Precision Standard Model boson masses in the pure $\overline{\text{MS}}$ scheme

Stephen P. Martin

Northern Illinois University

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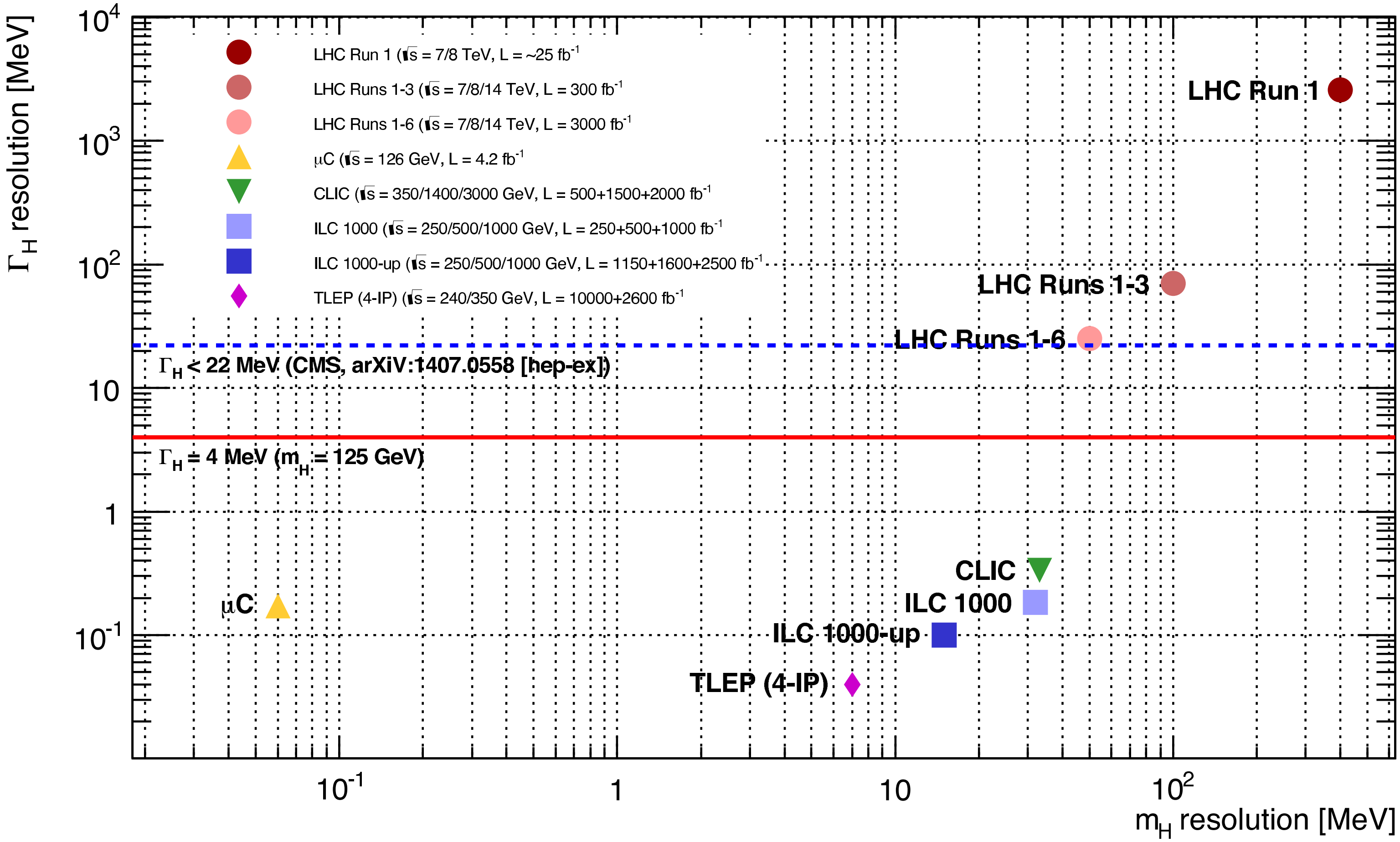
Snowmass Higgs mass and width resolution projections

Randle-Conde 1504.04302, based on Dawson et al 1310.8361



Theory calculations will need to keep up with future experimental results.

Ultimate goal: make pure theory uncertainties completely insignificant, so that all errors can be blamed on experimentalists.

This talk:

- “Higgs boson mass in the Standard Model at two-loop order and beyond” with D.G. Robertson, 1407.4336

  State-of-the-art calculation of Standard Model Higgs boson mass. Public computer code SMH implements results.

  See also Bezrukov, Kalmykov, Kniehl, Shaposhnikov 1205.2893 and Degrassi, Di Vita, Elias-Miro, Espinosa, Giudice, Isidori, Strumia 1205.6497 and Buttazzo, Degrassi, Giardino, Giudice, Sala, Salvio, Strumia 1307.3536

- “Z boson pole mass at two-loop order in the pure MS-bar scheme”, 1505.04833

- “Pole mass of the W boson at two-loop order in the pure MS-bar scheme”, 1503.03782

The pure $\overline{\text{MS}}$ scheme:

- Input parameters are λ, g, g', g_3, y_t, and the Higgs VEV v.
- Output parameters are pole masses M_h, M_W, M_Z, M_t, and other parameters G_μ, $\ldots$

This is an alternative to the on-shell and hybrid schemes, which use some or all of G_μ, $\Delta\alpha$, M_Z, Γ_Z, α_S, M_t, M_h as the input parameters. Calculations in on-shell scheme have already gone beyond 2-loop order. See other talks for (many) references.

Why should we want calculations in an alternative scheme?

“Let a hundred flowers bloom”

–Chairman Mao

If nothing else, an additional handle on theoretical error estimates.

As a matter of opinion, I believe the pure $\overline{\text{MS}}$ scheme is conceptually simpler, and in principle may be more easily pushed to higher orders, and to extensions of the Standard Model.

Principle and practice are two different things...

Definition of the Higgs VEV $v(Q)$: the minimum of the effective potential in Landau gauge.

Current state-of-the-art of the Landau gauge Standard Model effective potential:

- 2-loop: Ford, Jack and Jones hep-ph/0111190. (Paper actually from 1992.)
- 3-loop QCD and top Yukawa: SPM 1310.7553

  Terms proportional to $g_3^4 y_t^4$ and $g_3^2 y_t^6$ and y_t^8.

  These contributions change VEV by about 350 MeV, depending on choice of Q.
- Resummation of Goldstone boson contributions: SPM 1406.2355,

  J. Elias-Miro, J. R. Espinosa and T. Konstandin 1406.2652

  Conceptually significant, numerically small.

  Leads to much simpler formulas!

At least two other definitions of the VEV are commonly in use:

- VEV = $v_{\rm tree}$ = minimum of the **tree-level** potential. Drawbacks:
  - Need to include tadpole diagrams.
  - Expansion parameter for top loops is $\dfrac{N_c y_t^4}{16\pi^2 \lambda}$ rather than $\dfrac{N_c y_t^2}{16\pi^2}$. Converges more slowly.

- VEV = value that makes **Feynman gauge** Higgs tadpole vanish.

  Drawback:
  - Feynman gauge effective potential is much more complicated, not even known at 2-loop order.

Care is needed when comparing results of different groups.

The complex pole mass

$$s_{\text{pole}} = M^2 - i\Gamma M$$

is a physical observable. Does not depend on gauge-fixing or renormalization. However, for $V = W, Z$, the real part is slightly smaller than the Breit-Wigner masses that are usually quoted by experiment:

$$M_V = M_V^{\text{exp}}(1 - \Gamma_V^2/2M_V^2 + \ldots)$$

So, the real parts of the pole masses are, experimentally:

$$\begin{aligned} M_Z &= M_Z^{\text{exp}} - 34.1\,\text{MeV} = 91.1535 \pm 0.0021\,\text{GeV}, \\ M_W &= M_W^{\text{exp}} - 27\,\text{MeV} = 80.358 \pm 0.015\,\text{GeV}. \end{aligned}$$

2-loop Higgs pole mass

Obtained from the 2-loop self-energy function:

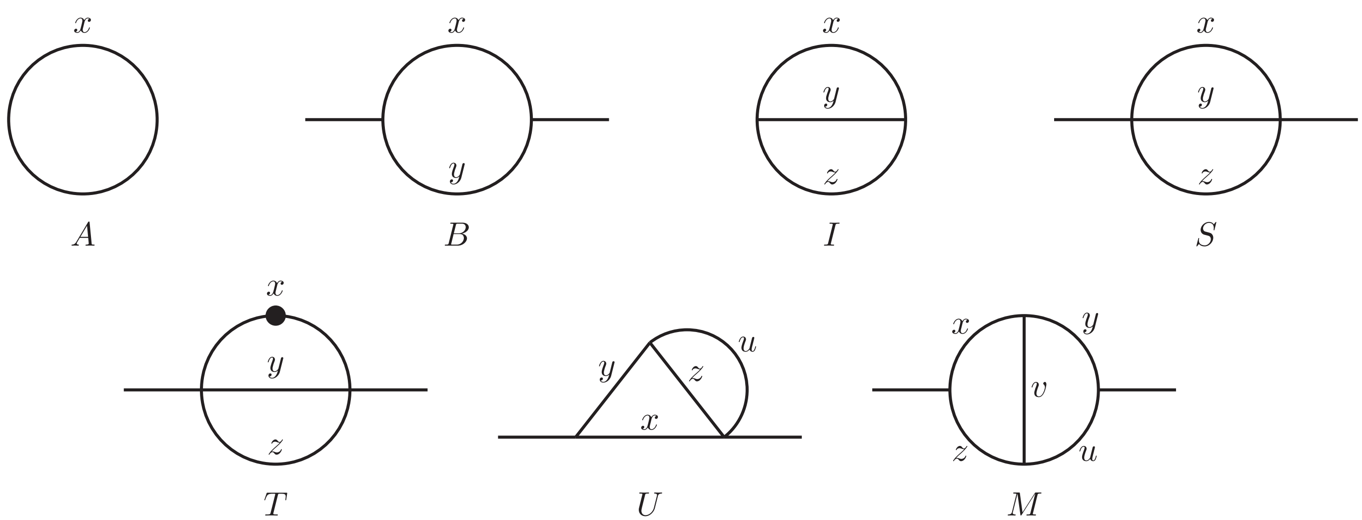
$$\Pi(s) = \frac{1}{16\pi^2}\Pi^{(1)}(s) + \frac{1}{(16\pi^2)^2}\Pi^{(2)}(s)$$

Sum of all 1-particle irreducible 2-point Feynman diagrams with $d = 4 - 2\epsilon$.

No counterterm diagrams! Instead, calculate in terms of bare quantities λ_B, y_{tB}, g_B, g'_B, g_{3B}, m^2_B, v_B. Then, just re-express in terms of $\overline{\text{MS}}$ quantities.

No tadpole diagrams! They sum to 0 automatically, by the definition of the VEV. Must use Landau gauge only.

Results are reduced to the Tarasov (hep-ph/9703319) basis of scalar integrals:



x, y, z, u, v are squared masses.

These are in turn evaluated by the TSIL computer library, SPM and D.G. Robertson hep-ph/0501132, using the differential equations methods.

Complete list of basis integrals needed:

$$I^{(1)} = \{B(t,t),\ B(h,h),\ B(W,W),\ B(Z,Z),\ A(t),\ A(h),\ A(W),\ A(Z)\}$$

$$\begin{aligned} I^{(2)} = \{ & M(h,h,h,h,h),\ U(h,h,h,h),\ S(h,h,h),\ M(h,Z,h,Z,Z),\ U(h,h,Z,Z), \\ & M(W,W,W,W,h),\ U(W,W,W,h),\ S(h,W,W),\ T(W,W,h),\ M(Z,Z,Z,Z,h), \\ & U(Z,Z,Z,h),\ S(h,Z,Z),\ T(Z,Z,h),\ M(W,W,W,W,Z),\ U(W,W,W,Z), \\ & S(W,W,Z),\ T(W,W,Z),\ T(Z,W,W),\ M(W,Z,W,Z,W),\ U(Z,Z,W,W), \\ & M(h,W,h,W,W),\ U(h,h,W,W),\ M(t,t,t,t,Z),\ U(t,t,t,Z),\ S(t,t,Z), \\ & T(t,t,Z),\ T(Z,t,t),\ M(t,t,t,t,h),\ U(t,t,t,h),\ S(h,t,t),\ T(t,t,h), \\ & M(t,Z,t,Z,t),\ U(Z,Z,t,t),\ M(t,h,t,h,t),\ U(h,h,t,t),\ M(t,W,t,W,0), \\ & U(W,W,0,t),\ U(t,t,0,W),\ S(0,t,W),\ T(W,0,t),\ T(t,0,W),\ M(t,t,t,t,0), \\ & T(t,0,t),\ \overline{T}(0,t,t),\ M(W,W,W,W,0),\ T(W,0,W),\ \overline{T}(0,W,W),\ U(W,W,0,0), \\ & S(0,0,W),\ T(W,0,0),\ U(Z,Z,0,0),\ S(0,0,Z),\ T(Z,0,0),\ I(h,h,h),\ I(t,t,Z), \\ & I(h,t,t),\ I(W,W,Z),\ I(h,W,W),\ I(h,Z,Z),\ I(0,t,W),\ I(0,h,W), \\ & I(0,h,Z),\ I(0,W,Z),\ I(0,0,W),\ I(0,0,Z),\ I(0,0,h),\ I(0,0,t)\}. \end{aligned}$$

`TSIL`: the 38 integrals in red have to be done numerically. The others reduce to polylogs. All necessary integrals obtained in a fraction of a second (total) on modern hardware, with relative accuracy $< 10^{-10}$.

Final result for 2-loop pole mass:

$$M_h^2 - i\Gamma_h M_h = 2\lambda v^2 + \frac{1}{16\pi^2}\Delta^{(1)}_{M_h^2} + \frac{1}{(16\pi^2)^2}\left[\Delta^{(2),\mathrm{QCD}}_{M_h^2} + \Delta^{(2),\mathrm{non-QCD}}_{M_h^2}\right],$$

This is a function of: v, λ, y_t, g, g', g_3, Q.

Explicit 1-loop part:

$$\begin{aligned}\Delta^{(1)}_{M_h^2} = \; & 3y_t^2(4t-s)B(t,t) - 18\lambda^2 v^2 B(h,h) \\ & + \frac{1}{2}(g^2+g'^2)\left[(s-3Z-s^2/4Z)B(Z,Z) - sA(Z)/2Z + 2Z\right] \\ & + g^2\left[(s-3W-s^2/4W)B(W,W) - sA(W)/2W + 2W\right],\end{aligned}$$

For s, plug in real part M_h^2, and solve by iteration.

Imaginary part $i\Gamma_h M_h$ is numerically negligible; makes a difference of order 1 MeV in M_h. The same is true for the bottom Yukawa coupling.

Explicit 2-loop QCD part:

$$\Delta^{(2),\mathrm{QCD}}_{M_h^2} = g_3^2 y_t^2 \Big[8(4t-s)(s-2t)M(t,t,t,t,0) + (36s-168t)T(t,0,t)$$
$$+16(s-4t)\overline{T}(0,t,t) + 14sB(t,t)^2 + (-176+36s/t)A(t)B(t,t)$$
$$+(80t-36s)B(t,t) - 28A(t)^2/t + 80t - 17s \Big].$$

2-loop non-QCD part is much more complicated:

$$\Delta^{(2),\mathrm{non-QCD}}_{M_h^2} = \sum_i c_i^{(2)} I_i^{(2)} + \sum_{j\leq k} c_{j,k}^{(1,1)} I_j^{(1)} I_k^{(1)} + \sum_j c_j^{(1)} I_j^{(1)} + c^{(0)}.$$

The coefficients $c_i^{(2)}$ and $c_{j,k}^{(1,1)}$ and $c_j^{(1)}$ and $c^{(0)}$ are available in electronic form in a file called `coefficients.txt`. They are ratios of polynomials in v, λ, y_t, g, and g'.

Leading 3-loop contributions to M_h

In the approximation $M_h^2 \ll M_t^2$, the self-energy function is given by derivatives of the effective potential, and

$$\delta M_h^2 \quad = \quad \left[\frac{\partial^2}{\partial v^2} - \frac{1}{v}\frac{\partial}{\partial v}\right] \delta V_{\rm eff}.$$

Using 3-loop resummed effective potential involving top quark:

$$\Delta M_h^2 \quad = \quad \frac{1}{(16\pi^2)^3}\left[\Delta_{M_h^2}^{(3),\text{leading QCD}} + \Delta_{M_h^2}^{(3),\text{leading non-QCD}}\right]$$

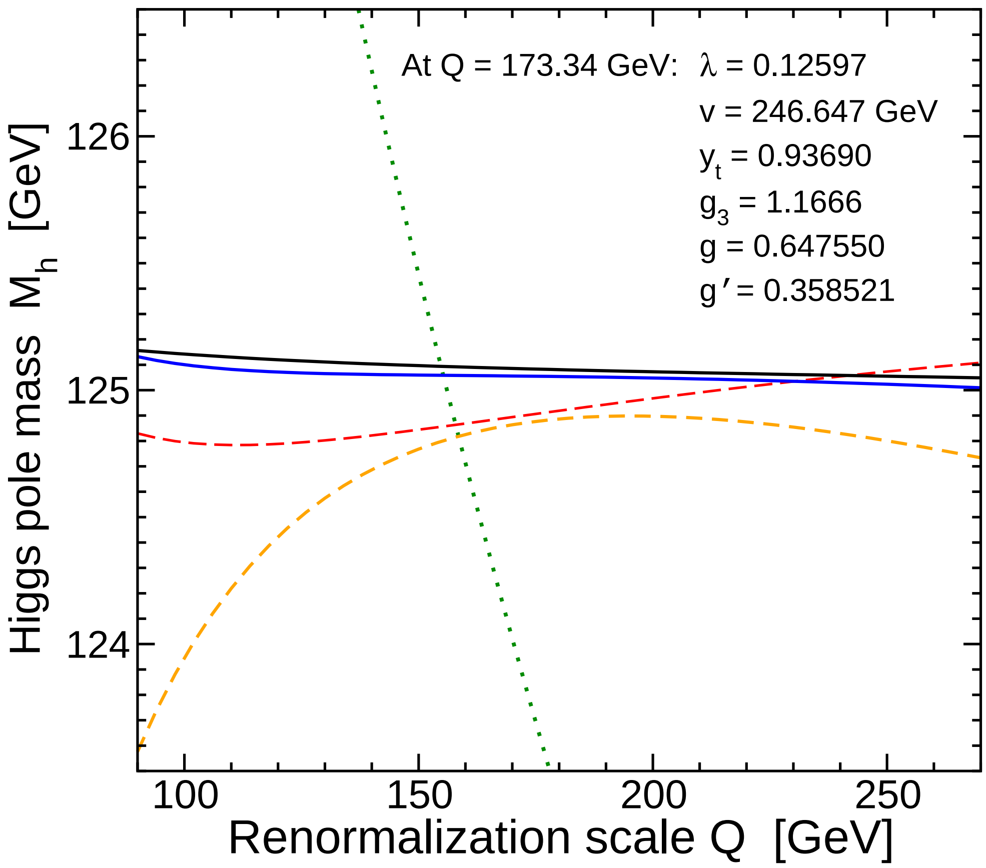
where

$$\begin{aligned}\Delta_{M_h^2}^{(3),\text{leading QCD}} \quad &= \quad g_3^4 y_t^2 t \left[248.1 + 839.2\overline{\ln}(t) + 160\overline{\ln}^2(t) - 736\overline{\ln}^3(t)\right] \\ &\quad + g_3^2 y_t^4 t \left[2764.4 + 1283.7\overline{\ln}(t) - 360\overline{\ln}^2(t) + 240\overline{\ln}^3(t)\right],\end{aligned}$$

and similarly for non-QCD part.

Here $\overline{\ln}(X) \equiv \ln(X/Q^2)$.

RG dependence of computed pole mass M_h. Run all input parameters from M_t to Q.



Green dotted: tree-level

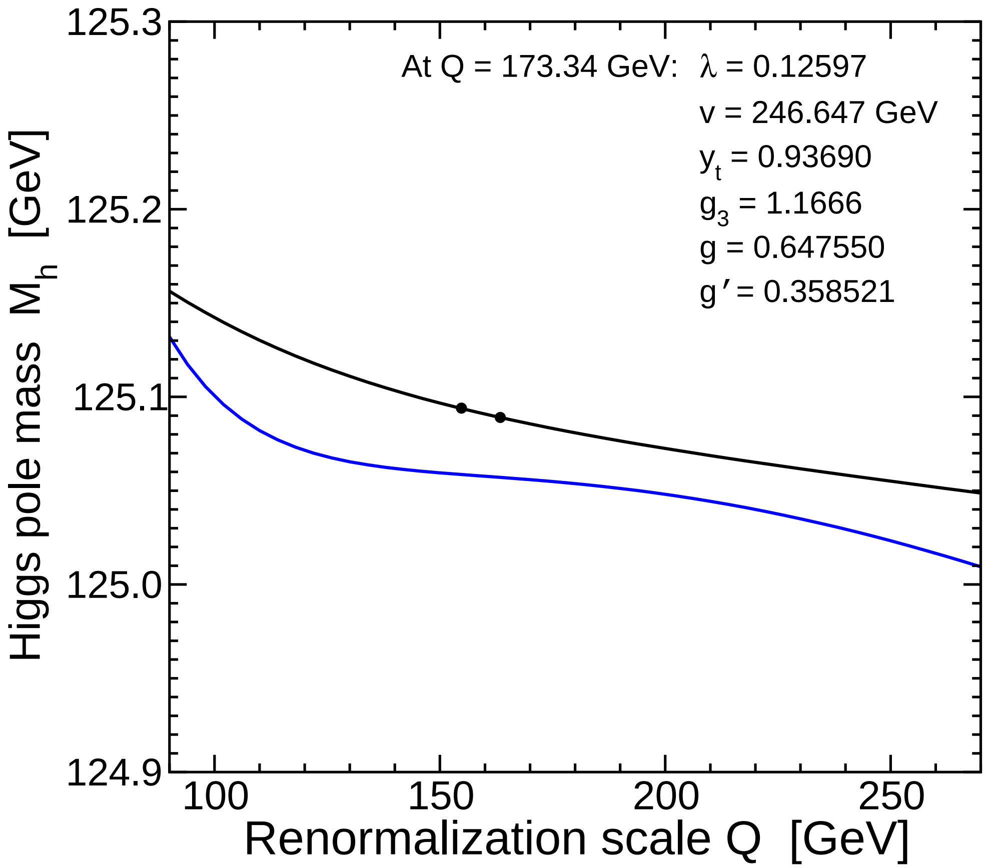
Orange short dashed: 1-loop

Red long-dashed: 1-loop + 2-loop QCD

Blue solid: 2-loop

Black solid: 2-loop + leading 3-loop

Closeup of RG dependence of computed pole mass M_h:



Blue: 2-loop

Black: 2-loop + leading 3-loop

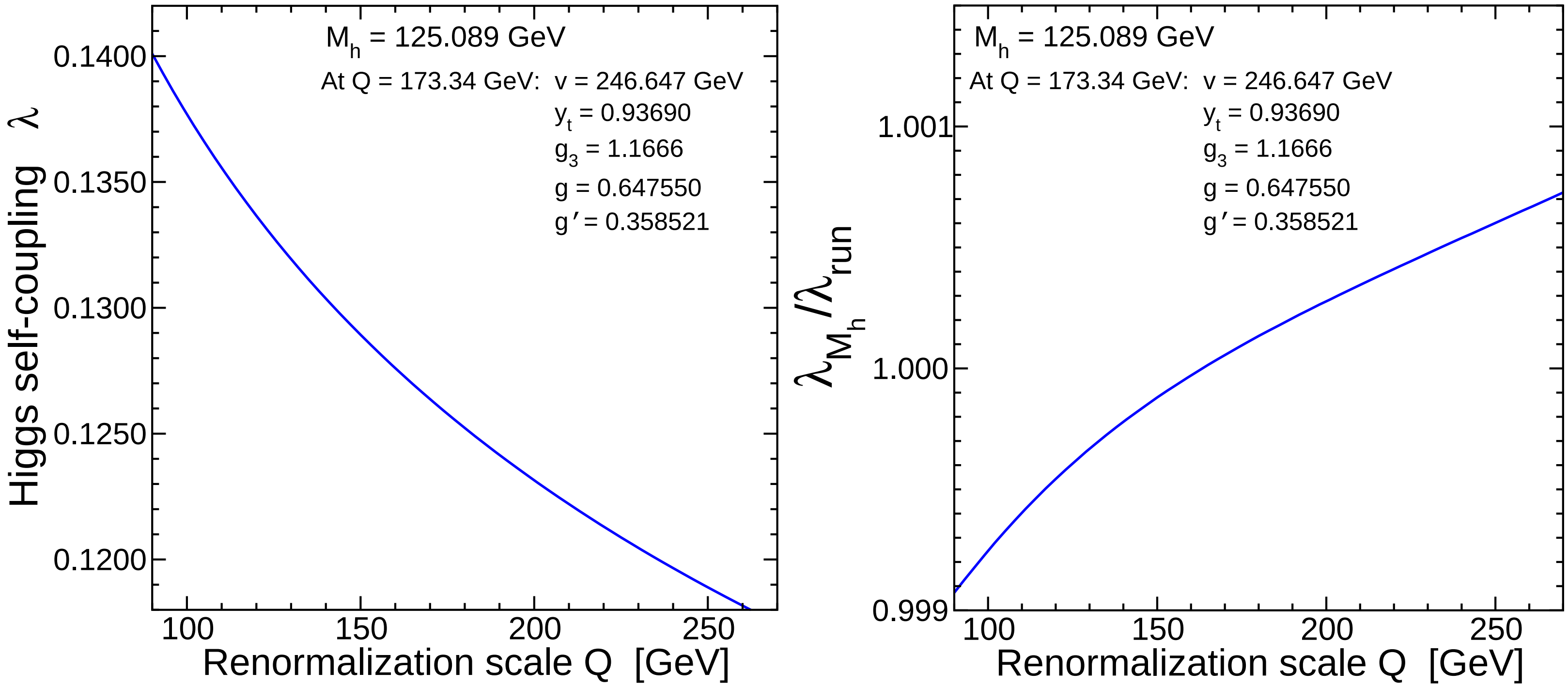
Left dot: scale Q at which $M_h^2 = 2\lambda v^2$

Right dot: scale $Q^2 = t = y_t^2 v^2/2$

From RG scale dependence, theory error is arguably well under 100 MeV.

Does not include parametric errors from experimental M_t, α_S, etc.

The inverse question: given M_h, what is the self-coupling λ ?



Left panel: $\lambda_{M_h}(Q)$ as determined from the fixed pole mass M_h, calculated at Q.

Right panel: Compare $\lambda_{M_h}(Q)$ obtained at Q to $\lambda_{\text{run}}(Q)$ obtained by running it from M_t to Q.

Scale dependence is well under 0.1%, for a reasonable range of Q.

Public software code implementation: SMH (with D.G. Robertson)

Library functions for inclusion in your own C, C++ or Fortran code:

- SMH_RGrun runs λ, y_t, g_3, g, g', v, m^2 from scale $Q_{\rm initial}$ to $Q_{\rm final}$.
- SMH_Find_vev minimizes $V_{\rm eff}$ to find v , given m^2, λ, y_t, g_3, g, g', Q.
- SMH_Find_m2 minimizes $V_{\rm eff}$ to find m^2 , given v, λ, y_t, g_3, g, g', Q.
- SMH_Find_Mh Computes M_h , given λ, v, y_t, g_3, g, g', Q.
- SMH_Find_lambda Computes λ , given M_h, v, y_t, g_3, g, g', Q.

Command line versions of these also exist:

```
$ ./calc_Mh 0.126 246.6 0.937 1.167 0.648 0.359 173.3 3
(* SMH(iggs) Version 1.01 *)

Mh(loops = 3.0) = 125.074162

Total calculation time (s): 0.382756
```

The loop order can be chosen at run time from:

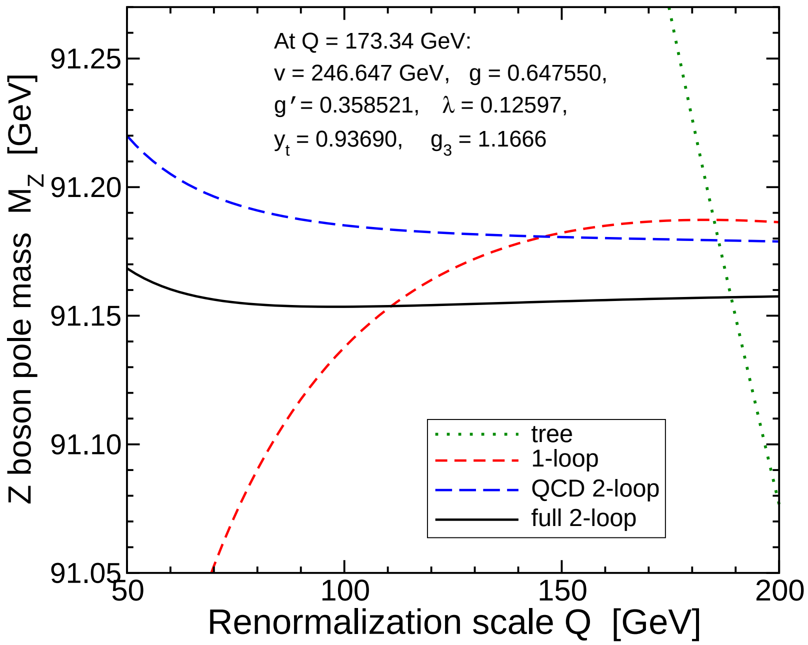
```
0    tree level
1    1-loop
1.5  1-loop plus 2-loop QCD
2    2-loop
2.5  2-loop plus leading 3-loop QCD
3    2-loop plus leading 3-loop
```

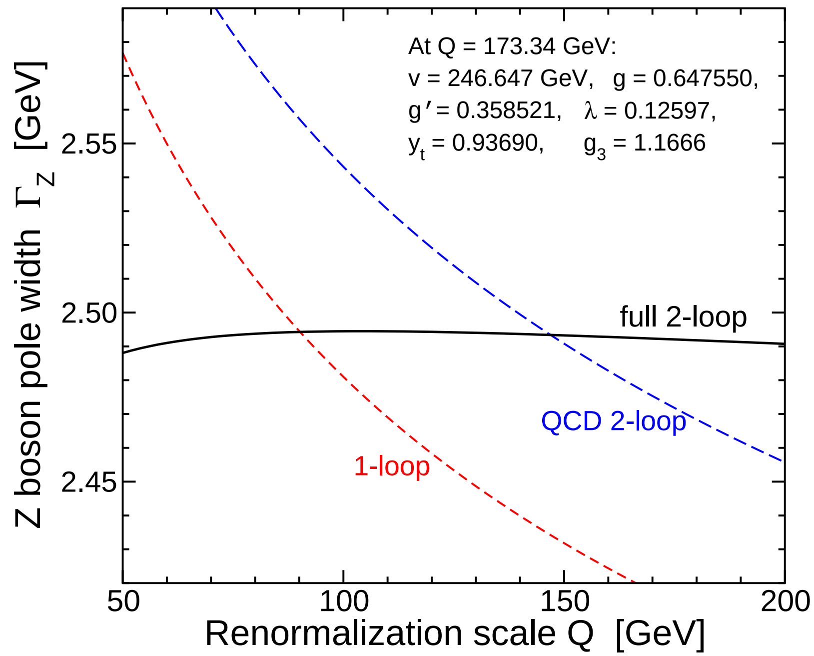
For much more information, see the `README.txt` file provided with SMH.

Coming soon: inclusion of Z, W, t pole masses in pure $\overline{\text{MS}}$ scheme.

In 1505.04833, did a similar calculation of the Z boson complex pole mass

$$M_Z^2 - i\Gamma_Z M_Z.$$



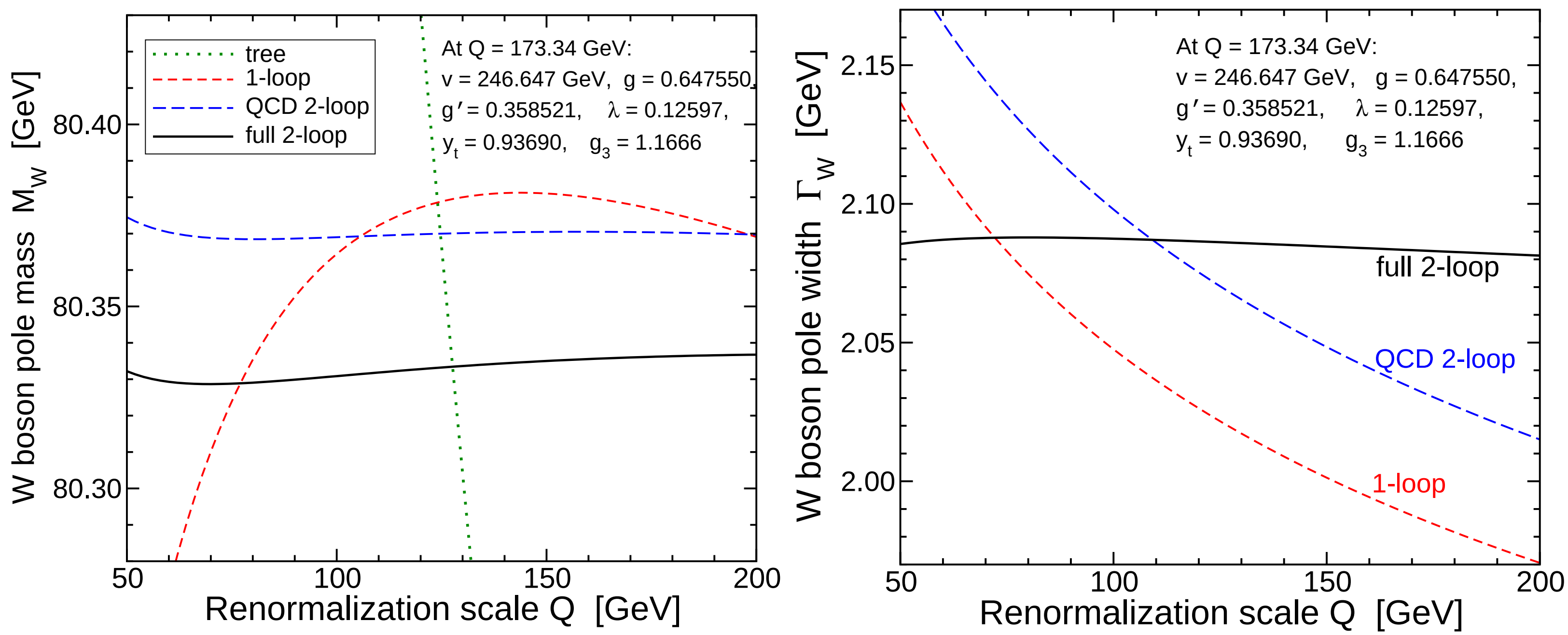


Recall that the usual experimental Breit-Wigner mass $M_Z^{\rm exp}$ is larger than this by 34.1 MeV, so with these benchmark parameters, $M_Z^{\rm exp} = 91.1876$ GeV.

Scale dependence of M_Z is $\pm$2 MeV from median value over the range 70 GeV $< Q <$ 200 GeV, but I believe the true theory error is likely worse.

In 1503.03782, did the same calculation for the W boson pole mass

$$M_W^2 - i\Gamma_W M_W.$$



Scale dependence for both M_W and Γ_W is roughly ± 4 MeV from their median values, but again I believe this is partly accidental, and true theory uncertainty is higher.

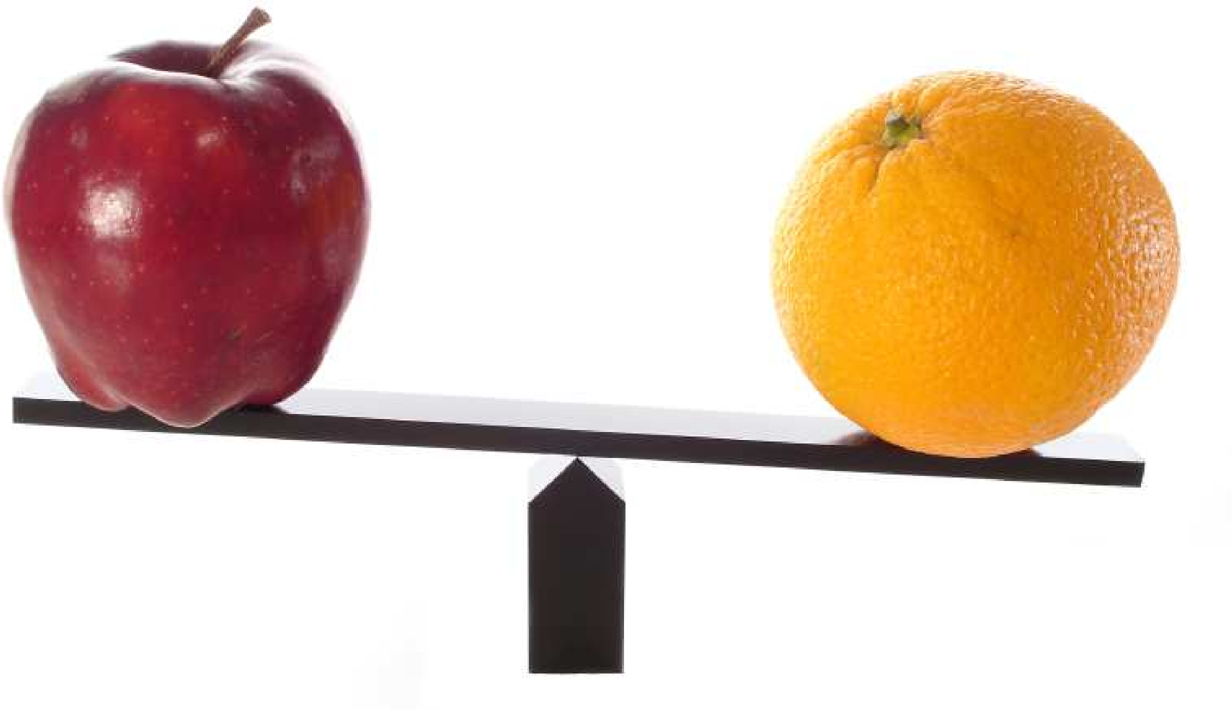
An amusing fact: the 2-loop QCD contributions have a larger **scale dependence**, but a smaller **magnitude**, than the 2-loop non-QCD effects.

This is partly, but not entirely, due to a 1% “fine-tuning”:

$$39 - 4\pi^2 = -0.478\ldots$$

in the leading non-logarithmic QCD term expanded in the limit of large top mass, which would naively be expected to give the largest contribution.

Comparisons to other multi-loop calculations of M_W ?



In on-shell or hybrid schemes, earlier calculations of W mass go beyond 2-loop order to include some 3-loop effects:

Awramik, Czakon, Freitas, Weiglein 0311148

DeGrassi, Gambino, Giardino 1411.7040

Uses **Feynman gauge** definition of VEV.

In the pure $\overline{\text{MS}}$ scheme, there are two earlier 2-loop calculations, but they are very difficult to compare to mine:

Jegerlehner, Kalmykov, Veretin 0105304, 0212319

Kniehl, Pikelner, Veretin 1503.02138

Both use VEV defined as minimum of **tree-level** potential. For large top masses, due to tadpole effects, their loop expansion parameter is effectively

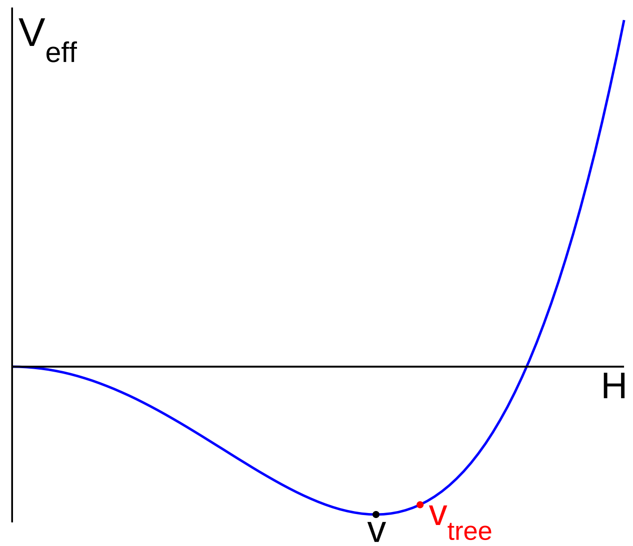
$$\frac{N_c y_t^4}{16\pi^2 \lambda}.$$

Boson pole mass results in pure $\overline{\text{MS}}$ scheme:

- M_h: Full 2-loop, plus 3-loop $\alpha_S^2\alpha_t$ and $\alpha_S\alpha_t^2$ and α_t^3 in the approximation $M_h^2/M_t^2 \ll 1$. Scale dependence: ± 50 MeV.
- M_W, M_Z Full 2-loop. Scale dependence: ± 4 MeV.
- True theory errors are likely larger.
- Perhaps adequate for LEP+LHC data, but not for FCCee (or other future lepton colliders).
- Need 3 loops, at least.
- Not obvious that 3-loop QCD-enhanced contributions will dominate over non-QCD. That did not happen at 2-loop order! QCD part has larger scale dependence but smaller magnitude.

Thank you!

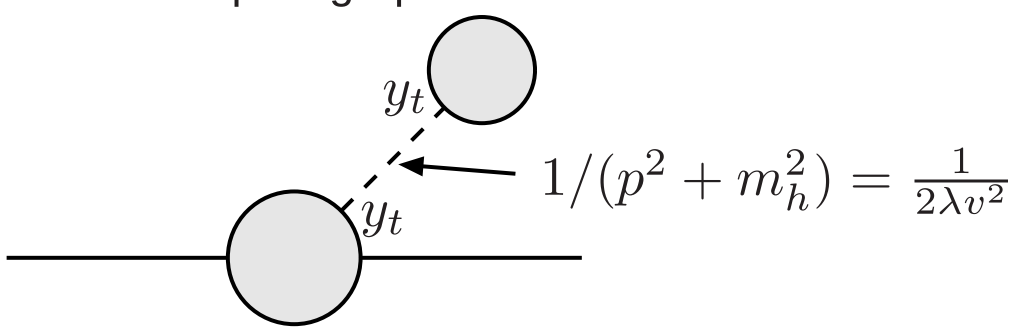
Backup slides:



v = true minimum of $V_{\rm eff}$, gives better perturbation expansion than expanding around $v_{\rm tree}$.

Condition for sum of tree tadpole + radiative tadpoles to vanish: $\frac{\partial}{\partial v} V_{\rm eff}(v) = 0$.

If one chooses to expand around $v_{\rm tree}$ instead, then one can maintain general gauge fixing, but must include tadpole graphs:



with expansion parameter $(N_c y_t^4/16\pi^2\lambda)^L$ at loop order L.

Complex pole squared mass

$$s_{\text{pole}} \equiv M_h^2 - i\Gamma_h M_h$$

is the solution of:

$$s_{\text{pole}} = m_B^2 + 3\lambda_B v_B^2 + \frac{1}{16\pi^2}\Pi^{(1)}(s_{\text{pole}}) + \frac{1}{(16\pi^2)^2}\Pi^{(2)}(s_{\text{pole}}).$$

In principle this is gauge invariant, but we compute it in Landau gauge, because only in that gauge do we know 2-loop relation of VEV to other parameters.

Expand the solution s_{pole} in a series in $1/\epsilon$.

Renormalizability says $1/\epsilon$ poles will cancel.

Relations between $\overline{\text{MS}}$ and bare quantities, for example:

$$\begin{aligned}
v_B^2 &= \mu^{-2\epsilon} v^2 \Big[1 + \frac{1}{16\pi^2}\frac{c_{1,1}^{\phi}}{\epsilon} + \frac{1}{(16\pi^2)^2}\Big(\frac{c_{2,2}^{\phi}}{\epsilon^2} + \frac{c_{2,1}^{\phi}}{\epsilon}\Big) + \ldots\Big],\\
\lambda_B &= \mu^{2\epsilon} \Big[\lambda + \frac{1}{16\pi^2}\frac{c_{1,1}^{\lambda}}{\epsilon} + \frac{1}{(16\pi^2)^2}\Big(\frac{c_{2,2}^{\lambda}}{\epsilon^2} + \frac{c_{2,1}^{\lambda}}{\epsilon}\Big) + \ldots\Big],\\
y_{tB} &= \mu^{\epsilon} \Big[y_t + \frac{1}{16\pi^2}\frac{c_{1,1}^{y_t}}{\epsilon} + \ldots\Big],\\
g_B &= \mu^{\epsilon} \Big[g + \frac{1}{16\pi^2}\frac{c_{1,1}^{g}}{\epsilon} + \ldots\Big], \qquad \text{etc.}
\end{aligned}$$

Dimensional regularization scale μ is related to $\overline{\text{MS}}$ scale Q by:

$$Q^2 = 4\pi e^{-\gamma_E} \mu^2$$

Counterterms c_{ij}^X are known already from beta functions and Higgs anomalous dimension. Cancellations of $1/\epsilon$ poles provides a check!

Examples:

$$
\begin{aligned}
c^{\phi}_{1,1} &= -3y_t^2 + \frac{9}{4}g^2 + \frac{3}{4}g'^2, \\
c^{\phi}_{2,2} &= 12g_3^2 y_t^2 - \frac{9}{4}y_t^4 - \frac{27}{8}y_t^2 g^2 - \frac{1}{8}y_t^2 g'^2 - \frac{33}{32}g^4 + \frac{27}{16}g^2 g'^2 + \frac{91}{32}g'^4, \\
c^{\phi}_{2,1} &= -10g_3^2 y_t^2 + \frac{27}{8}y_t^4 - \frac{45}{16}y_t^2 g^2 - \frac{85}{48}y_t^2 g'^2 + \frac{271}{64}g^4 \\
&\quad -\frac{9}{32}g^2 g'^2 - \frac{431}{192}g'^4 - 3\lambda^2, \\
c^{\lambda}_{1,1} &= -3y_t^4 + 6\lambda y_t^2 + 12\lambda^2 - \frac{9}{2}\lambda g^2 - \frac{3}{2}\lambda g'^2 + \frac{9}{16}g^4 + \frac{3}{8}g^2 g'^2 + \frac{3}{16}g'^4,
\end{aligned}
$$

etc.

Checks:

- All $1/\epsilon$ and $1/\epsilon^2$ terms cancel in $s_{\rm pole} = M_h^2 - i\Gamma_h M_h$.
- Some individual coefficients $c_i^{(2)}$ and $c_{j,k}^{(1,1)}$ are singular in the formal limits $g, g' \rightarrow 0$ or $\lambda \rightarrow 0$, but the whole expression is well-behaved, due to redundancy relations for basis integrals when squared mass arguments approach 0.
- Logs of m_G^2 cancel when Goldstone boson contributions resummed.
- Cancellations between Landau gauge vector propagators with poles at squared mass equal to 0 and the corresponding Goldstone propagators.
- Imaginary part $-i\Gamma_h M_h$ checks precisely with 3-body decay widths $\Gamma(h \rightarrow W f \overline{f}')$ and $\Gamma(h \rightarrow Z f \overline{f})$, from Keung and Marciano, 1984.

Checks (continued):

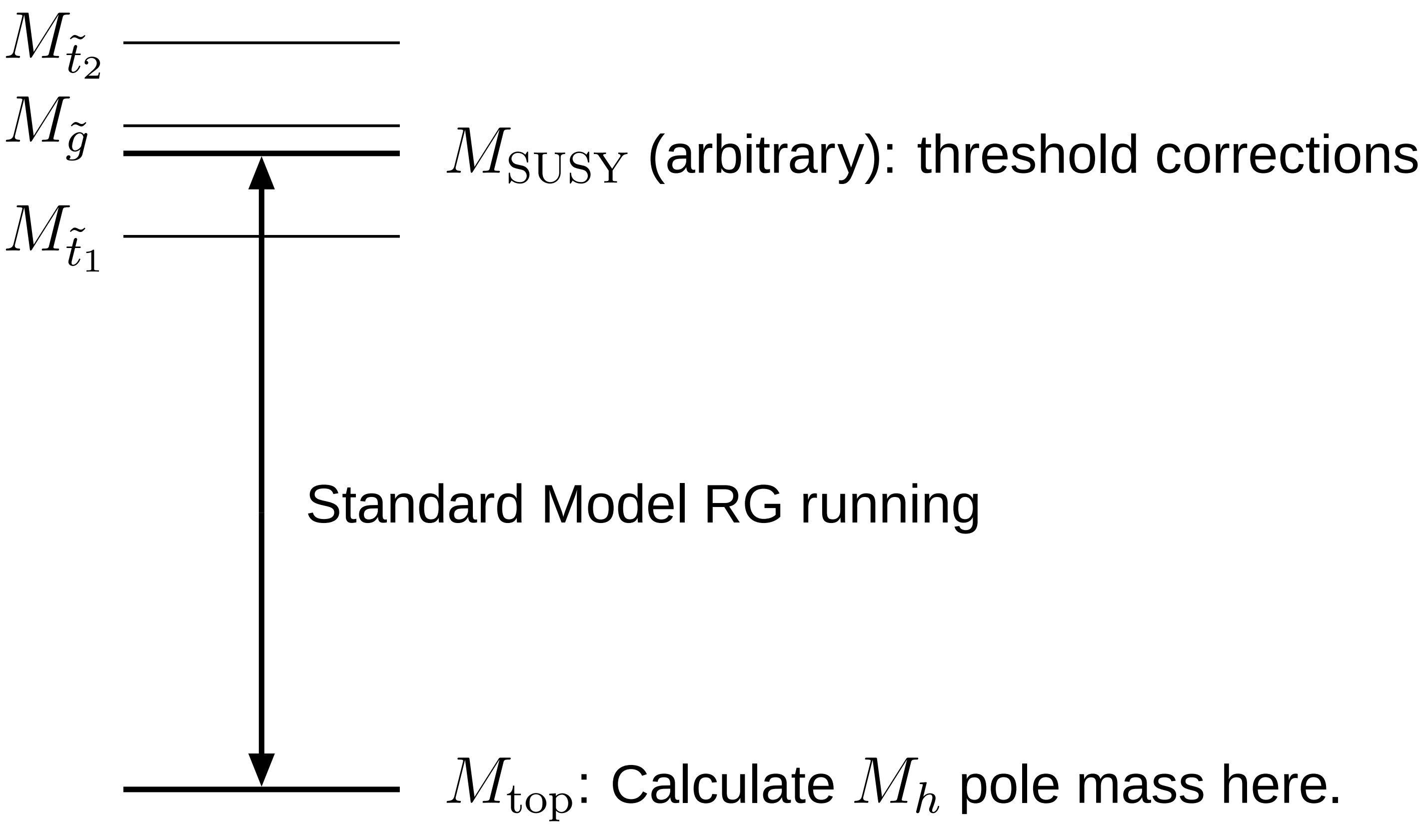
- M_h^2 is renormalization group scale invariant through terms of 2-loop order. In principle, equivalent to the $1/\epsilon$ check, but in practice tests the validity of many intermediate steps.

$$0 \;=\; Q\frac{d}{dQ}M_h^2 \quad = \quad \left[Q\frac{\partial}{\partial Q} - \gamma_\phi v\frac{\partial}{\partial v} + \sum_X \beta_X \frac{\partial}{\partial X}\right] M_h^2,$$

where $X = \{\lambda, y_t, g, g', g_3\}$.

If one puts an arbitrary coefficient in front of each Feynman diagram, the checks above are enough to fix most of them to be unity!

Motivation for SUSY people: the effective field theory approach to M_h



My opinion: this is clearly the best way forward for M_h in SUSY.

Modular approach, needs M_h computation **in the Standard Model**.

Might need modification, perhaps: