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The Standard Model of particle physics

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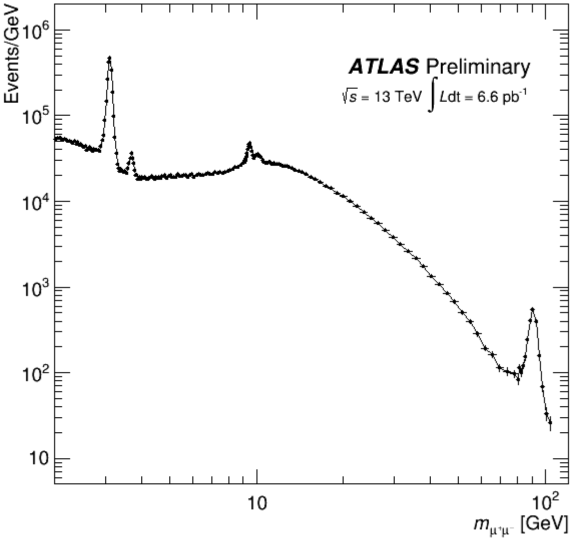

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- R.K. Ellis, W.J. Stirling and B.R. Webber, "QCD And Collider Physics," Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol. 8 (1996) 1.
- D. E. Soper, Basics of QCD perturbation theory, arXiv:hep-ph/0011256.
- Lectures by Keith Ellis, Douglas Ross, Adrian Signer, Robert Thorne and Bryan Webber (thanks!).

The Standard Model Lagrangian is determined by symmetries

- space-time symmetry: global Poincaré-symmetry
- internal symmetries: local $SU(n)$ gauge symmetries

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SM}} = \; & -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\bar{\psi} \not{D} \psi && \text{gauge sector} \\
& + |DH|^2 - V(H) && \text{EWSB sector} \\
& + \psi_i \lambda_{ij} \psi_j H + \mathrm{h.c.} && \text{flavour sector} \\
& + N_i M_{ij} N_j && \nu\text{-mass sector}
\end{aligned}
$$

. . . requiring renormalizability and ignoring the strong CP-problem.

Outline

- QED and QCD as gauge theories
- QCD for the LHC
- Breaking gauge symmetries:

  the Englert-Brout-Higgs-Guralnik-Hagen-Kibble mechanism

The Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

"A great deal more was hidden in the Dirac equation than the author had expected when he wrote it down in 1928. Dirac himself remarked in one of his talks that his equation was more intelligent than its author." (Weisskopf on Dirac)

[We use natural units: $c = \hbar = 1$, so that [mass] $=$ [length]$^{-1}$ $=$ [time]$^{-1}$ $=$ (Giga) electron volt]

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What would you do to reconcile quantum theory and special relativity:

$$i\frac{\partial}{\partial t}\phi = H\phi \quad \text{and} \quad E^2 = \vec{p}^2 + m^2\,?$$

Iterate the Schrödinger equation to arrive at

$$\left(i\frac{\partial}{\partial t}\right)^2 \phi = H^2\phi = (-\vec{\nabla}^2 + m^2)\phi$$

or

$$(\Box + m^2)\phi = 0$$

where $\Box = \partial^2/\partial t^2 - \vec{\nabla}^2$.

Dirac was not satisfied with the Klein-Gordon equation $(\Box + m^2)\phi = 0$ since it contains

- solutions with negative energy $E < 0$;
- a second order derivative in time, and can thus lead to negative probability densities $|\phi|^2 < 0$.

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Bohr: "What are you working on Mr. Dirac?"
Dirac: "I am trying to take the square root of something."

Dirac wanted an equation that is Lorentz covariant and first order in the time derivative:

$$i\frac{\partial\psi}{\partial t} = H_{\mathrm{Dirac}}\psi = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m)\psi = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$

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Iterating the equation on both sides yields

$$\begin{aligned} E^2\psi &= \left(i\frac{\partial}{\partial t}\right)^2\psi = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi \\ &= \left(-\alpha^i\alpha^j\nabla^i\nabla^j - i(\beta\alpha^i + \alpha^i\beta)m\nabla^i + \beta^2 m^2\right)\psi \\ &= (p^2 + m^2)\psi = (-\nabla^i\nabla^i + m^2)\psi\,. \end{aligned}$$

The α_i and β must satisfy

$$\begin{aligned}
\alpha_i\alpha_j + \alpha_j\alpha_i &= 2\delta_{ij} \\
\beta\alpha_i + \alpha_i\beta &= 0 \\
\beta^2 = 1
\end{aligned}$$

so they cannot be numbers.

Dirac proposed that the α_i and β are 4×4 matrices, and that ψ is a 4-component column vector, known as Dirac spinor.

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One choice of matrices is

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli matrices.

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The Dirac spinor describes particles and antiparticles with spin 1/2:

$$\psi = \begin{pmatrix} \psi^{\uparrow} \\ \psi^{\downarrow} \\ \psi^{\uparrow} \\ \psi^{\downarrow} \end{pmatrix}$$

There is a more compact way to write the Dirac equation.

Define the γ-matrices

$$\gamma^0 \equiv \beta \quad \text{and} \quad \vec{\gamma} \equiv \beta\vec{\alpha}$$

so that

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\ .$$

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With the previous choice of α_i and β one has

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where $\vec{\sigma}$ are the Pauli matrices.

Using the γ-matrices, the Dirac equation becomes:

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \quad \text{or} \quad (i\not{\partial} - m)\psi(x) = 0$$

where we have introduced $\partial_\mu \equiv \partial/\partial x^\mu = (\partial/\partial t, -\vec{\nabla})$ and $\not{\partial} \equiv \gamma^\mu\partial_\mu$.

The beauty and magic of the Dirac equation

The Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

- is form-invariant (covariant) under Lorentz transformations;
- describes particles with spin $1/2$;
- predicts the correct magnetic moment $g = 2$;
- predicts the existence of anti-particles!

The Dirac Lagrangian

One can construct Lorentz scalars and vectors from Dirac spinors and the γ-matrices, e.g.

$$\begin{aligned} \overline{\psi}\psi &\xrightarrow{LT} \overline{\psi}\psi \\ \overline{\psi}\gamma^\mu\psi &\xrightarrow{LT} \Lambda^\mu_{\ \nu}\overline{\psi}\gamma^\nu\psi \end{aligned}$$

where we have defined $\overline{\psi} \equiv \psi^\dagger\gamma^0$.

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Using $\overline{\psi}, \psi$ and γ^μ one can thus construct a Lorentz covariant Lagrangian

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi\,,$$

which leads to the Dirac equation through the usual Euler-Lagrange equations,

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial\overline{\psi}/\partial x_\mu)}\right) - \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0 \quad \text{and} \quad \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_\mu)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0\,.$$

Gauge transformations: QED

Consider the Lagrangian for a free Dirac field ψ:

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi\,.$$

The Lagrangian is invariant under a phase transformation of the fermion field:

$$\psi \to e^{-i\omega}\psi, \quad \overline{\psi} \to e^{i\omega}\overline{\psi},$$

where ω is a constant (i.e. independent of x).

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The set of numbers $e^{-i\omega}$ form a group. This particular group is "abelian" which is to say that any two elements of the group commute:

$$e^{-i\omega_1} e^{-i\omega_2} = e^{-i\omega_2} e^{-i\omega_1}\,.$$

This particular group is called $U(1)$, i.e. the group of all unitary 1×1 matrices. (A unitary matrix satisfies $U^\dagger = U^{-1}$.)

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Thus the Dirac Lagrangian is invariant under global $U(1)$ transformations.

We now require invariance under local $U(1)$ transformations, i.e.

$$\psi \to e^{-i\omega(x)}\psi, \quad \overline{\psi} \to e^{i\omega(x)}\overline{\psi},$$

where $\omega(x)$ now depends on the space-time point.

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$$\psi \to e^{-i\omega(x)}\psi, \quad \overline{\psi} \to e^{i\omega(x)}\overline{\psi},$$

where $\omega(x)$ now depends on the space-time point.

Note that $\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi$ is not invariant under local $U(1)$ transformations:

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} = \mathcal{L} \; + \overline{\psi}\gamma^\mu[\partial_\mu\omega(x)]\psi\,,$$

where we consider infinitesimal transformations

$$\psi \to \psi + \delta\psi = \psi - i\omega(x)\psi \quad \text{and} \quad \overline{\psi} \to \overline{\psi} + \delta\overline{\psi} = \overline{\psi} + i\omega(x)\overline{\psi}\,.$$

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We can restore invariance under local $U(1)$ transformations if we introduce a vector field $A_\mu(x)$ with the interaction

$$-e\overline{\psi}\gamma^\mu A_\mu\psi\,,$$

so that the Lagrangian density becomes

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\,\psi\,.$$

The new Lagrangian

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is invariant under local $U(1)$ transformations if we require that

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We need to add a Lorentz- and gauge invariant kinetic term for the field A_μ:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\,\psi\,,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

[We have fixed the coefficient of the term $\propto F_{\mu\nu}F^{\mu\nu}$ so that we recover the standard form of Maxwell's equations.]

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A mass term for the new field $\propto m_A^2 A_\mu A^\mu$ is not invariant under gauge transformations,

$$\delta\mathcal{L} = \frac{2m_A^2}{e}A^\mu \partial_\mu \omega(x) \neq 0\,,$$

and thus not allowed.

It is useful to introduce the concept of a "covariant derivative" D_μ as

$$D_\mu = \partial_\mu + ieA_\mu \, .$$

With

$$\psi \to \psi + \delta\psi = \psi - i\omega(x)\psi \quad \text{and} \quad A_\mu \to A_\mu + \delta A_\mu = A_\mu + \frac{1}{e}[\partial_\mu \omega(x)]$$

one finds

$$D_\mu\psi \to D_\mu\psi + \delta(D_\mu\psi) = D_\mu\psi - i\omega(x)D_\mu\psi$$

so that

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$$

is gauge invariant.

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One can express $F_{\mu\nu}$ in terms of the covariant derivative:

$$F_{\mu\nu} = -\frac{i}{e}[D_\mu, D_\nu] = \ldots = \partial_\mu A_\nu - \partial_\nu A_\mu \, .$$

Gauge transformations: Summary

- The Dirac Lagrangian is invariant under local $U(1)$ transformations if we add a vector field A_μ and an interaction $-e\overline{\psi}\gamma^\mu A_\mu \psi$.
- The interaction is obtained by replacing the derivate ∂_μ with the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$.
- The gauge-invariant kinetic term for the vector field is $\propto F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} \propto [D_\mu, D_\nu]$.
- The new vector (gauge) field is massless, since a term $\propto A_\mu A^\mu$ is not gauge-invariant.
- The Lagrangian resulting from local $U(1)$ gauge-invariance is identical to that of QED.

Gauge transformations: non-abelian gauge groups

We now apply the idea of local gauge invariance to the case where the transformation is "non-abelian", i.e. different elements of the group do not commute with each other.

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We focus on the group $SU(n)$, i.e. the group of special unitary transformations.

To specify an $SU(n)$ matrix, we need $n^2 - 1$ real parameters, so we can write

$$e^{-i\omega^a T^a}$$

where the $\omega^a, a \in \{1, ..., n^2 - 1\}$ are real parameters, and the T^a are called generators of the group. [If you are unfamiliar with the concept of a group generator, you can think of the T^a as traceless, hermitian $n \times n$ matrices.]

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The crucial new feature is that the elements of $SU(n)$ do not commute,

$$e^{-i\omega_1^a T^a} e^{-i\omega_2^a T^a} \neq e^{-i\omega_2^a T^a} e^{-i\omega_1^a T^a} \,,$$

because the generators do not commute:

$$[T^a, T^b] = if^{abc} T^c \neq 0 \,.$$

Recall that the $SU(n)$ transformations act on the fermion fields, so ψ carries an index i, with $i \in \{1, ..., n\}$:

$$\psi \to \left(e^{-i\omega^a T^a}\right)\psi \quad \text{or} \quad \psi_i \to \left(e^{-i\omega^a T^a}\right)_i^{\ j}\psi_j$$

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Considering infinitesimal transformations

$$\delta\psi_i = -i\omega^a (T^a)_i^j \psi_j \quad \text{and} \quad \delta\overline{\psi}^i = i\omega^a \overline{\psi}^j (T^a)_j^i$$

one finds that the Lagragian is not invariant under local $SU(n)$ transformations:

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We can restore local $SU(n)$ gauge-invariance by introducing $n^2 - 1$ new vector particles A_μ^a, one for each generator of the group.

They should transform as

$$\delta A_\mu^a(x) = -f^{abc} A_\mu^b(x) \omega^c(x) + \frac{1}{g} \left[\partial_\mu \omega^a(x)\right] \,.$$

The interaction of the new vector particles and the fermions is obtained by replacing the ordinary derivative with the covariant derivative

$$D_\mu = (\partial_\mu + igT^a A_\mu^a) \,.$$

Note that in this case D_μ is a $n \times n$ matrix.

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The $SU(n)$ invariant Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \overline{\psi}^i \,(i\gamma^\mu D_\mu - m)^j_i \,\psi_j$$

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The field strength tensor is constructed from

$$F^{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$$

with $F^{\mu\nu} = T^a F^a_{\mu\nu}$.

This gives $\quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A^b_\mu A^c_\nu.$

Non-abelian gauge transformations: Summary

- A non-abelian gauge theory is a theory in which the Lagrangian is invariant under local transformations of a non-abelian group.
- This invariance is achieved by introducing a gauge boson, A^a_μ, for each generator of the group. The interaction between the gauge bosons and the fermions is obtained by replacing the partial derivative ∂_μ with the covariant derivative $D_\mu = (\partial_\mu + igT^a A^a_\mu)$.
- The kinetic term for the vector field is $\propto F^a_{\mu\nu} F^{a\,\mu\nu}$, where $F^a_{\mu\nu}$ is constructed from the commutator of the covariant derivative.
- $F^a_{\mu\nu} F^{a\,\mu\nu}$ contains terms which are cubic and quartic in the gauge boson fields, indicating that the gauge bosons interact with each other.
- The gauge bosons are massless, since a term $\propto A^a_\mu A^{a\,\mu}$ is not invariant under local gauge transformations.