Electroweak precision observables in the SM: present & future

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1. **Electroweak precision observables**

2. **Current status of SM loop results**

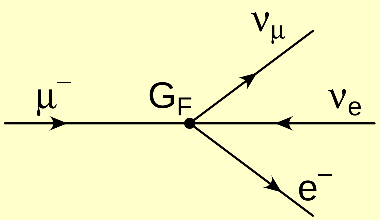
3. **Renormalization/parametrization**

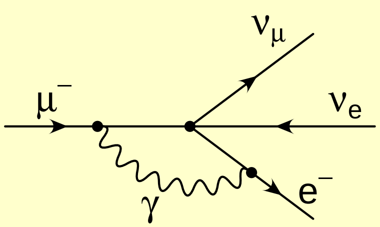
4. **Future projections**

Electroweak precision observables

W mass

μ decay in Fermi Model



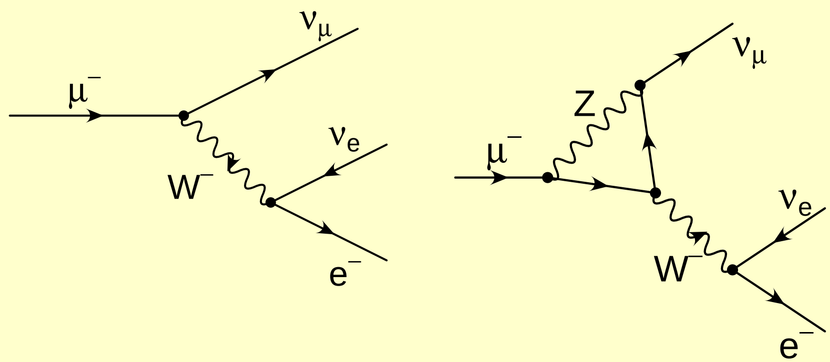


← QED corr. (2-loop)

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} F\left(\frac{m_e^2}{m_\mu^2}\right)(1 + \Delta q)$$

Ritbergen, Stuart '98
Pak, Czarnecki '08

μ decay in Standard Model



$$\frac{G_F^2}{\sqrt{2}} = \frac{e^2}{8 s_W^2 M_W^2}(1 + \Delta r)$$

electroweak corrections

Z-pole observables

- Deconvolution of initial-state QED radiation:

$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{\rm ini}(s,s') \otimes \sigma_{\rm hard}(s')$$

- Subtraction of γ-exchange, γ–Z interference, box contributions:

$$\sigma_{\rm hard} = \sigma_{\rm Z} + \sigma_\gamma + \sigma_{\gamma \rm Z} + \sigma_{\rm box}$$

- Z-pole contribution:

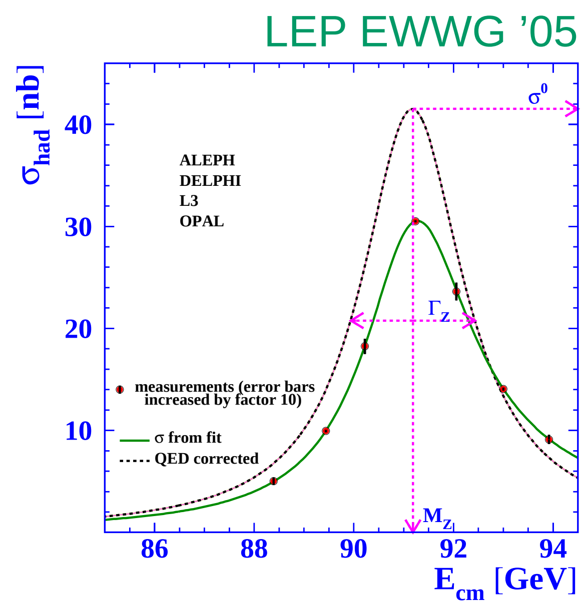
$$\sigma_{\rm Z} = \frac{R}{(s-\overline{M}_{\rm Z}^2)^2 + \overline{M}_{\rm Z}^2\overline{\Gamma}_{\rm Z}^2} + \sigma_{\rm non-res}$$

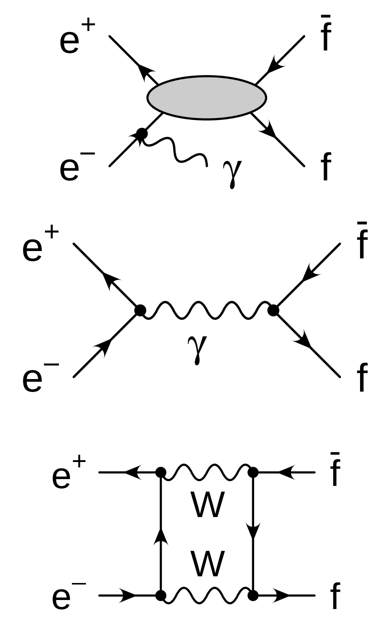
- In experimental analyses:

$$\sigma \sim \frac{1}{(s-M_{\rm Z}^2)^2 + s^2\Gamma_{\rm Z}^2/M_{\rm Z}^2}$$

$$\overline{M}_{\rm Z} = M_{\rm Z}\Big/\sqrt{1+\Gamma_{\rm Z}^2/M_{\rm Z}^2} \approx M_{\rm Z} - 34\ {\rm MeV}$$

$$\overline{\Gamma}_{\rm Z} = \Gamma_{\rm Z}\Big/\sqrt{1+\Gamma_{\rm Z}^2/M_{\rm Z}^2} \approx \Gamma_{\rm Z} - 0.9\ {\rm MeV}$$





Z-pole observables

Z width:

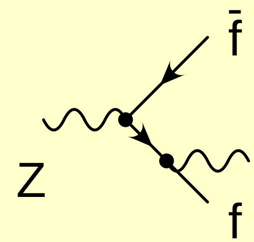
$$\overline{\Gamma}_Z = \frac{1}{\overline{M}_Z} \mathrm{Im}\,\Sigma_Z(s_0).$$

Optical theorem:

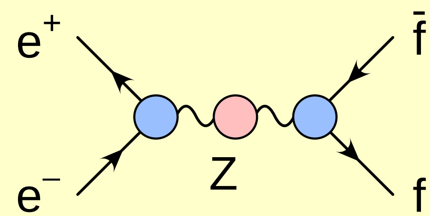
$$\overline{\Gamma}_Z = \sum_f \overline{\Gamma}_f, \qquad \overline{\Gamma}_f \approx \frac{N_c \overline{M}_Z}{12\pi}\left[\left(\mathcal{R}_V^f |g_V^f|^2 + \mathcal{R}_A^f |g_A^f|^2\right)\frac{1}{1+\mathrm{Re}\,\Sigma'_Z}\right]_{s=\overline{M}_Z^2}$$

$\mathcal{R}_V^f$, $\mathcal{R}_A^f$: Final-state QED/QCD radiation;
known to $\mathcal{O}(\alpha_s^4)$, $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha\alpha_s)$

Kataev '92
Chetyrkin, Kühn, Kwiatkowski '96
Baikov, Chetyrkin, Kühn, Rittinger '12



g_V^f, g_A^f, Σ'_Z: Electroweak corrections



Z-pole observables

Peak cross section:

$$\sigma^0_{\text{had}} = \sigma_Z(s = \overline{M}_Z^2)$$

(agrees with result from running-width BW with $s = M_Z^2$)

Explicit calculation:

$$\sigma^0_{\text{had}} = \frac{12\pi}{\overline{M}_Z^2} \sum_q \frac{\overline{\Gamma}_e \overline{\Gamma}_q}{\overline{\Gamma}_Z^2} (1 + \delta X)$$

Correction term first at NNLO:

Grassi, Kniehl, Sirlin '01
Freitas '13

$$\delta X_{(2)} = -(\text{Im}\,\Sigma'_{Z(1)})^2 - 2\overline{\Gamma}_Z \overline{M}_Z \,\text{Im}\,\Sigma''_{Z(1)}$$

Branching ratios:

e.g. $R_b = \Gamma_b / \Gamma_{\text{had}}$ (probes third quark generation)

Z-pole observables

Effective weak mixing angle:

Z-pole asymmetries:

$$A_{\mathsf{FB}}^{f} \equiv \frac{\sigma(\theta < \frac{\pi}{2}) - \sigma(\theta > \frac{\pi}{2})}{\sigma(\theta < \frac{\pi}{2}) + \sigma(\theta > \frac{\pi}{2})} = \frac{3}{4}\mathcal{A}_e\mathcal{A}_f$$

$$A_{\mathsf{LR}} \equiv \frac{\sigma(\mathcal{P}_e > 0) - \sigma(\mathcal{P}_e < 0)}{\sigma(\mathcal{P}_e > 0) + \sigma(\mathcal{P}_e < 0)} = \mathcal{A}_e$$

$$\mathcal{A}_f = 2\frac{g_{Vf}/g_{Af}}{1 + (g_{Vf}/g_{Af})^2} = \frac{1 - 4|Q_f|\sin^2\theta_{\mathsf{eff}}^{f}}{1 - 4|Q_f|\sin^2\theta_{\mathsf{eff}}^{f} + 8(|Q_f|\sin^2\theta_{\mathsf{eff}}^{f})^2}$$

Most precisely measured for $f = \ell$ (also $f = b, c$)

Current status of SM loop results

Known corrections to Δr, $\sin^2\theta^f_{\rm eff}$, g_{Vf}, g_{Af}:

- Complete NNLO corrections (Δr, $\sin^2\theta^\ell_{\rm eff}$) Freitas, Hollik, Walter, Weiglein '00
  Awramik, Czakon '02; Onishchenko, Veretin '02
  Awramik, Czakon, Freitas, Weiglein '04; Awramik, Czakon, Freitas '06
  Hollik, Meier, Uccirati '05,07; Degrassi, Gambino, Giardino '14

- "Fermionic" NNLO corrections (g_{Vf}, g_{Af}) Czarnecki, Kühn '96
  Harlander, Seidensticker, Steinhauser '98
  Freitas '13,14

- Partial 3/4-loop corrections to ρ/T-parameter
  $\mathcal{O}(\alpha_t\alpha_s^2)$, $\mathcal{O}(\alpha_t^2\alpha_s)$, $\mathcal{O}(\alpha_t\alpha_s^3)$ Chetyrkin, Kühn, Steinhauser '95
  Faisst, Kühn, Seidensticker, Veretin '03
  Boughezal, Tausk, v. d. Bij '05
  Schröder, Steinhauser '05; Chetyrkin et al. '06
  $(\alpha_t \equiv \frac{y_t^2}{4\pi})$ Boughezal, Czakon '06

Current uncertainties

	Experiment	Theory error	Main source
M_{W}	80.385 ± 0.015 MeV	4 MeV	α^3, $\alpha^2\alpha_{\mathrm{s}}$
Γ_Z	2495.2 ± 2.3 MeV	0.5 MeV	α^2_{bos}, α^3, $\alpha^2\alpha_{\mathrm{s}}$, $\alpha\alpha_{\mathrm{s}}^2$
σ^0_{had}	41540 ± 37 pb	6 pb	α^2_{bos}, α^3, $\alpha^2\alpha_{\mathrm{s}}$
$R_b \equiv \Gamma^b_{\mathrm{Z}}/\Gamma^{\mathrm{had}}_{\mathrm{Z}}$	0.21629 ± 0.00066	0.00015	α^2_{bos}, α^3, $\alpha^2\alpha_{\mathrm{s}}$
$\sin^2\theta^{\ell}_{\mathrm{eff}}$	0.23153 ± 0.00016	4.5×10^{-5}	α^3, $\alpha^2\alpha_{\mathrm{s}}$

Methods for theory error estimates:

- Parametric factors, $i.e.$ factors of α, N_c, N_f, ...
- Geometric progression, $e.g.$ $\frac{\mathcal{O}(\alpha^3)}{\mathcal{O}(\alpha^2)} \sim \frac{\mathcal{O}(\alpha^2)}{\mathcal{O}(\alpha)}$
- Renormalization scale dependence (often underestimates error)
- Renormalization scheme dependence (may underestimate error)

Example: Error estimation for Γ_Z

- Goemetric perturbative series

$\alpha_t = \alpha m_t^2$

$$\mathcal{O}(\alpha^3) - \mathcal{O}(\alpha_t^3) \sim \frac{\mathcal{O}(\alpha^2) - \mathcal{O}(\alpha_t^2)}{\mathcal{O}(\alpha)} \mathcal{O}(\alpha^2) \sim 0.26 \text{ MeV}$$

$$\mathcal{O}(\alpha^2\alpha_s) - \mathcal{O}(\alpha_t^2\alpha_s) \sim \frac{\mathcal{O}(\alpha^2) - \mathcal{O}(\alpha_t^2)}{\mathcal{O}(\alpha)} \mathcal{O}(\alpha\alpha_s) \sim 0.30 \text{ MeV}$$

$$\mathcal{O}(\alpha\alpha_s^2) - \mathcal{O}(\alpha_t\alpha_s^2) \sim \frac{\mathcal{O}(\alpha\alpha_s) - \mathcal{O}(\alpha_t\alpha_s)}{\mathcal{O}(\alpha)} \mathcal{O}(\alpha\alpha_s) \sim 0.23 \text{ MeV}$$

$$\mathcal{O}(\alpha\alpha_s^3) - \mathcal{O}(\alpha_t\alpha_s^3) \sim \frac{\mathcal{O}(\alpha\alpha_s) - \mathcal{O}(\alpha_t\alpha_s)}{\mathcal{O}(\alpha)} \mathcal{O}(\alpha\alpha_s^2) \sim 0.035 \text{ MeV}$$

$$\mathcal{O}(\alpha_{bos}^2) \sim \mathcal{O}(\alpha_{bos})^2 \sim 0.1 \text{ MeV}$$

- Parametric prefactors:

$$\mathcal{O}(\alpha_{bos}^2) \sim \Gamma_Z \alpha^2 \sim 0.1 \text{ MeV}$$

$$\mathcal{O}(\alpha\alpha_s^2) - \mathcal{O}(\alpha_t\alpha_s^2) \sim \frac{\alpha n_{lq}}{\pi} \alpha_s^2 \sim 0.29 \text{ MeV}$$

Total: $\delta\Gamma_Z \approx 0.5$ MeV

Example: Error estimation for M_W

- Renormalization scheme dependence:

  a) Uncertainty of $\mathcal{O}(\alpha^2)$ corrections beyond leading $\alpha^2 m_\mathsf{t}^4$ and $\alpha^2 m_\mathsf{t}^2$ from comparison of $\overline{\mathsf{MS}}$ and OS schemes: Degrassi, Gambino, Sirlin '96

  $\delta M_\mathsf{W} \sim 2$ MeV (for $M_\mathsf{H} \sim 100$ GeV)

  Actual remaining $\mathcal{O}(\alpha^2)$ corrections: Freitas, Hollik, Walter, Weiglein '00

  $\delta M_\mathsf{W} \sim 3$ MeV (for $M_\mathsf{H} \sim 100$ GeV)

  b) Estimate of missing $\mathcal{O}(\alpha^3)$ corrections from comparison of $\overline{\mathsf{MS}}$ and OS results: Awramik, Czakon, Freitas, Weiglein '03; Degrassi, Gambino, Giardino '14

  $\delta M_\mathsf{W} \sim 4...5$ MeV (after accounting for $\mathcal{O}(\alpha_\mathsf{t}\alpha_\mathsf{s}^3)$ corrections)

  $\longrightarrow$ Saturates previous δM_W estimate!

  Note: Differences in (implicitly) resummed higher-order contributions

Renormalization/parametrization

Renormalization

Use of $\overline{\text{MS}}$ renormalization for m_t reduces h.o. QCD corrections of $\mathcal{O}(\alpha_\text{t}\alpha_\text{s}^n)$:

loops $(n{+}1)$	$\Delta\rho^{\overline{\text{MS}}}_{(n)} \Big/ \left(\frac{3G_F\overline{m}_\text{t}^2}{8\sqrt{2}\pi^2}\right)$	$\Delta\rho^{\text{OS}}_{(n)} \Big/ \left(\frac{3G_F m_\text{t}^2}{8\sqrt{2}\pi^2}\right)$	
2	$-0.193\left(\frac{\alpha_\text{s}}{\pi}\right)$	$-3.970\left(\frac{\alpha_\text{s}}{\pi}\right)$	Djouadi, Verzegnassi '87 Kniehl '90
3	$-2.860\left(\frac{\alpha_\text{s}}{\pi}\right)^2$	$-14.59\left(\frac{\alpha_\text{s}}{\pi}\right)^2$	Avdeev, Fleischer, et al. '94 Chetyrkin, Kühn, Steinhauser '95
4	$-1.680\left(\frac{\alpha_\text{s}}{\pi}\right)^3$	$-93.15\left(\frac{\alpha_\text{s}}{\pi}\right)^3$	Schöder, Steinhauser '05 Chetyrkin, Faisst, Kühn, et al. '06 Boughezal, Czakon '06

No clear pattern of this kind known for $\mathcal{O}(\alpha^n)$

$\longrightarrow$ Only few results available that allow direct comparison

e.g. Faisst, Kühn, Seidensticker, Veretin '03

Parametrization: Resummation

Parametrization of perturbation series: $\boldsymbol{\alpha}$ **vs.** $\boldsymbol{G_F}$**?**

G_F can resum some leading one-loop terms

$$\Delta\alpha \equiv 1 - \frac{\alpha(0)}{\alpha(M_Z)} \approx 0.059 \qquad\qquad \Delta\rho = \frac{3\alpha}{16\pi s^2 c^2} \frac{m_t^2}{M_Z^2}$$

But: Strong cancellations between $\Delta\alpha$ and $\Delta\rho$ terms beyond one-loop:

$$\begin{aligned} \Delta r_{\text{res}}^{(3)} &= (\Delta\alpha)^3 \quad -3(\Delta\alpha)^2\left(\tfrac{c^2}{s^2}\Delta\rho\right) \quad +6(\Delta\alpha)\left(\tfrac{c^2}{s^2}\Delta\rho\right)^2 \quad -5\left(\tfrac{c^2}{s^2}\Delta\rho\right)^3 \\ &\approx (2.05 \quad -3.40 \quad +3.74 \quad -1.72) \times 10^{-4} \\ &= 0.68 \times 10^{-4} \end{aligned}$$

$\longrightarrow$ Not *the* numerically leading contribution anymore

Future projections

ILC: $\sqrt{s} \approx M_Z$ with 30 fb^{-1}

FCC-ee: $\sqrt{s} \approx M_Z$ with 4×3000 fb^{-1}

	Current exp.	ILC	FCC-ee	Current perturb.
M_W [MeV]	15	3–5	~ 1	4
Γ_Z [MeV]	2.3	~ 1	~ 0.1	0.5
R_b [10^{-5}]	66	15	$\lesssim 5$	15
$\sin^2 \theta^{\ell}_{eff}$ [10^{-5}]	16	1.3	0.3	4.5

$\longrightarrow$ Existing theoretical calculations adequate for LEP/SLC/LHC, but not ILC/FCC-ee!

Theory and parametric uncertainties

	ILC	FCC-ee	perturb. error with 3-loop[†]	Param. error ILC*	Param. error FCC-ee**
$M_{\rm W}$ [MeV]	3–5	~ 1	1	2.6	1
Γ_Z [MeV]	~ 1	~ 0.1	$\lesssim 0.2$	0.5	0.06
R_b $[10^{-5}]$	15	$\lesssim 5$	5–10	< 1	< 1
$\sin^2\theta^{\ell}_{\rm eff}$ $[10^{-5}]$	1.3	0.3	1.5	2	2

[†] **Theory scenario:** $\mathcal{O}(\alpha\alpha_{\rm s}^2)$, $\mathcal{O}(N_f\alpha^2\alpha_{\rm s})$, $\mathcal{O}(N_f^2\alpha^2\alpha_{\rm s})$

(N_f^n = at least n closed fermion loops)

Parametric inputs:

*** ILC:** $\delta m_t = 100$ MeV, $\delta\alpha_{\rm s} = 0.001$, $\delta M_{\rm Z} = 2.1$ MeV

****FCC-ee:** $\delta m_t \lesssim 50$ MeV, $\delta\alpha_{\rm s} = 0.0001$, $\delta M_{\rm Z} = 0.1$ MeV

also: $\delta(\Delta\alpha) = 5 \times 10^{-5}$

Theory uncertainties in extraction of pseudo-observables

- Subtraction of QED radiation contributions

  $\longrightarrow$ Known to $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha^3 L^3)$ for **ISR**, $\mathcal{O}(\alpha^2)$ for **FSR** and $\mathcal{O}(\alpha^2 L^2)$ for **A_{FB}**

  $(L = \log \frac{s}{m_e^2})$

  Berends, Burgers, v.Neerven '88
  Kniehl, Krawczyk, Kühn, Stuart '88
  Beenakker, Berends, v.Neerven '89
  Skrzypek '92; Montagna, Nicrosini, Piccinini '97

  $\longrightarrow$ $\mathcal{O}(0.1\%)$ uncertainty on σ_Z, A_{FB}

  $\longrightarrow$ Improvement needed for ILC/FCC-ee

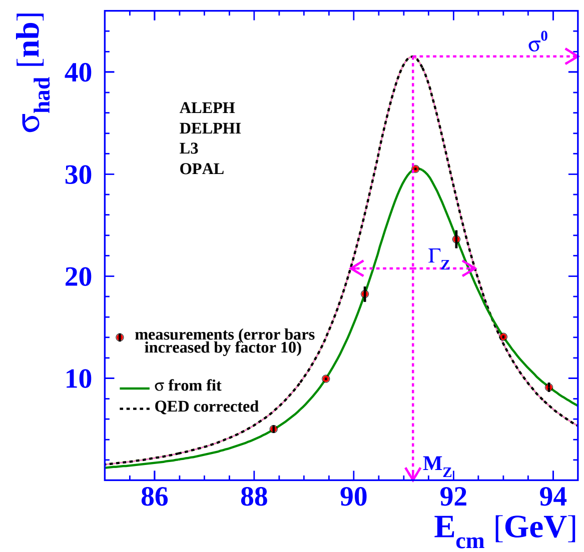
- Subtaction of non-resonant γ-exchange, γ–Z interf., box contributions, Bhabha scattering

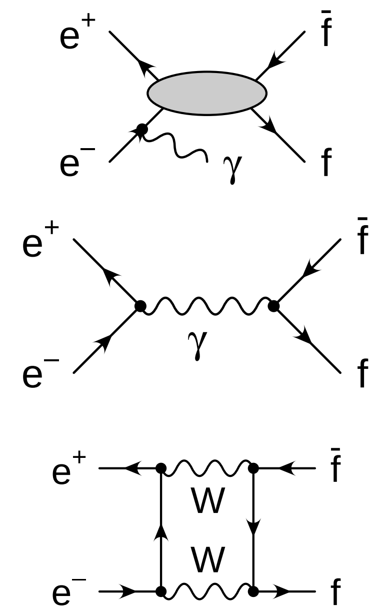
  see, e.g., Bardin, Grünewald, Passarino '99

  $\longrightarrow$ $\mathcal{O}(0.01\%)$ uncertainty within SM (improvements may be needed)

  $\longrightarrow$ Sensitivity to some NP beyond EWPO

LEP EWWG '05

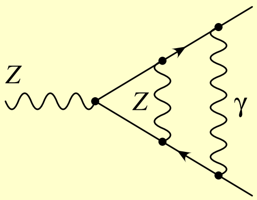




Non-factorizable contributions

Factorization of massive and QED/QCD FSR:

$$\overline{\Gamma}_f \approx \frac{N_c \overline{M}_Z}{12\pi} \left[\left(\mathcal{R}_V^f |g_V^f|^2 + \mathcal{R}_A^f |g_A^f|^2 \right) \frac{1}{1 + \mathrm{Re}\, \Sigma'_Z} \right]_{s = \overline{M}_Z^2}$$



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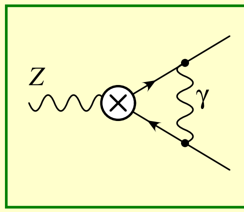
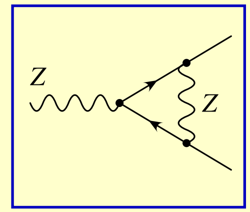
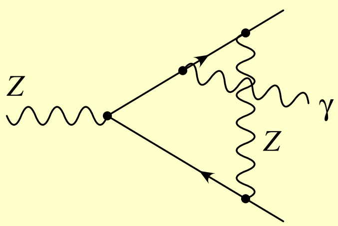


$+$ finite, with $\otimes =$



Additional non-factorizable contributions, e.g.



$\longrightarrow$ Known at $\mathcal{O}(\alpha\alpha_\mathrm{s})$ Czarnecki, Kühn ’96
Harlander, Seidensticker, Steinhauser ’98

$\longrightarrow$ Currently not known at $\mathcal{O}(\alpha^2)$ and beyond

$\longrightarrow$ $\mathcal{O}(0.01\%)$ uncertainty on Γ_Z, σ_Z, maybe larger for A_b
(improvements may be needed)

Theory challenges

Full SM corrections at >2-loop:

- Large number of diagrams and tensor integrals, $\mathcal{O}(1000) - \mathcal{O}(10000)$
  $\longrightarrow$ Use of computer algebra tools

- Many different scales (masses and ext. momenta)
  $\longrightarrow$ In general not possible analytically
  $\longrightarrow$ Numerical methods must be automizable, stable, fastly converging
  $\longrightarrow$ Need procedure for isolating divergent pieces

Analytic calculations

- Useful for diagrams with up to two scales
  ($e.\,g.$ M_{W} & m_{t} or M_{W} & M_{Z})
- Reduce to master integrals with integration-by-parts and Lorentz-invariance identities   Chetyrkin, Tkachov '81; Gehrmann, Remiddi '00; Laporta '00; ...
- Evaluate master integrals with differential equations or Mellin-Barnes representations   Kotikov '91; Remiddi '97; Smirnov '00,01; ...

Current status:

Single-scale problems: $Zf\bar{f}$ QED/QCD vertex corrections up to 4-loop
Gorishnii, Kataev, Larin '88,91; Chetyrkin, Kühn, Kwiatkowski '96
Baikov, Chetyrkin, Kühn, Rittinger '12

Two-scale problems: $Zf\bar{f}$ electroweak 2-loop vertex diagrams with $m_f = 0$
Awramik, Czakon, Freitas, Weiglein '04

Extendability: Possible, but much work needed

Asymptotic expansions

- Exploit large mass ratios, $e.\,g.\ M_Z^2/m_t^2 \approx 1/4$
- Fast numerical evaluation

Current status:

Two-scale problems: $\mathcal{O}(\alpha\alpha_s^n)$ corr. to $\Delta\rho$, Δr, ...

$\longrightarrow$ Several expansion terms up to 3-loop, leading term up to 4-loop

Djouadi, Verzegnassi '87; Bardin, Chizhov '88
Chetyrkin, Kühn, Steinhauser '95
Faisst, Kühn, Seidensticker, Veretin '03
...

Up to three-scale problems: $Zf\bar{f}$ ew. 2-loop vertex corrections

Barbieri et al. '92,93; Fleischer, Tarasov, Jegerlehner '93,95
Degrassi, Gambino, Sirlin '97; Awramik, Czakon, Freitas, Weiglein '04

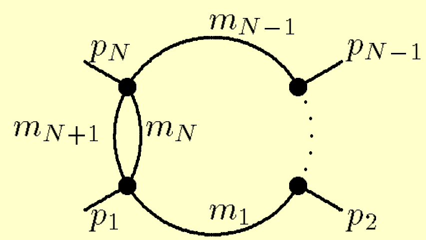
Extendability: Promising, mostly limited by computing/algorithmic power

Dispersion relations

- Numerically integrate over cuts
- High precision, but no known path towards full automatization
- Subtraction of UV-divergencies by hand

Example: Topologies with **self-energy sub-loop** S. Bauberger et al. '95

$$B_0(p^2, m_1^2, m_2^2) = -\int_{(m_1+m_2)^2}^{\infty} \mathrm{d}s \, \frac{\Delta B_0(s, m_1^2, m_2^2)}{s - p^2}$$



$$T_{N+1}(p_i; m_i^2) = -\int_{s_0}^{\infty} \mathrm{d}s \, \Delta B_0(s, m_N^2, m_{N+1}^2)$$

$$\times \int \mathrm{d}^4 q \, \frac{1}{q^2 - s} \, \frac{1}{(q+p_1)^2 - m_1^2} \cdots \frac{1}{(q+p_1+\cdots+p_{N-1})^2 - m_{N-1}^2}$$

Dispersion relations

- Numerically integrate over cuts
- High precision, but no known path towards full automatization
- Subtraction of UV-divergencies by hand

Current status:

Self-energy and vertex diagrams with arbitrary number of scales

Freitas, Hollik, Walter, Weiglein ’00; Awramik, Czakon ’02; Awramik, Czakon, Freitas ’04

Extendability: Only for certain applications

Direct numerical integration

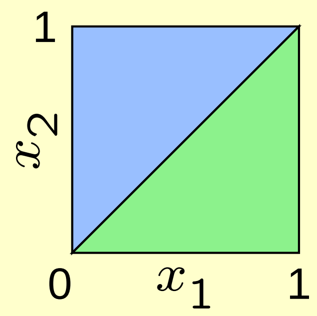
General form of Feynman integral:

$$I = \int_0^1 dx_1 \dots dx_n \, \delta(1 - \sum_i x_i) \, \frac{N(x_i)}{D(x_i)^{r+\varepsilon}}$$

$\longrightarrow$ Can be integrated numerically (if finite)

Alternatives: Integration in momentum space, Mellin-Barnes space

Treatment of divergencies:



- **Sector decomposition:** Sub-divide integration space such that divergent terms factorize
  Binoth, Heinrich '00,03
- **Subtraction terms:** Remove divergencies with simple terms that can be integrated analytically
  Nagy, Soper '03
  Becker, Reuschle, Weinzierl '10; Freitas '12

Direct numerical integration

- Automizable, but computing intensive
- Internal thresholds reduce numerical convergence

Current status:

Several 2-loop applications with many scales

Anastasiou et al., Petriello et al., Borowka et al., ...

Individual 3-loop integrals

Extendability: Likely, but more work needed

Conclusions

- **Current SM predictions** for electroweak precision observables under good control (compared to experimental uncertainties)

- **LHC** will provide independent results for $\sin^2\theta_{\text{eff}}$ and M_{W},
but overall precision not substantially improved

- **ILC/FCC-ee** with $\sqrt{s} \sim M_{\text{Z}}$ will reduce exp. error of some EWPO by $\mathcal{O}(10)$
$\longrightarrow$ 3-loop (and maybe some 4-loop) corrections needed!

- **Asymptotic expansion and numerical integration techniques** are promising but more work needed

- **Open questions** in evaluation of theory errors, resummation and optimal choice of inputs

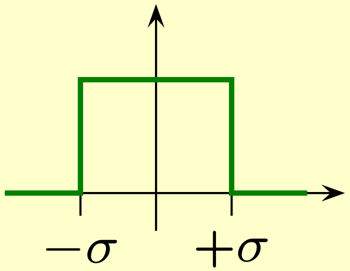
Backup slides

Combination of theory errors

- Add theory errors from each source **linearly**:

  Idea: each value within error range in equally likely

  $\longrightarrow$ Use flat prior in global fits



- Add theory errors from each source **quadratically**:

  Idea: different error sources are uncorrelated

  $\longrightarrow$ Use Gaussian prior in global fits (central limit theorm)