Theoretical Concepts in Particle Physics (1)

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What is HEP?
What is HEP

Find the basic laws of Nature

More formally

\[ \mathcal{L} = ? \]

- We have quite a good answer
- It is very elegant, it is based on axioms and symmetries
- We use particles to answer this question
- Particle physics is a tool to understand Nature
What is mechanics?

- Answer the question: what is $x(t)$?
- A system can have many DOFs, and then we seek to find $\vec{x}(t) \equiv x_1(t), x_2(t),...$
- Once we know $\vec{x}(t)$ we know any observable
- Solving for $q_1 \equiv x_1 + x_2$ and $q_2 \equiv x_1 - x_2$ is the same as solving for $x_1$ and $x_2$
- The idea of generalized coordinates is very important

How do we solve mechanics?
How do we find $x(t)$?

$x(t)$ minimizes something

- This is an axiom
- The thing that $x(t)$ minimized is called “the action” and is denoted by $S$
- There is one action for the whole system
- Similar to a minimum of a function

\[ \min[f(x)] \Rightarrow x_0, \quad \min[S(x(t))] \Rightarrow x_0(t), \]

- The condition for a minimum of a function is $\frac{df(x)}{dx} = 0$. What is the equivalent one for a minimum of an action?
What is $S$?

$$S = \int_{t_1}^{t_2} L(x, \dot{x}) dt, \quad \dot{x} \equiv \frac{dx}{dt} = v$$

The solution of the requirement that $S$ is minimal is given by the E-L equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

Once we know $L$ we can find $x(t)$ up to initial conditions.

To find a minimum of function we solve an algebraic eq. For the action we have a differential eq.

Mechanics is reduced to the question “what is $L$?”
An example: Newtonian mechanics

We assume particle with one DOF and

\[ L = \frac{mv^2}{2} - V(x) \]

- We use the E-L equation

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \quad L = \frac{mv^2}{2} - V(x) \]

- The solution is \(-V'(x) = m\dot{v}\), aka \(F = ma\)
- Here \(F = ma\) is the output, not the starting point!
- So how do we find what is L?
What is $L$?

$L$ is the most general one that is invariant under some symmetries.

We (again!) rephrase the question. Now we ask what are the symmetries of the system that lead to $L$. 
Example of symmetries

What are the symmetries that give

\[ L = \frac{mv^2}{2} - V(x) \]

- In 1d, if we require \( x \rightarrow -x \) invariant what can we say about \( V(x) \)?
- In 3d, if we require rotation invariant?
- In 1d with two particles, if we require \( x_1 \rightarrow x_1 + C \) and \( x_2 \rightarrow x_2 + C' \) invariant?
- What about the kinetic term, \( mv^2/2 \)?
- (homework) \( x_1 \rightarrow -3x_2 \) and \( x_2 \rightarrow -x_1/3 \)?
What is field theory
What is a field?

- In math: something that has a value in each point. We can denote it as $\phi(x, t)$
  - Temperature (scalar field)
  - Wind (vector field)
  - Mechanical string (?)
  - Density of people (?)
  - Electric and magnetic fields (vector fields)

How good is the field description of each of these?

- In physics a field used to be associated with a source, but now we know that fields are fundamental
An (familiar) example: the EM field

Consider $E(x, t)$. It obeys the wave equation

$$\frac{\partial^2 E(x, t)}{\partial t^2} = c^2 \frac{\partial^2 E(x, t)}{\partial x^2}$$

The solution is ($\varphi_0$ depends on IC)

$$E(x, t) = A \cos(\omega t - kx + \varphi_0), \quad \omega = ck$$

Some important implications of the result

- Each mode has its own amplitude, $A(\omega)$
- The energy in each $\omega$ is conserved
- The superposition principle

Are the statements above exact?
How to deal with generic field theories

- \( \phi(x, t) \) has an infinite number of DOFs. It can be an approximation for many (but finite) DOFs
- To solve mechanics of fields we need to find \( \phi(x, t) \)
- Here \( \phi \) is the generalized coordinate, while \( x \) and \( t \) are treated the same (nice!)
- We still need to minimize \( S \)

\[
S = \int \mathcal{L} dx dt \quad \mathcal{L}[\phi(x, t), \phi'(x, t), \dot{\phi}(x, t)]
\]

- We usually require Lorentz invariant (and use \( c = 1 \))

\[
S = \int \mathcal{L} d^4 x \quad \mathcal{L}[\phi(x, t), \partial_\mu \phi(x_\mu)]
\]
Solving field theory

We also have an E-L equation for field theories

\[ \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \]

- We have a way to solve field theory, just like mechanics. Give me \( \mathcal{L} \) and I can know everything!
- Just like in Newtonian mechanics, we want to get \( \mathcal{L} \) from symmetries!
Free field theory

- The "kinetic term" is promoted

\[ T \propto \left( \frac{dx}{dt} \right)^2 \Rightarrow T \propto \left( \frac{d\phi}{dt} \right)^2 - \left( \frac{d\phi}{dx} \right)^2 \equiv (\partial_\mu \phi)^2 \]

- Free particles, and thus free fields, only have kinetic terms

\[ \mathcal{L} = (\partial_\mu \phi)^2 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2} \]

- An \( \mathcal{L} \) of a free field gives a wave equation

- As in Newtonian mechanics, what used to be the starting point, here is the final result

- Why did we get it?
Harmonic oscillator
The harmonic oscillator

Why do we care so much about harmonic oscillators?

- Because we really care about springs?
- Because we really care about pendulums?
The harmonic oscillator

Why do we care so much about harmonic oscillators?

- Because we really care about springs?
- Because we really care about pendulums?

Because almost any function around its minimum can be approximated as a harmonic function!

- Indeed, we usually expand the potential around one of its minima
- We identify a small parameter, and keep only few terms in a Taylor expansion
Classic harmonic oscillator

\[ V = \frac{kx^2}{2} \]

We solve and get

\[ x(t) = A \cos(\omega t) \quad k = m\omega^2 \]

- The period does not depend on the amplitude
- Energy is conserved

Which of the above two statements is a result of the approximation of keeping only the harmonic term in the expansion?
Coupled oscillators
Coupled oscillators

- There are normal modes
- The normal modes are not “local” as in the case of one oscillator
- The energy of each mode is conserved
- This is an approximation!
- Once we keep non-harmonic terms energy moves between modes

\[ V(x, y) = \frac{k_1 x^2}{2} + \frac{k_2 y^2}{2} + \alpha x^2 y \]

- What determines the rate of energy transfer?
Things to think about

- http://lepp.cornell.edu/~yuvalg/CERNsummer/
- Relations harmonic oscillators and fields?
- When can we treat an oscillator as harmonic?