Superstatistical cluster decay

Grzegorz Wilk [*] and **Zbigniew Włodarczyk[#]**

[*] NCNR, Warsaw, Poland, [#] UJK, Kielce, Poland

XI Workshop on Particle Correlations and Femtoscopy
3-7 November 2015, Warsaw, Poland

Content:

(1) Introduction

(2) *Superststistics - Scale parameter fluctuations*

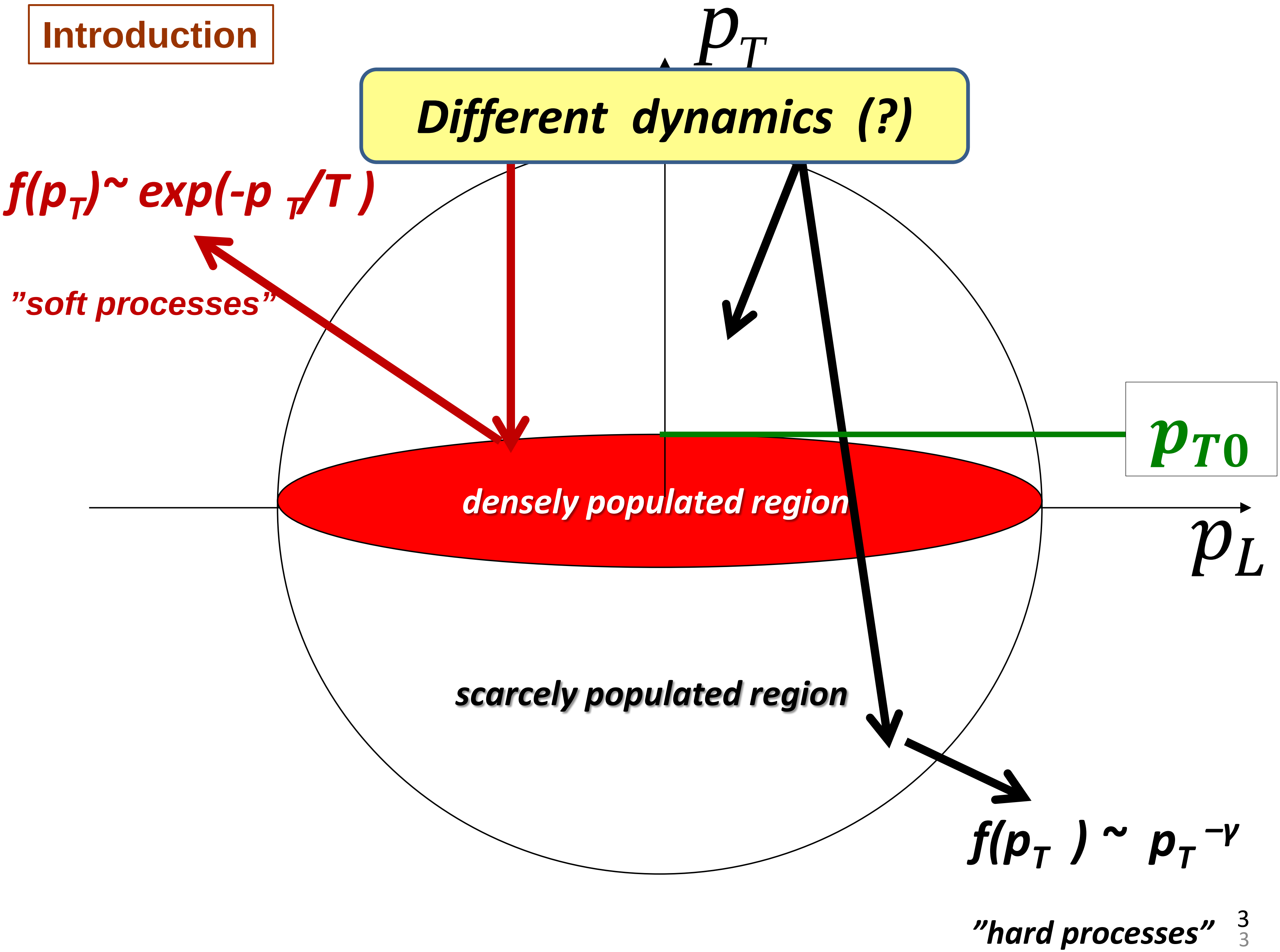
(3) *Approximate description of cluster decay*

(4) Fluctuations of relativistic temperature

(5) Conclusions

G.Wilk&G.Z.Włodarczyk, Phys.Let.A379(2015)2941

Introduction



Introduction

p_T

Different dynamics (?)

$f(p_T) \sim \exp(-p_T/T)$

"soft pr

One can describe both regions at the same time using a single formula interpolating between both regimes:

„Hagedorn" (1977-1984) $\quad h(p_T) = C\left(1 + \frac{p_T}{nT}\right)^{-n}$

Tsallis (1988- ...) $\quad f(p_T) = C\left[1 - (1-q)\frac{p_T}{T}\right]^{\frac{1}{1-q}}$

Notice: $\quad h(p_T) = f(p_T) \;\; for \;\; n = \frac{1}{q-1}$

$f(p_T) \sim p_T^{-\gamma}$

"hard processes"

Introduction

p_T

Different dynamics (?)

$f(p_T) \sim \exp(-p_T/T)$

”soft pr

One can describe both regions at the same time using a single formula interpolating between both regimes:

„Hagedorn” (1977-1984) $$h(p_T) = C\left(1 + \frac{p_T}{nT}\right)^{-n}$$

Tsallis (1988- ...) $$f(p_T) = C\left[1 - (1-q)\frac{p_T}{T}\right]^{\frac{1}{1-q}}$$

Examples of quasi-power law distributions

$f(p_T) \sim p_T^{-\gamma}$

”hard processes”

Introduction

Explanation?

(*) The most natural (in community of multiparticle production reactions) is that Tsallis distribution is of thermodynamical origin and follows from the replacement of BG statistics by Tsallis statistics (based on Tsallis entropy)

(*) On the other hand, this distribution also describes data which are not likely to follow a thermal approach but rather come from some kind of hard collisions (described by, for example, quantum chromodynamics, QCD) . In such cases its justification must be different .

(*) Among many possibilities the most interesting from our point of view is:

- *superstatistics,*
- *stochastic network approach*
- *connection with multiplicative noise*
- *order statistics*
- *……………*

(*) However: essentially all distributions of interest, including Tsallis, can be derived from information theory based on Shannon entropy.

Introduction

Explanation?

(*) The most natural (in community of multiparticle production reactions) is that Tsallis distribution is of thermodynamical origin and follows from the replacement of BG statistics by Tsallis statistics (based on Tsallis entropy)

(*) On the other hand, this distribution also describes data which are not likely to follow a thermal approach but rather come from some kind of hard collisions (described by, for example, quantum chromodynamics, QCD) . In such cases its justification must be different .

(*) Among many possibilities the most interesting from our point of view is:

- *superstatistics,* ➔ ***Scale parameter fluctuations***
- *stochastic network approach*
- *connection with multiplicative noise*
- *order statistics*
- *……………*

(*) However: essentially all distributions of interest, including Tsallis, can be derived from information theory based on Shannon entropy.

Superststistics – scale parameter fluctuations

$$f(E) = \frac{1}{T'} exp\left(-\frac{E}{T'}\right)$$

Boltzmann-Gibbs

convoluted with.

$$g(T') = \frac{1}{n\Gamma(n)T}\left(\frac{nT}{T'}\right)^{n+1} exp\left(-\frac{nT}{T'}\right),$$

$$n = \frac{1}{q-1}; \qquad q = 1 + \frac{Var(T)}{\langle T\rangle^2}$$

gamma distribution

results in

$$h(E) = \frac{n-1}{nT}\left(1 + \frac{E}{nT}\right)^{-n}$$

Tsallis

GW and ZW, PRL 84 (2000) 2770 and EPJA40(2009)299

It should be mentioned at this point that:

(*) Depending on the statistical properties of the fluctuations, one obtains different effective statistical mechanical descriptions.

(*) Tsallis statistics follow from the above gamma distribution of intensive variable.

(*) Other classes of generalized statistics can be obtained as well and, for small variance of the fluctuations, they all behave in a universal way.

GW and ZW, PRL 84 (2000) 2770 and EPJA40(2009)299

It should be mentioned at this point that:

(*) Thermal perspective: system considered is not homogeneous, it has different temperatures in different parts, which are fluctuating around some mean temperature T.

(*) It must therefore be described by two parameters: a mean temperature T and the mean strength of the fluctuations defined by q.

(*) The inevitable exchange of heat between any selected region of the system and the rest leads to the equilibration of the temperature in the whole system.

(*) The corresponding process of heat conductance leads to the Langevin equation with multiplicative noise term resulting in fluctuations of the temperature T given in the form of the above gamma function.

GW and ZW, PRL 84 (2000) 2770 and EPJA40(2009)299

Some remarks worth to be remembered (1):

$$g(T') = \frac{1}{nT\Gamma(n)} \cdot \left(\frac{nT}{T'}\right)^{n+1} \cdot exp\left(-\frac{nT}{T'}\right) \sim$$

$$\sim \quad \left(\frac{1}{T'}\right)^{\kappa} \cdot exp\left(-\frac{nT}{T'}\right) = g_1(T') \cdot g_2(T')$$

Some remarks worth to be remembered (1):

$$g(T') = \frac{1}{nT\Gamma(n)} \cdot \left(\frac{nT}{T'}\right)^{n+1} \cdot exp\left(-\frac{nT}{T'}\right) \sim$$

$$\sim \left(\frac{1}{T'}\right)^{\kappa} \cdot exp\left(-\frac{nT}{T'}\right) = g_1(T') \cdot g_2(T')$$

Fluctuating exp(-E/T') using only $g_1(T')$ results in a scale free distribution:

$$h_1(E) = \int_0^{\infty} dT' g_1(T') exp\left(-\frac{E}{T'}\right) \propto E^{-\kappa+1}$$

Some remarks worth to be remembered (1):

$$g(T') = \frac{1}{nT\Gamma(n)} \cdot \left(\frac{nT}{T'}\right)^{n+1} \cdot exp\left(-\frac{nT}{T'}\right) \sim$$

$$\sim \left(\frac{1}{T'}\right)^{\kappa} \cdot exp\left(-\frac{nT}{T'}\right) = g_1(T') \cdot g_2(T')$$

Fluctuating exp(-E/T') using only $g_1(T')$ results in a scale free distribution:

$$h_1(E) = \int_0^{\infty} dT' g_1(T') exp\left(-\frac{E}{T'}\right) \propto E^{-\kappa+1}$$

The scale appears when one cuts-off the small values of T'.

***Some remarks worth to be remembered (2)*:**

For example, sharp cut-off of the small *T′* results in distribution numerically very near to Tsallis distribution:

$$h_2(E) = \int_T^{\infty} dT' g_1(T') exp\left(-\frac{E}{T'}\right) \propto E^{-\kappa+1}\left[\Gamma(\kappa-1) - \Gamma\left(\kappa-1,\frac{E}{T}\right)\right] =$$

$$= \frac{1}{\kappa} + \sum_{i=1}^{\infty} \frac{\Gamma(i+\kappa)}{\Gamma(i+\kappa+1)\Gamma(i+1)}\left(-\frac{E}{T}\right)^i$$

***Some remarks worth to be remembered (2)*:**

For example, sharp cut-off of the small *T'* results in distribution numerically very near to Tsallis distribution:

$$h_2(E) = \int_T^{\infty} dT' g_1(T') exp\left(-\frac{E}{T'}\right) \propto E^{-\kappa+1}\left[\Gamma(\kappa-1) - \Gamma\left(\kappa-1, \frac{E}{T}\right)\right] =$$

$$= \frac{1}{\kappa} + \sum_{i=1}^{\infty} \frac{\Gamma(i+\kappa)}{\Gamma(i+\kappa+1)\Gamma(i+1)}\left(-\frac{E}{T}\right)^i$$

but not identical with it because its expansion reads:

$$\left(1+\frac{E}{\kappa T}\right)^{-\kappa} = \frac{1}{\kappa} + \sum_{i=1}^{\infty} \frac{\Gamma(i+\kappa)}{\Gamma(1+\kappa)\Gamma(i+1)}\left(-\frac{E}{\kappa T}\right)^i$$

Some remarks worth to be remembered (2):

For example, sharp cut-off of the small *T'* results in distribution numerically very near to Tsallis distribution:

$$h_2(E) = \int_T^{\infty} dT' g_1(T') exp\left(-\frac{E}{T'}\right) \propto E^{-\kappa+1}\left[\Gamma(\kappa-1) - \Gamma\left(\kappa-1, \frac{E}{T}\right)\right] =$$

$$= \frac{1}{\kappa} + \sum_{i=1}^{\infty} \frac{\Gamma(i+\kappa)}{\Gamma(i+\kappa+1)\Gamma(i+1)}\left(-\frac{E}{T}\right)^i$$

but not identical with it because its expansion reads:

$$\left(1 + \frac{E}{\kappa T}\right)^{-\kappa} = \frac{1}{\kappa} + \sum_{i=1}^{\infty} \frac{\Gamma(i+\kappa)}{\Gamma(1+\kappa)\Gamma(i+1)}\left(-\frac{E}{\kappa T}\right)^i$$

whereas smoothing the cutt-off and using instead the exponential dumping provided by the $g_2(T')$ results in proper Tsallis distribution:

$$h\,(E) = \int_0^{\infty} dT' g_1(T') g_2(T') exp\left(-\frac{E}{T\prime}\right) \propto \left(1 + \frac{E}{\kappa T}\right)^{-\kappa}$$

Approximate description of cluster decay

(*) In *([AB]: A. Białas, Phys. Lett. B 747 (2015) 150)* a new statistical cluster decay model of hadronization has been analyzed numerically, showing that even without resorting to approaches of the kind mentioned above, the resulting distribution of transverse momenta follows rather closely a Tsallis distribution, albeit not identically so.

(*) The hadronic clusters produced were supposed to decay purely thermally (following the **exponential** Boltzmann–Gibbs (BG) statistics) but, at the same time, were supposed to move in the transverse direction with a **fluctuating** (transverse) Lorentz factor distributed according to the assumed **power law**.

(*) It turns out that the combination of both distributions follows (at least numerically) a quasi-power like distribution, closely resembling a Tsallis distribution.

(*) Note that, according to what was said before regarding superstatistics, production and decay of such clusters can be regarded as an example of superstatistics at work (and not necessarily resulting in a Tsallis distribution).

(*) Le’ts investigate this phenomenon in more detail, aiming to obtain its analytical justification and a deeper understanding from the nonextensive statistical point of view.

Approximate description of cluster decay

The distribution of transverse momenta proposed:

$$f(p_T) \propto \int d\gamma_T\, K_0\left(\gamma_T \frac{m_T}{T}\right) I_0\left(\sqrt{\gamma_T{}^2 - 1}\,\frac{p_T}{T}\right) \cdot g(\gamma_T); \qquad g(\gamma_T) \sim \gamma_T{}^{-\kappa}$$

γ_T - transverse Lorentz factor

Numerical calculations (performed using Tsallis distribution) show that whereas for large p_T one has $f(p_T) \sim p_T{}^{-\kappa-1}$ there is a suppression for small values of p_T. Why:

$$\tilde{f}(p_T) \backsim K_0\left(\gamma_T \frac{m_T}{T}\right) I_0\left(\sqrt{\gamma_T{}^2 - 1}\,\frac{p_T}{T}\right) \sim \begin{cases} exp(-p_T) & for\ \ \gamma_T = 1 \\ \dfrac{1}{p_T} & for\ large\ \ \gamma_T \end{cases}$$

Because:

$$\begin{cases} K_0(x) I_0(y) \cong \dfrac{exp(-x+y)}{2\sqrt{xy}} & for\ x, y \gg \dfrac{1}{4} \\ \gamma_T - \sqrt{\gamma_T{}^2 - 1} \cong \dfrac{1}{2\gamma_T} & for\ large\ \gamma_T \\ m_T \cong p_T & for\ p_T \gg m \end{cases}$$

Approximate description of cluster decay

$$f(p_T) \propto \int d\gamma_T \, K_0\left(\gamma_T \frac{m_T}{T}\right) I_0\left(\sqrt{\gamma_T{}^2 - 1}\, \frac{p_T}{T}\right) \cdot g(\gamma_T) \propto$$

$$\propto \int_1^{\infty} d\gamma_T \, exp\left(-\frac{p_T}{T\gamma_T}\right) p_T{}^{-1}\, \gamma_T{}^{-\kappa-1} =$$

$$= p_T{}^{-\kappa-1}\left[\Gamma(\kappa) - \Gamma\left(\kappa, \frac{p_T}{T}\right)\right]$$

Limitation of $\gamma_T > 1$ leads to a result which is very near to the numerical results presented in {AB} paper, and not coinciding with a Tsallis distribution (but be so close to it that they can be fitted using a proper Tsallis distribution).

Approximate description of cluster decay

Question:

In the above the important factor are fluctuations of the Lorentz factor described by

$$g(\gamma_T) \sim {\gamma_T}^{-\kappa}$$

Is it possible to obtain them from the quark structure of the colliding nucleons?

Answer: **Yes, it is.**

Assume that he fraction of transverse momentum of the parton, $\boldsymbol{p_T}$, with respect to the momentum of hadron, $\boldsymbol{p_h}$, is $\boldsymbol{x = p_T/p_h}$ and that the density of the parton distribution is

$$w(x) = A\, x^a\, (1-x)^b$$

In the center of mass system the Lorentz factor of the cluster formed by the collision of partons with fractions of momenta $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, is given by

$$\gamma_T = \frac{x_1 + x_2}{2\sqrt{x_1 x_2}}$$

Then we have

$$\frac{1}{\sigma}\frac{d\sigma}{d\gamma_T} \propto \frac{1}{{\gamma_T}^{\kappa-1}} \cong {\gamma_T}^{-\kappa}$$ (for a=1 and b=3 we have $\kappa = 5$)

Fluctuations of relativistic temperature

Note:

(*) In formula

$$f(p_T) \propto \int_1^\infty d\gamma_T \, exp\left(-\frac{p_T}{T\gamma_T}\right) p_T{}^{-1} \, \gamma_T{}^{-\kappa-1}$$

the scale factor is $T\gamma_T$ not γ_T alone

(*) $T\gamma_T = T^*$ can be regarded to be a a relativistic transverse temperature

(*) We know that to get a proper Tsallis distribution when fluctuating the scale parameter, one has to use the full gamma function. Therefore, **fluctuating $T\gamma_T$ should bring the distribution of the transverse momenta to the desired final Tsallis form.**

Fluctuations of relativistic temperature

$$T\gamma_T = T^*$$

Assuming:

$$b\left(\frac{1}{\gamma_T}\right) = \frac{1}{\Gamma(\kappa+1)\Gamma(\alpha+1)}\left(\frac{1}{\gamma_T}\right)^{\kappa}\left(1-\frac{1}{\gamma_T}\right)^{\alpha}$$

$$g\left(\frac{1}{T}\right) = \frac{nT_0}{\Gamma(n)}\left(\frac{nT_0}{T}\right)^{n-1} exp\left(-\frac{nT_0}{T}\right)$$

Wpisz tutaj równanie.

one gets:

$$g'\left(\frac{1}{T},\frac{1}{\gamma_T}\right) = b\left(\frac{1}{\gamma_T}\right)\cdot g\left(\frac{1}{T}\right)$$

$$T\gamma_T = T^*$$ →

$$g'\left(\frac{1}{T^*},\frac{1}{T}\right) = \frac{(nT_0)^n}{\Gamma(n)\Gamma(\kappa+1)\Gamma(\alpha+1)}\left(\frac{1}{T^*}\right)^{\kappa} T^{\kappa+\alpha-n+1}\left(\frac{1}{T}-\frac{1}{T^*}\right)^{\alpha} exp\left(-\frac{nT_0}{T}\right)$$

Fluctuations of relativistic temperature

$T\gamma_T = T^*$

In the case when the parameters of the two components of this joint distribution are related in a certain way, for example if

α=*n* −1 −κ,

One gets

$$g'\left(\frac{1}{T^*},\frac{1}{T}\right) = \frac{(nT_0)^n}{\Gamma(n)\Gamma(\kappa+1)\Gamma(\alpha+1)}\left(\frac{1}{T^*}\right)^{\kappa}\left(\frac{1}{T}-\frac{1}{T^*}\right)^{\alpha} exp\left(-\frac{nT_0}{T}\right)$$

and fluctuations of the relativistic temperature T^* are again given by a gamma distribution (but this time with a changed shape parameter):

$$g'\left(\frac{1}{T^*}\right) = \int_{1/T^*}^{\infty} g'\left(\frac{1}{T^*},\frac{1}{T}\right) d\left(\frac{1}{T}\right) = \frac{n-1-\kappa}{\Gamma(\kappa+1)\Gamma(n+1)}\left(\frac{nT_0}{T^*}\right)^{\kappa} exp\left(-\frac{nT_0}{T^*}\right)$$

Fluctuations of relativistic temperature

$$T\gamma_T = T^*$$

$$g'\left(\frac{1}{T^*}\right) = \int_{1/T^*}^{\infty} g'\left(\frac{1}{T^*},\frac{1}{T}\right) d\left(\frac{1}{T}\right) = \frac{n-1-\kappa}{\Gamma(\kappa+1)\Gamma(n+1)}\left(\frac{nT_0}{T^*}\right)^{\kappa} exp\left(-\frac{nT_0}{T^*}\right)$$

This means that one could therefore consider the existence of fluctuations of the relativistic temperature $1/T*$, which are again given by a gamma distribution with parameter κ defining the size of these fluctuations. Relative fluctuations:

$$\omega(z) = \frac{Var(1/z)}{\langle 1/z\rangle^2}$$

satisfy relation

$$\omega(T^*) = \frac{\omega(T) + \omega(\gamma_T) + 2\omega(T)\omega(\gamma_T)}{1 + 3\omega(T)}$$

which connects fluctuations of T and γ_T.

The corresponding nonextensivity parameter is

$$q^* - 1 = \frac{1}{n^*} = \omega(T^*)$$

Conclusions - 1

(*) The statistical cluster decay mechanism discussed above is yet another example of superstatistics (which is not necessarily connected with a Tsallis distribution).

(*) Fluctuations of the Lorentz factor fully specify the slope parameter of the transverse momentum distribution in the region of large values of p_T

(i) If they are given by a scale free power-like distribution than the resultant distribution of p_T is also a scale-free, power-like one.

(ii) Its behavior for small values of p_T is dictated by the fact that the Lorentz factor is defined only for $\gamma_T \geq 1$, therefore there is always a natural cut-off in $g(\gamma_T)$, eliminating $\gamma_T < 1$.

As a result one gets a distribution which is not a Tsallis distribution, remaining, however, quite close to it numerically.

Conclusions - 2

(*) *One can invent a more general distribution, which is a product of a beta distribution in 1/γ_T and a gamma distribution for the parameter 1/T.*

(*) *If the parameters defining these distributions are related in some specific way, the resultant distribution is again a gamma distribution with a modified shape and one gets the true Tsallis distribution for the spectrum of the transverse momenta with the slope parameter defined by:* $\boldsymbol{q^* - 1 = \frac{1}{n^*} = \omega(T^*)}$

(*) *In this case it is determined by fluctuations of the relativistic temperature*

$$\boldsymbol{T\gamma_T = T^*}$$

Thank you