corners ?

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Example: Chiral symmetry

- Freedom in the discretisation of the Dirac operator
- Different discretisation (=action) should be equivalent in the continuum
- We know how preserve chiral-flavour symmetry at finite lattice spacing
- But numerically very expensive
- Popular choices for fermions: Wilson-like fermion, Twisted mass, Staggered
- In general, for heavy quark, we don't have finite lattice-spacing chiral-flavour symmetry
- But should recover the right symmetry in the continuum limit

Wish-list

We want to simulate QCD with both physical light and heavy quarks

Light sector

- u, d, s in the sea
- Physical pion mass $m_\pi = 140 MeV$
- Hadron should "fit in the box": $m_\pi L > 3$ (rule of thumb) $\rightarrow L \sim 4\ \mathrm{fm}$

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Heavy sector

- c quark in the sea (quenching the b is probably fine)
- Valence b-quark $\Rightarrow$ Lattice spacing fine enough to "resolve" the b

  Discretisation errors $\sim (am_q)^n$, $n = 1, 2$ depends on the lattice action

$$am_b \ll 1 \Rightarrow a^{-1} \gg 5 GeV \Rightarrow a \ll 0.04\ \mathrm{fm}$$

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Putting them together, number of points per space/time direction:

$$L/a \ggg 4\ \mathrm{fm} \times 5\ \mathrm{GeV} \sim 100$$

A challenge

Way to go ?

- Brute force: 100^4 lattice with $a^{-1} = 5\ \mathrm{GeV} \Rightarrow a \sim 0.04\ \mathrm{fm}$ and four flavour

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- Famous example:

$$m_B^{stat} \sim E^{stat} + m_b$$

Static binding energy linearly divergent with the inverse lattice spacing [Eichten & Hill '90]

$$E^{stat} = \frac{19.95}{12\pi^2} \times \frac{g_0^2}{a} + \dots$$

Get worse at higher order, $E^{kin} \sim 1/a^2 \dots$

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- Existence of the continuum limit requires a non-perturbative subtraction of the divergences
- Lattice results in the heavy-quark sector have to be taken with care !

$\Rightarrow$ Look at the details of computation

Different choices of lattice implementation (non-exhaustive list)

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The errors can be very different

Lattice Methods and results for Heavy Quark

This is a non-exhaustive selection (apologies if I don't mention your favourite method/results)

Define a correlator from pseudo-scalar density of a heavy quark $j_5 = \bar{\psi}_h \gamma_5 \psi_h$

$$G(t) = a^6 \sum_{\mathbf{x}} (am_h^{bare})^2 \langle 0 | j_5(\mathbf{x}, t) j_5(0) | 0 \rangle$$

Well defined continuum limit $a \to 0$

Moments are defined by $G_n = \sum_t (t/a)^n G(t)$, symmetrised such that odd moments vanish

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Assume perturbation theory is applicable for low momenta $n \geq 4$

$$G_n = \frac{g_n(\alpha_{\overline{\mathrm{MS}}}(\mu), \mu/m_h)}{(am_h(\mu))^{n-4}} + \mathcal{O}((am_h)^m)$$

$\Rightarrow$ Compute G_n on the lattice and g_n are computed in continuum perturbation theory

Introduce reduce moments

$$R_n = \begin{cases} G_4/G_4(0) & \text{for } n = 4, \\ \frac{am_{\eta_h}}{2am_h^{bare}}(G_n/G_n^{(0)})^{1/(n-4)} & \text{for } n \geq 6, \end{cases}$$

where

- $G_n^{(0)}$ is the n-th moment at lowest order in PT
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Match to the continuum expression (again valid up to $\mathcal{O}((am_h)^m\alpha_s)$)

$$R_n = \begin{cases} r_4(\alpha_{\overline{\rm MS}}, \mu/m_h) & \text{for } n = 4, \\ z(\mu/m_h, m_{\eta_h})r_n(\alpha_{\overline{\rm MS}}, \mu/m_h) & \text{for } n \geq 6, \end{cases}$$

where $z(\mu/m_h, m_{\eta_h}) = m_{\eta_h}/(2m_h(\mu))$ and r_n ratio of $g_n/g_n^{(0)}$

Use lattice simulations (*staggered fermions*) with multiple heavy quark masses

- MILC configurations, $n_f = 2 + 1$ ASQTAD sea quarks
- HISQ valence quarks (up to $m_{\eta_h} \sim 8~\mathrm{GeV}$)
- Smallest lattice 0.044 fm

In 2010, was only possible with staggered fermions

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Fit the lattice data using an Ansatz

$$z(\mu/m_h, m_{\eta_h}) = \sum_j z_j(\mu/m_h) \left(\frac{2\Lambda}{m_{\eta_h}}\right)^j$$

to obtain the mass dependence of the z

(In practice, very sophisticated Bayesian Fit procedure to remove the lattice artefacts)

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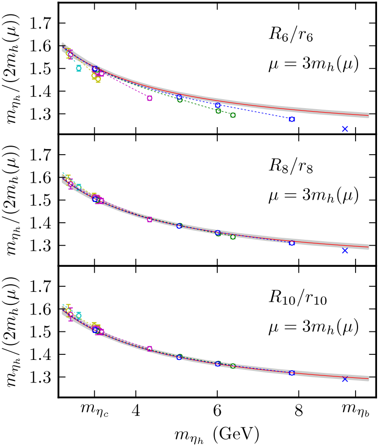
And the experimental value at physical point

$$m_b(\mu) = \frac{m_{\eta_b}^{exp}}{2z(\mu/m_b, m_{\eta_b})}$$

Result from HPQCD



HPQCD, McNeile et al '10

$$m_b^{\overline{\rm MS}}(m_b, n_f = 5) \quad = \quad 4.164(23)\ \text{GeV} \qquad (1)$$

(2)

Result from HPQCD

- This method is not based on effective theory
- Instead, combined Lattice data with Pertubation Theory
- Very small error bars

$n_f = 3$ flavours HPQCD, McNeile et al '10

$$m_b^{\overline{\mathrm{MS}}}(m_b, n_f = 5) = 4.164(23)\ \mathrm{GeV}$$

New result $n_f = 4$ HPQCD, Chakraborty et al '15

$$m_b^{\overline{\mathrm{MS}}}(m_b, n_f = 5) = 4.162(48)\ \mathrm{GeV}$$

The ratio method ETMc '10

- $m_q = Z_m m_{bare}$ is a renormalised heavy quark mass, say $\overline{\mathrm{MS}}$ at a given scale
- Conversion between $\overline{\mathrm{MS}}$ and pole mass given by $\rho(m_q)m_q = m_{pole}$
- m_P is the mass of a pseudo-scalar meson made of m_q and a light quark.

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- Consider a pair of heavy quark of masses $m_q^{(1)}$ and $m_q^{(2)}$, with fixed ratio

$$\frac{m_q^{(2)}}{m_q^{(1)}} = \frac{m_{bare}^{(2)}}{m_{bare}^{(1)}} \equiv \lambda$$

- The ratio of heavy-light pseudo-scalar meson masses is then

$$\lim_{m_q^{(2)}\to\infty} \left[\frac{m_q^{(1)}}{m_q^{(2)}} \frac{\rho(m_q^{(1)})}{\rho(m_q^{(2)})} \frac{m_P(m_q^{(2)})}{m_P(m_q^{(1)})} \right] = 1$$

So define

$$y(m_q^{(n)}) \equiv \frac{m_q^{(n-1)}}{m_q^{(n)}} \frac{\rho(m_q^{(n-1)})}{\rho(m_q^{(n)})} \frac{m_P(m_q^{(n)})}{m_P(m_q^{(n-1)})}$$

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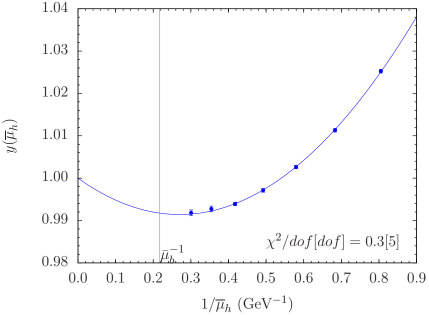
Compute the product

$$y(m_q^{(N)}) \times y(m_q^{(N-1)}) \times \ldots \times y(m_q^{(2)}) = \frac{m_q^{(1)}}{m_q^{(N)}} \frac{\rho(m_q^{(1)})}{\rho(m_q^{(N)})} \frac{m_P(m_q^{(N)})}{m_P(m_q^{(1)})}$$

for several quark masses and use the static point to interpolate

Continuum $n_f = 2$ $\quad m_b^{\overline{\rm MS}}(m_b) = 4.29(12)$



Effective theories for heavy quark

Momentum of a heavy quark (inside a hadron) $p = m_Q v + k$

Interaction with light dof $k \sim \Lambda_{\mathrm{QCD}} \ll m_Q$

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Separate the higher and lower components of the heavy quark, and find an effective Lagrangian (see eg [Grozin '02])

$$\mathcal{L}_{\rm eff}^{\rm heavy} = \bar{\psi}_{\rm h}(x) \left[iv.D + \frac{(iD_\perp)^2}{2m_Q} + \frac{g\sigma.G}{4m_Q} + ... \right] \psi_{\rm h}(x)$$

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Depending on the physics, different kinds of expansion

- Expansion in $\Lambda_{\rm QCD}/m_Q$ at 0-velocity: HQET
- Expansion in v and $1/am_Q$: NRQCD

Original motivation: HQET (static) for heavy-light and NRQCD for heavy-heavy

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Note that leading order

- the HQET Lagrangian only has a static part D_0
- whereas in NRQCD we include the kinetic part $\sim D_\perp^2/m_Q$

$\Rightarrow$ This has crucial consequences on the lattice

Non perturbative HQET Alpha '04

In infinite volume $m_B = m_{bare} + E^{stat}$, where m_{bare} cancels the $1/a$ divergence

- Simulate QCD in small volume $L_1 \sim 0.5\ \mathrm{fm}$ with $am_b \ll 1$.

  Compute a "finite volume mass" $\Gamma(L_1, m_q) = \lim_{a\to 0} \Gamma(L_1, m_q, a)$

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$$\begin{aligned}\Gamma(L_2, m_{\rm q}) &= \lim_{a\to 0} \left(\Gamma^{\rm stat}(L_2, a) + m_{\rm bare}(m_{\rm q}, a)\right) \\ &= \lim_{a\to 0} \left(\Gamma^{\rm stat}(L_2, a) - \Gamma^{\rm stat}(L_1, a)\right) + \Gamma^{\rm QCD}(L_1, m_{\rm q})\end{aligned}$$

(note that the divergence cancels out in the difference)

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- Re-iterate until the volume is large enough

Strategy [Heitger & Sommer 03]

For an observable Φ with an additive renormalization (in our case $\Phi = L\Gamma = L(\Gamma + m_{\rm bare})$)

$$\begin{aligned}
\Phi(L_\infty, m_{\rm q}) &= \lim_{a\to 0}\left[\Phi^{\rm HQET}(L_\infty, a) - \Phi^{\rm HQET}(L_2, a)\right] & a \sim 0.1 \text{ fm} \quad (1)\\
&+ \lim_{a\to 0}\left[\Phi^{\rm HQET}(L_2, a) - \Phi^{\rm HQET}(L_1, a)\right] & a \sim 0.05 \text{ fm} \quad (2)\\
&+ \lim_{a\to 0} \Phi^{\rm QCD}(L_1, m_{\rm q}, a) & a \sim 0.025 \text{ fm} \quad (3)
\end{aligned}$$

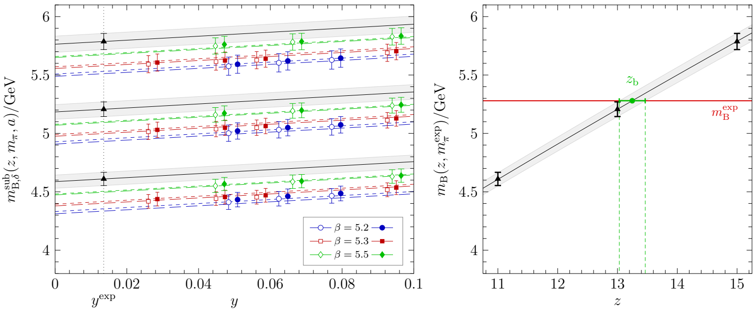
To extract the b-quark mass, we match the B mesons mass to its experimental value, and solve

$$\Phi(m_{\rm b}, L_\infty) = \Phi^{\rm exp} = L_\infty m_{\rm B}^{\rm exp}$$

by an interpolation in the quark mass

- HQET at $1/m$ order non-perturbatively matched to QCD
- $n_f = 2$ flavours
- Continuum limit

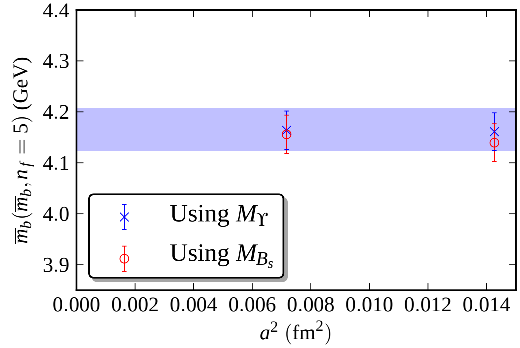
$$m_b^{\overline{\mathrm{MS}}}(m_b) = 4.21(11)\ \mathrm{GeV}$$



NRQCD HPQCD, Lee et al. '13

- NRQCD $\mathcal{O}(v^4)$ perturbatively matched to QCD
- $n_f = 2+1$ MILC configurations
- 2 lattice spacings: $a \sim 0.12, 0.09$ fm

$$m_b^{\overline{\mathrm{MS}}}(m_b) = 4.166(43)\ \mathrm{GeV}$$



Conclusions and Outlook

- Lattice QCD has reached a golden age in the light quark sectors
  Chiral symmetry, physical quark masses, $2+1+1$ flavours, QCD+QED
- In the heavy quark sector the situation is different
- No ab-initio computation of the *b*-quark mass available yet
- Ingenious methods have been developed
- Various computation with very different errors
- Either combine LQCD with EFT or with PT
- Details of the computation are important to understand the error
- Future interesting direction: design specific action for heavy quark

Lattice methods for heavy quark (II)

Main advantages and disadvantages of the different methods

method	pros	cons
NRQCD	heavy-heavy possible many results available	non-renormalizable (no continuum limit)
HQET	theoretically well-defined continuum limit non-perturbative	only heavy-light only $n_f = 2$ so far
Fermilab	from light to heavy many results available	partially perturbative matching
RHQ	from light to heavy, np matching	numerically hard