Model independent analysis of nearly Levy correlations in 1, 2 and 3 dimensions

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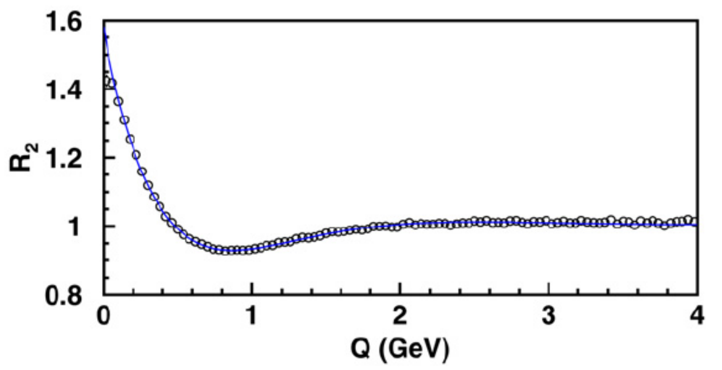


Fig. 1. The Bose–Einstein correlation function R_2 for events generated by PYTHIA. The curve corresponds to a fit of the one-sided Lévy parametrization, Eq. (13).

T. Csörgő et al. / Physics Letters B 663 (2008) 214–216

OUTLINE

Model-independent shape analysis:

- General introduction
- Edgeworth,
- Laguerre,
- Levy expansions

Summary

MODEL - INDEPENDENT SHAPE ANALYIS I.

experimental properties:

i) The correlation function tends to a constant for large values of the relative momentum Q.

ii) The correlation function has a non-trivial structure at a certain value of its argument.

The location of the non-trivial structure in the correlation function is assumed for simplicity to be close to $Q = 0$.

Model-independent but experimentally testable:

- $w(t)$ measure in an abstract H-space
- approximate form of the correlations
- t: dimensionless scale variable

$$\int dt w(t) h_n(t) h_m(t) = \delta_{n,m},$$

$$f(t) = \sum_{n=0}^{\infty} f_n h_n(t),$$

$$f_n = \int dt w(t) f(t) h_n(t).$$

e.g. $t = Q_I R_I$

MODEL - INDEPENDENT SHAPE ANALYIS II.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)},$$

$$R_2(\mathbf{k}_1, \mathbf{k}_2) = C_2(\mathbf{k}_1, \mathbf{k}_2) - 1.$$

Let us assume, that the function $g(t) = R_2(t)/w(t)$ is also an element of the Hilbert space H. This is possible, if

$$\int dt\, w(t) g^2(t) = \int dt\, \left[R_2^2(t)/w(t)\right] < \infty, \qquad (6)$$

Then the function *g* can be expanded as

$$g(t) = \sum_{n=0}^{\infty} g_n h_n(t),$$

$$g_n = \int dt\, R_2(t) h_n(t).$$

From the completeness of the Hilbert space and from the assumption that *g*(*t*) is in the Hilbert space:

$$R_2(t) = w(t) \sum_{n=0}^{\infty} g_n h_n(t).$$

MODEL - INDEPENDENT SHAPE ANALYIS III.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)},$$

$$C_2(t) = \mathcal{N} \left\{ 1 + \lambda_w\, w(t) \sum_{n=0}^{\infty} g_n h_n(t) \right\}$$

Model-independent AND experimentally testable:

- method for any approximate shape $w(t)$
- the core-halo intercept parameter of the CF is $$\lambda_* = \lambda_w \sum_{n=0}^{\infty} g_n h_n(0)$$
- coefficients by numerical integration (fits to data) $$g_n = \int dt\, R_2(t) h_n(t)$$
- condition for applicability: experimentally testabe $$\int dt \left[R_2^2(t) / w(t) \right] < \infty$$

EDGEWORTH EXPANSION: ~ GAUSSIAN

$$t = \sqrt{2} Q R_E,$$
$$w(t) = \exp(-t^2/2),$$
$$\int_{-\infty}^{\infty} dt\, \exp(-t^2/2) H_n(t) H_m(t) \propto \delta_{n,m},$$

$$H_n(t) = \exp(t^2/2) \left(-\frac{d}{dt}\right)^n \exp(-t^2/2).$$

$$H_1(t) = t,$$
$$H_2(t) = t^2 - 1,$$
$$H_3(t) = t^3 - 3t,$$
$$H_4(t) = t^4 - 6t^2 + 3, \; \ldots$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_E \exp(-Q^2 R_E^2) \times \left[1 + \frac{\kappa_3}{3!} H_3(\sqrt{2} Q R_E) + \frac{\kappa_4}{4!} H_4(\sqrt{2} Q R_E) + \ldots \right] \right\}.$$

3d generalization straightforward

- Applied by NA22, L3, STAR, PHENIX, ALICE, CMS (LHCb?)

LAGUERRE EXPANSIONS: ~ EXPONENTIAL

Model-independent but experimentally tested:

- *w*(*t*) exponential
- *t*: dimensionless
- Laguerre polynomials

$$t = QR_L,$$
$$w(t) = \exp(-t)$$
$$\int_0^\infty dt\, \exp(-t) L_n(t) L_m(t) \propto \delta_{n,m},$$

$$L_n(t) = \exp(t) \frac{d^n}{dt^n} (-t)^n \exp(-t).$$

$$L_0(t) = 1,$$
$$L_1(t) = t - 1,$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_L \exp(-QR_L) \left[1 + c_1 L_1(QR_L) + \frac{c_2}{2!} L_2(QR_L) + ... \right] \right\}$$

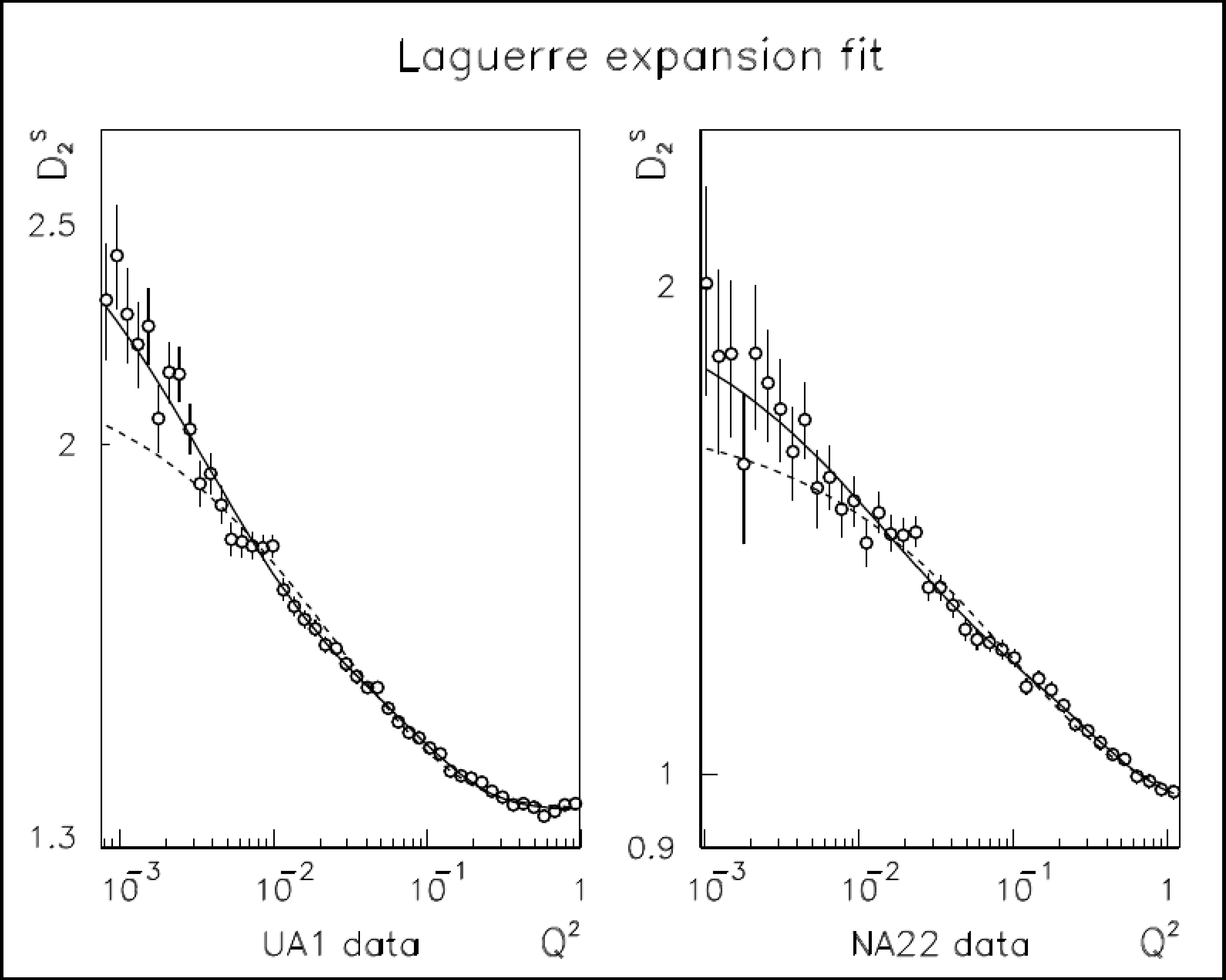
First successful tests

- NA22, UA1 data
- convergence criteria satisfied
- intercept parameter ~ 1

$$\int_0^\infty dt\, R_2^2(t) \exp(+t) < \infty,$$

$$\lambda_* = \lambda_L [1 - c_1 + c_2 - ...],$$
$$\delta^2 \lambda_* = \delta^2 \lambda_L \left[1 + c_1^2 + c_2^2 + ... \right] + \lambda_L^2 \left[\delta^2 c_1 + \delta^2 c_2 + ... \right]$$

LAGUERRE EXPANSIONS: ~ superEXPONENTIAL



T. Csörgő and S: Hegyi, hep-ph/9912220, T. Csörgő, hep-ph/001233

MINIMAL MODEL ASSUMPTION: LEVY

experimental conditions:
(i) The correlation function tends to a constant for large values of the relative momentum Q.
(ii) The correlation function deviates from its asymptotic, large Q value in a certain domain of its argument.

(iii) The two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

Model-independent but:

- Assumes that Coulomb can be corrected
- No assumptions about analyticity yet
- For simplicity, consider 1d case first
- For simplicity, consider factorizable *x k*
- Normalizations :
  - density
  - multiplicity
  - single-particle spectra

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)}$$

$$S(x,k) = f(x)\, g(k)$$

$$\int \mathrm{d}x\, f(x) = 1, \qquad \int \mathrm{d}k\, g(k) = \langle n \rangle,$$

$$N_1(k) = \int \mathrm{d}x\, S(x,k) = g(k).$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MINIMAL MODEL ASSUMPTION: LEVY

Model-independent but:

- not assumes analyticity
- C_2 measures a modulus squared Fourier-transform vs relative momentum
- Correlations non-Gaussian
- Radius not a variance
- $0 < \alpha \leq 2$

$$C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int \mathrm{d}x \, \exp(\mathrm{i}q_{12}x) \, f(x),$$

$$C(q;\alpha) = 1 + \lambda \exp\left(-|qR|^{\alpha}\right).$$

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

UNIVARIATE LEVY EXAMPLES

Include some well known cases:

- $\alpha = 2$
  - Gaussian source, Gaussian C_2

$$f(x) = \frac{1}{(2\pi R^2)^{1/2}} \exp\left[-\frac{(x-x_0)^2}{2R^2}\right]$$

$$C(q) = 1 + \exp\left(-q^2 R^2\right)$$

- $\alpha = 1$
  - Lorentzian source, exponential C_2

$$f(x) = \frac{1}{\pi} \frac{R}{R^2 + (x-x_0)^2},$$

$$C(q) = 1 + \exp\left(-|q\,R|\right).$$

- asymmetric Levy:
  - asymmetric support
  - Streched exponential

$$f(x) = \sqrt{\frac{R}{8\pi}} \frac{1}{(x-x_0)^{3/2}} \exp\left(-\frac{R}{8(x-x_0)}\right)$$

$$x_0 < x < \infty,$$

$$C(q) = 1 + \exp\left(-\sqrt{|q\,R|}\right).$$

T. Cs, hep-ph/0001233, T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

LEVY EXPANSIONS: ~ 1d LEVY

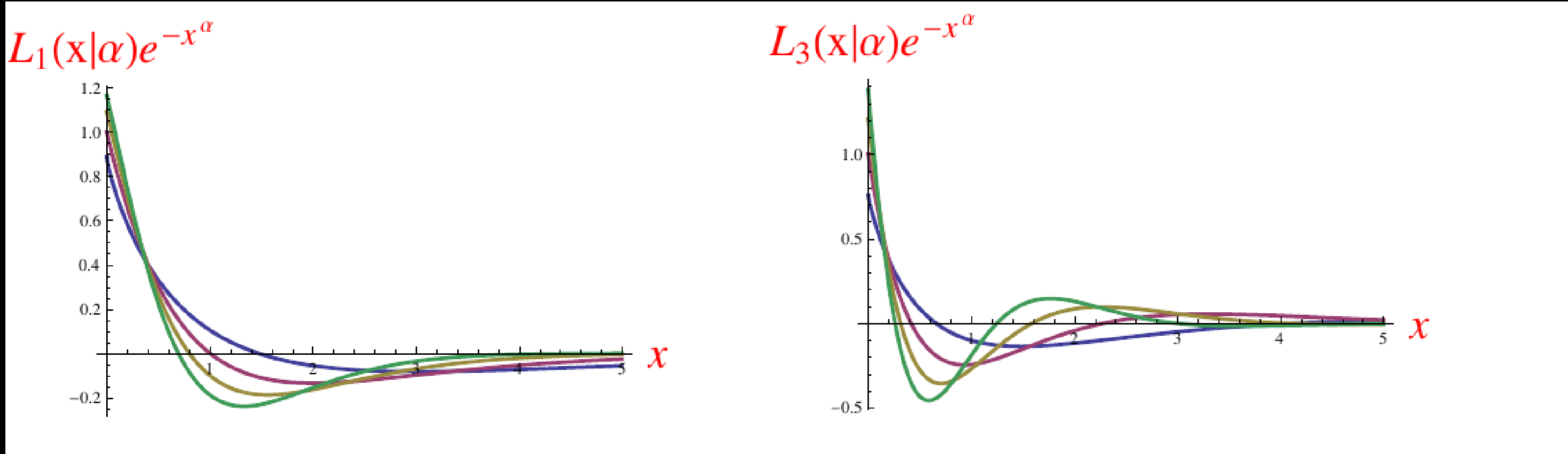
Model-independent but:

- Levy generalizes exponentials and Gaussians
- ubiquoutous in nature
- How far from a Levy?
- Need new set of polynomials orthonormal to a Levy weight

$$L_1(x\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

$$L_2(x\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & x & x^2 \end{pmatrix}$$

$$\mu_{r,\alpha} = \int_0^\infty dx\;\; x^r f(x\,|\,\alpha) = \frac{1}{\alpha}\Gamma\left(\frac{r+1}{\alpha}\right)$$



Lévy polynomials of first and third order times the weight function e^{-x^α} for $\alpha = 0.8, 1.0, 1.2, 1.4$.

1st-order Lévy polynomial	$\gamma\left[1 + \lambda e^{-R^\alpha Q^\alpha}\left[1 + c_1 L_1(Q\vert\alpha, R)\right]\right]$
3rd-order Lévy polynomial	$\gamma\left[1 + \lambda e^{-R^\alpha Q^\alpha}\left[1 + c_1 L_1(Q\vert\alpha, R) + c_3 L_3(Q\vert\alpha, R)\right]\right]$

M. de Kock, H. C. Eggers, T. Cs: arXiv:1206.1680v1 [nucl-th]

LEVY EXPANSIONS: ~ 1d LEVY

In case of α = 1 Laguerre is ok

$$\begin{aligned} L_0(t\,|\,\alpha=1) &= 1, \\ L_1(t\,|\,\alpha=1) &= t-1, \\ L_2(t\,|\,\alpha=1) &= t^2-4t+2. \end{aligned}$$

These reduce to the Laguerre expansions and Laguerre polynomials.

LEVY EXPANSIONS: ~ 1d LEVY

In case of α = 2 instead of Edgeworth new formulae for one-sided Gaussian:

$$
\begin{aligned}
L_0(t\,|\,\alpha=2) &= 1, \\
L_1(t\,|\,\alpha=2) &= \frac{1}{2}\left\{\sqrt{\pi}t - 1\right\}, \\
L_2(t\,|\,\alpha=2) &= \frac{1}{16}\left\{2(\pi-2)t^2 - 2\sqrt{\pi}t + (4-\pi)\right\}.
\end{aligned}
$$

Provides a new expansion around a Gaussian shape that is defined for the non-negative values of *t* only.

MULTIVARIATE LEVY DISTRIBUTIONS

The characteristic function is $f(t) = e^{-t^{\alpha}}$, where

$t = \left(\sum_{i,j=1,3} {R_{i,j}}^2 q_i q_j\right)^{1/2}$

$$C_2(k_1, k_2) = 1 + \lambda \exp\left[-\left(\sum_{i,j=1}^{3} R_{ij}^2 q_i q_j\right)^{\alpha/2}\right]$$

Model-independent but:

- A new parameter alpha generalizes Gauss
- Solved only for symmetric Levy distributions $\left({R_{i,j}}^2 = {R_{j,i}}^2\right)$
- Deep open problems in mathematical statistics

T. Cs, S. Hegyi, W.A. Zajc, EPJ C36, 67 (2004)

MULTIVARIATE LEVY EXPANSIONS

$$L_1(x\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

$$\mu_{r,\alpha} = \int_0^\infty dx\ x^r f(x\,|\,\alpha) = \frac{1}{\alpha}\Gamma(\frac{r+1}{\alpha})$$

$$L_1(t|\alpha) = \frac{t}{\alpha}\Gamma\left(\frac{1}{\alpha}\right) - \frac{1}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)$$

$$L_2(t\,|\,\alpha) = \det\begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & t & t^2 \end{pmatrix},$$

$$L_2(t\,|\,\alpha) = \frac{1}{\alpha^2}\left\{\left[\Gamma(\frac{1}{\alpha})\Gamma(\frac{3}{\alpha}) - \Gamma^2(\frac{2}{\alpha})\right]t^2 - \left[\Gamma(\frac{1}{\alpha})\Gamma(\frac{4}{\alpha}) - \Gamma(\frac{3}{\alpha})\Gamma(\frac{2}{\alpha})\right]t + \left[\Gamma(\frac{2}{\alpha})\Gamma(\frac{4}{\alpha}) - \Gamma^2(\frac{3}{\alpha})\right]\right\}.$$

1st-order Levy expansion

$$C_2(Q) = N\left\{1 + \lambda\exp\left(-\left(\sum_{i,j=1}^{3} R_{i,j}^2 q_i q_j\right)^{\alpha/2}\right)\left[1 + \frac{c_1}{\alpha}\left(\left(\sum_{i,j=1}^{3} R_{i,j}^2 q_i q_j\right)^{1/2}\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{2}{\alpha}\right)\right)\right]\right\}$$

$$t = \left(\sum_{i,j=1}^{3} R_{i,j}^2 q_i q_j\right)^{1/2}$$

M. de Kock, H. C. Eggers, T. Cs: arXiv:1206.1680v1 [nucl-th]

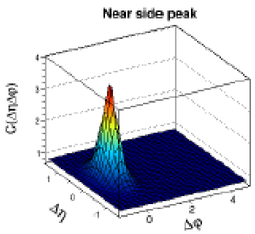
POSSIBLE APPLICATIONS I

Malgorzata's talk at WPCF2014

$$e_M = \alpha/2$$

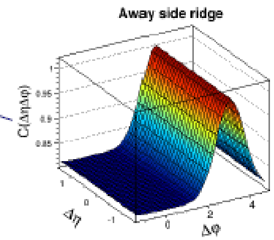
Levy expansion term could be added.

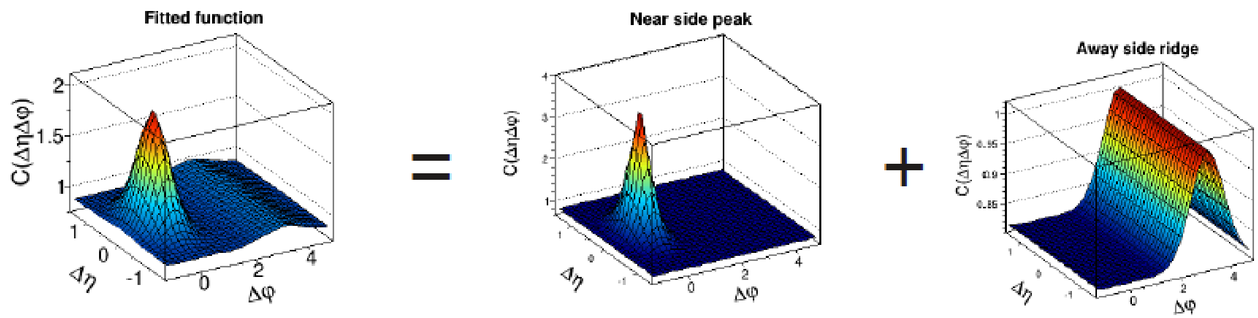
$$C(\Delta\eta,\Delta\varphi) = N$$



$$+ M_M\left\{\exp\left[-\left(\frac{\Delta\varphi^2}{2\sigma^2_{M_\varphi}} + \frac{\Delta\eta^2}{2\sigma^2_{M_\eta}}\right)^{e_M}\right] + \exp\left[-\left(\frac{\Delta(\varphi-2\pi)^2}{2\sigma^2_{M_\varphi}} + \frac{\Delta\eta^2}{2\sigma^2_{M_\eta}}\right)^{e_M}\right]\right\}$$

$$+ M_A\left\{\exp\left[-\left(\frac{\Delta(\varphi-\pi)^2}{2\sigma^2_{A_\varphi}}\right)\right] + \exp\left[-\left(\frac{\Delta(\varphi+\pi)^2}{2\sigma^2_{A_\varphi}}\right)\right]\right\}$$





26/08/2014, WPCF '14 Małgorzata Janik – Warsaw University of Technology 16/23

POSSIBLE APPLICATIONS II

Felix's talk at WPCF2015 (background substraction)

The background is modeled as a stretched exponential in q_{inv}:

$$\Omega(q_{\mathrm{inv}}) = 1 + \lambda_{\mathrm{bkgd}} e^{-|R_{\mathrm{bkgd}} q_{\mathrm{inv}}|^{\alpha_{\mathrm{bkgd}}}}$$

- **Could be multivariate Levy and expansion term could be added.**
- **But the form of the background may modify the signal.**

SUMMARY AND CONCLUSIONS

Several model-independent methods:

- Based on matching an abstract measure in *H* to the approximate shape of data
- Gaussian: Edgeworth expansions
- Exponential: Laguerre expansions
- Levy $(0 < \alpha \leq 2)$: Levy expansions
- In case of alpha = 1 Laguerre ok
- In case of alpha = 2 new formulae for Gaussian
- New directions: multivariate Levy expansions