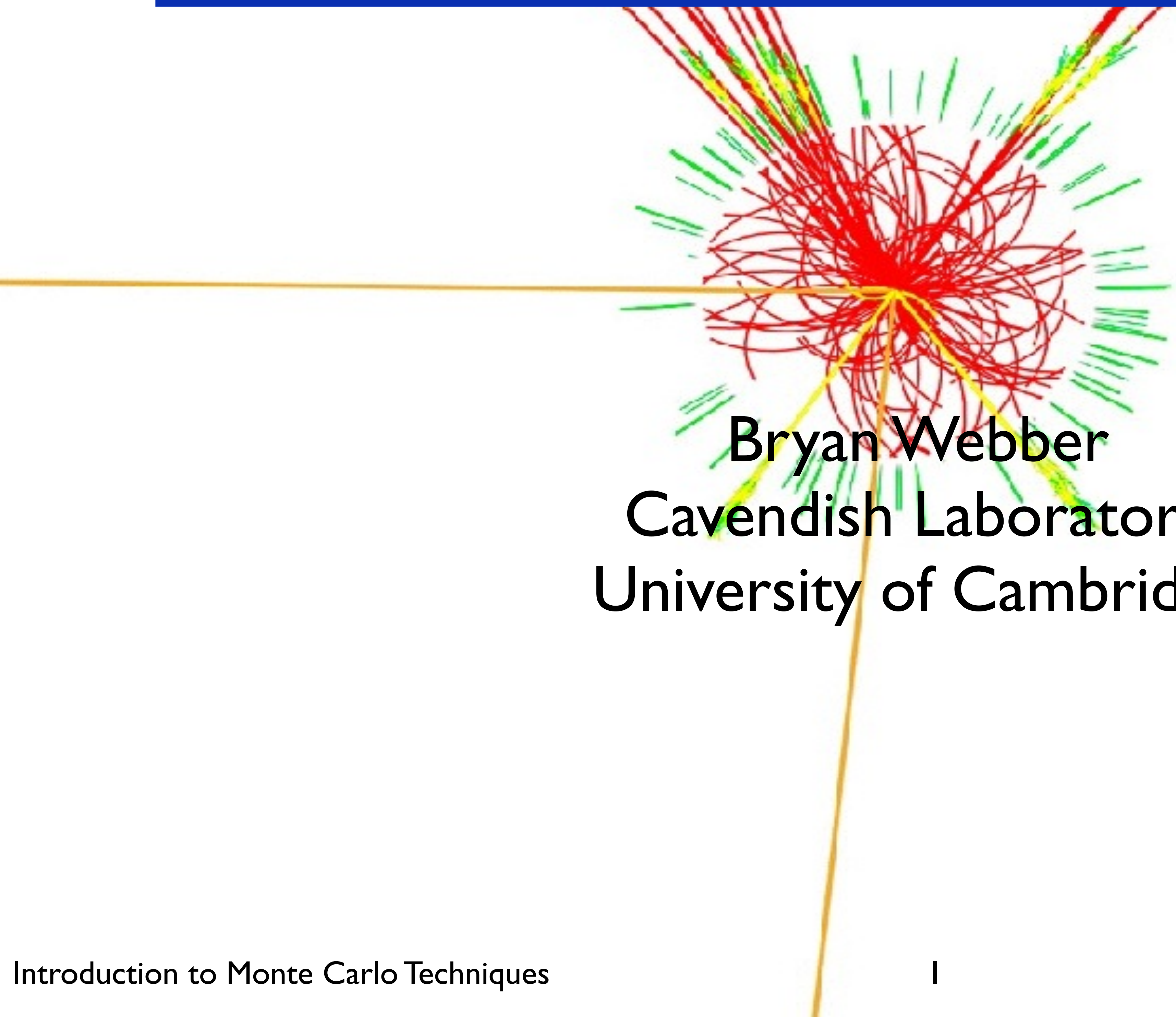
Introduction to Monte Carlo Techniques



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Introduction to Monte Carlo

- Lecture 1: The Monte Carlo method
  - theoretical foundations and limitations
  - parton-level event generation
- Lecture 2: Hadron-level event generation
  - parton showering
  - hadronization and underlying event
  - sample of results

Why Monte Carlo?

- Something to do with gambling?
- Not a place but a method …

Monte Carlo Event Generation

Monte Carlo Event Generation

- Aim is to produce simulated (particle-level) datasets like those from real collider events
  - i.e. lists of particle identities, momenta, ...
  - simulate quantum effects by (pseudo)random numbers
- Essential for:
  - Designing new experiments and data analyses
  - Correcting for detector and selection effects
  - Testing the SM and measuring its parameters
  - Estimating new signals and their backgrounds

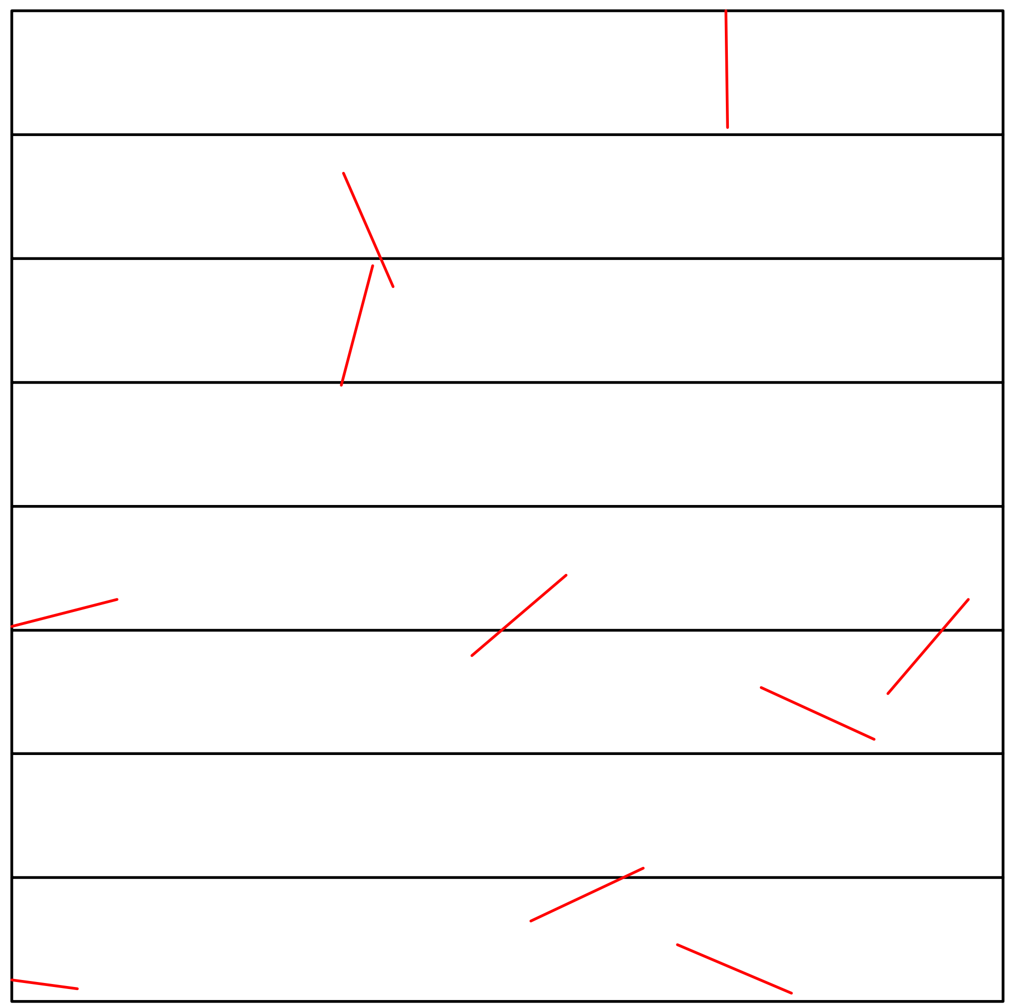
References

- A. Buckley et al., "General-purpose event generators for LHC physics", Phys.Rept. 504 (2011) 145 (MCNET-11-01, arXiv:1101.2599)
- M.H. Seymour & M. Marx, "Monte Carlo Event Generators", MCNET-13-05, arXiv:1304.6677
- A. Siódmok, "LHC event generation with general-purpose Monte Carlo tools", Acta Phys. Polon. B44 (2013)1587

Monte Carlo Method

Buffon’s needle

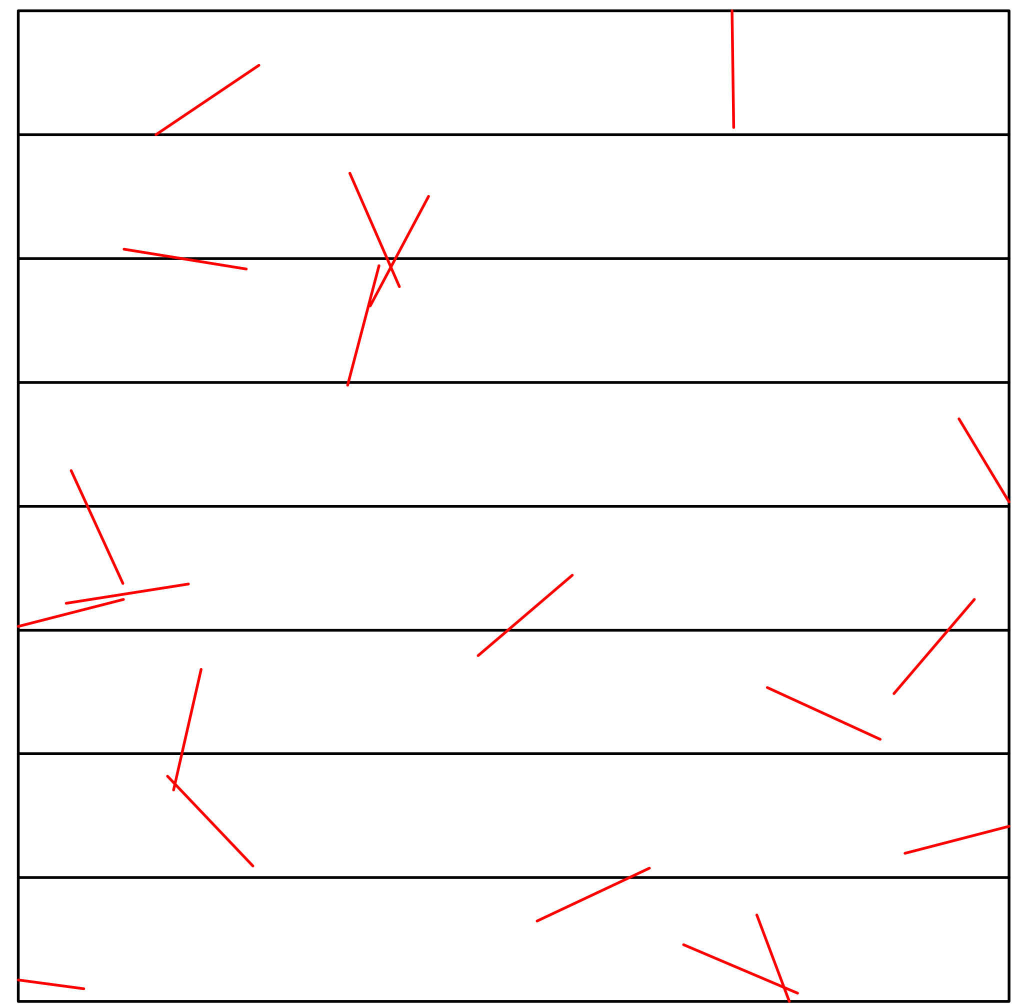
G-L Leclerc, Comte de Buffon, 1707-1788



$2 \times 10/5 = 4.000$

Buffon's needle

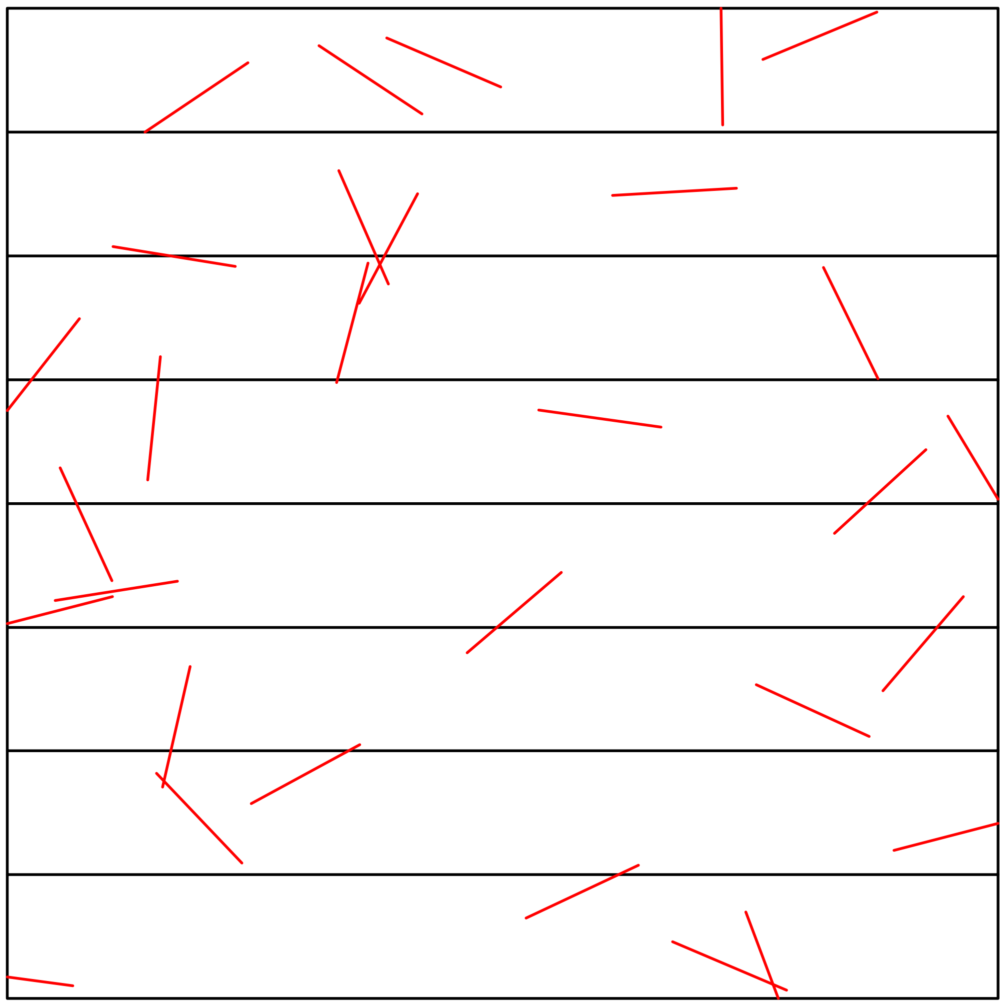
G-L Leclerc, Comte de Buffon, 1707-1788



2x20/12 = 3.333

Buffon's needle

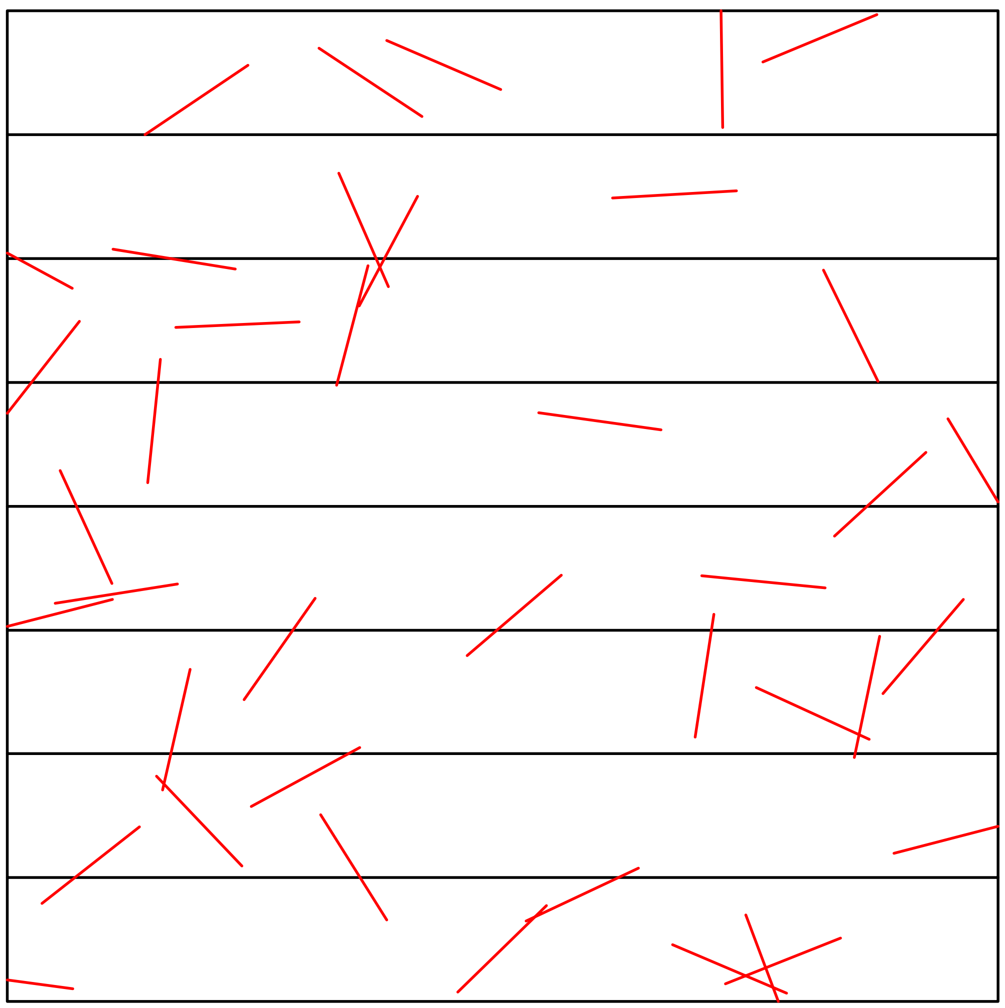
G-L Leclerc, Comte de Buffon, 1707-1788



2x30/16 = 3.750

Buffon's needle

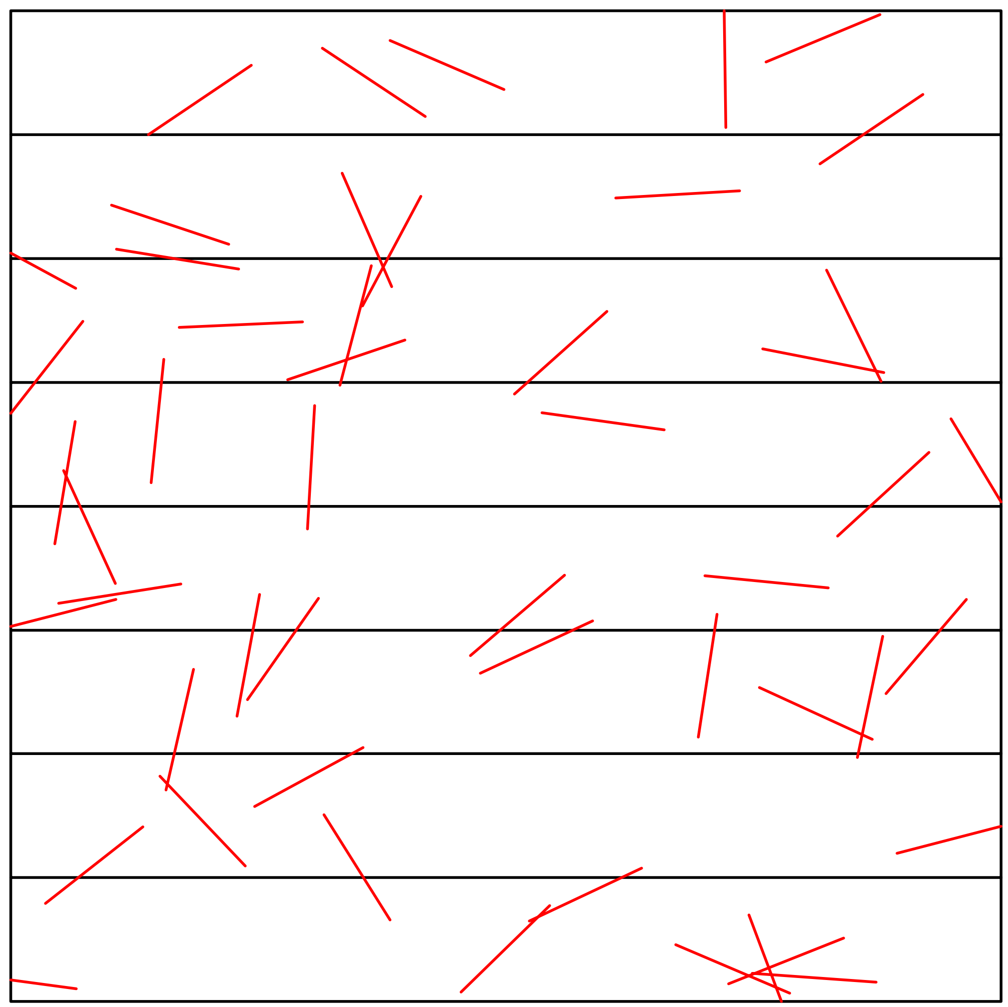
G-L Leclerc, Comte de Buffon, 1707-1788



$$2 \times 40/22 = 3.636$$

Buffon's needle

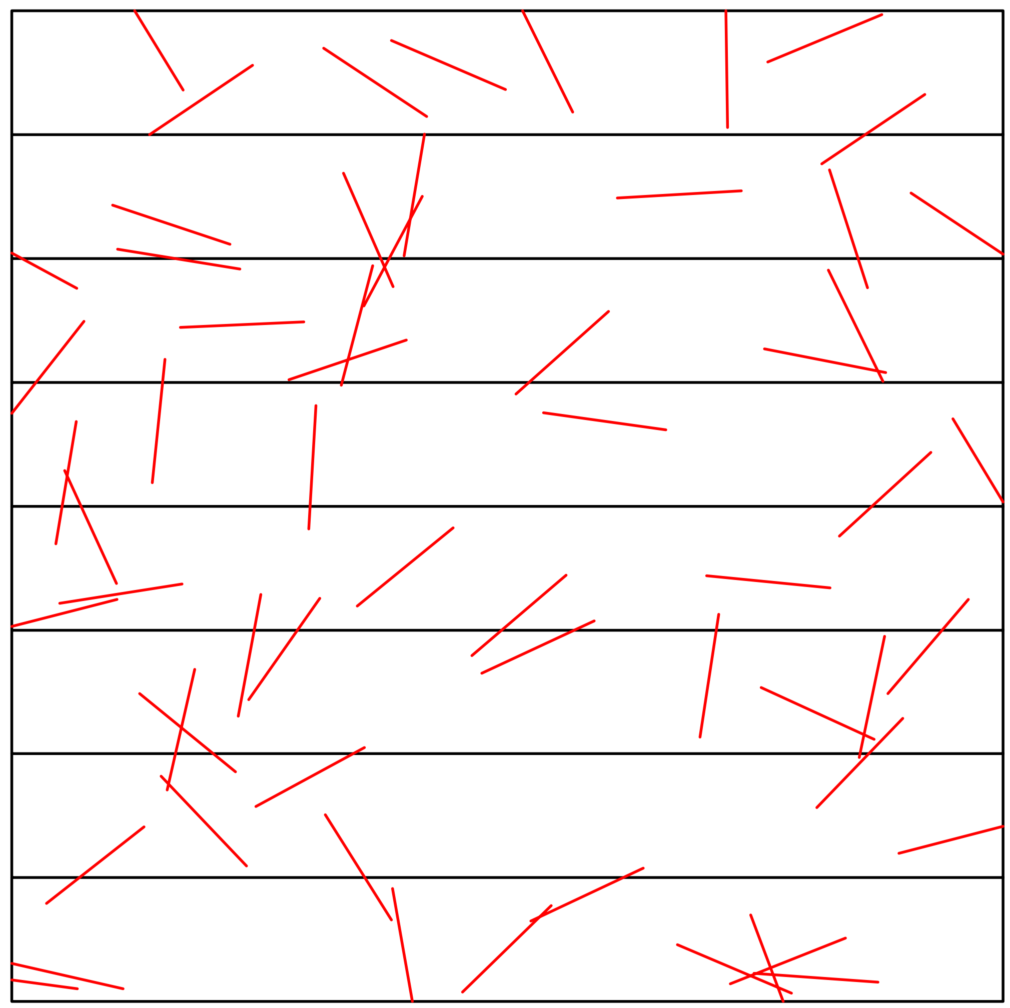
G-L Leclerc, Comte de Buffon, 1707-1788



2x50/28 = 3.571

Buffon's needle

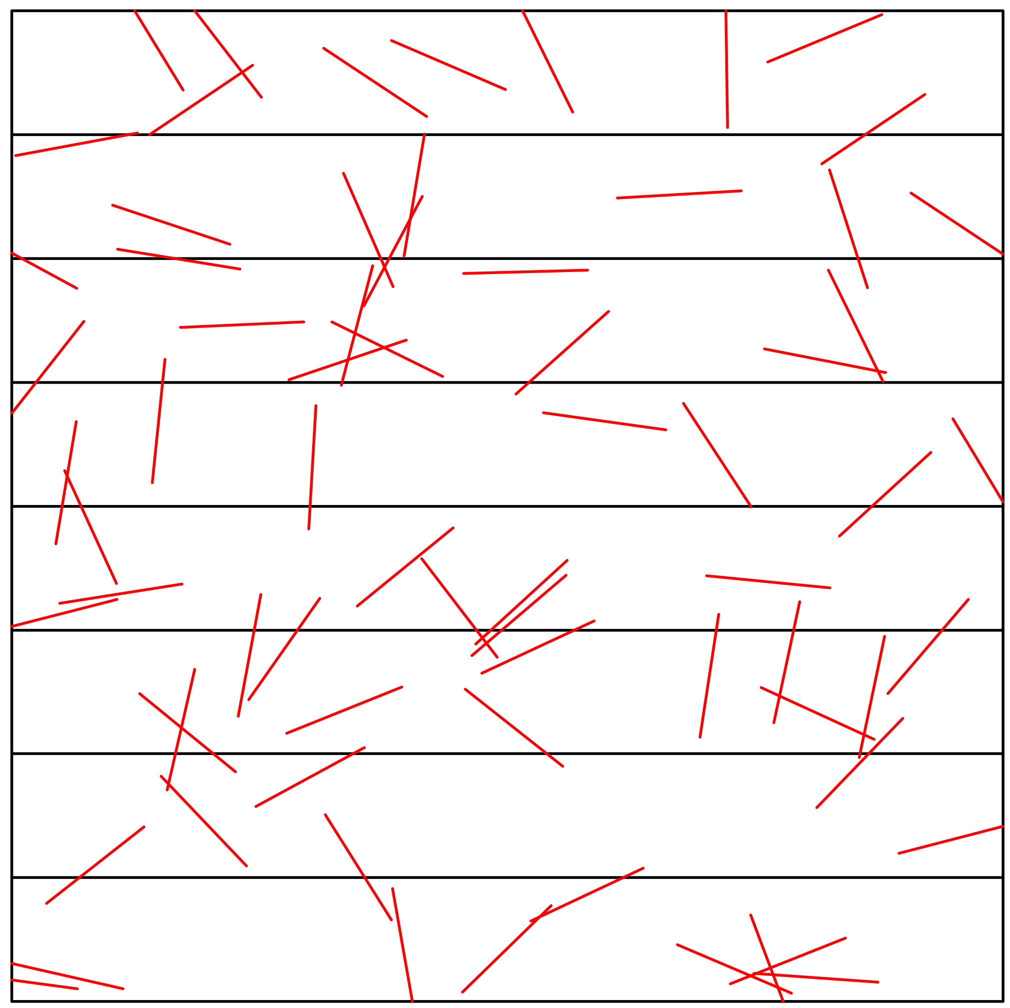
G-L Leclerc, Comte de Buffon, 1707-1788



$2 \times 60/34 = 3.529$

Buffon's needle

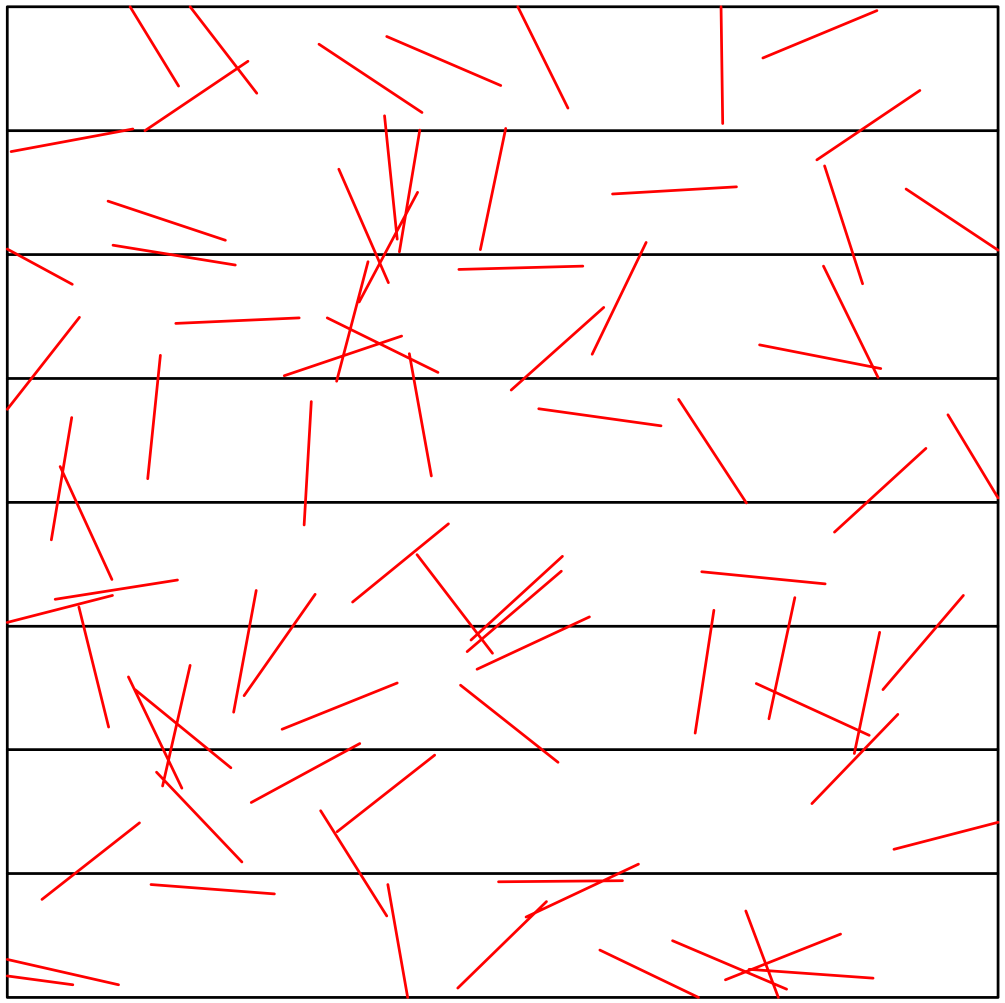
G-L Leclerc, Comte de Buffon, 1707-1788



$$2 \times 70/40 = 3.500$$

Buffon's needle

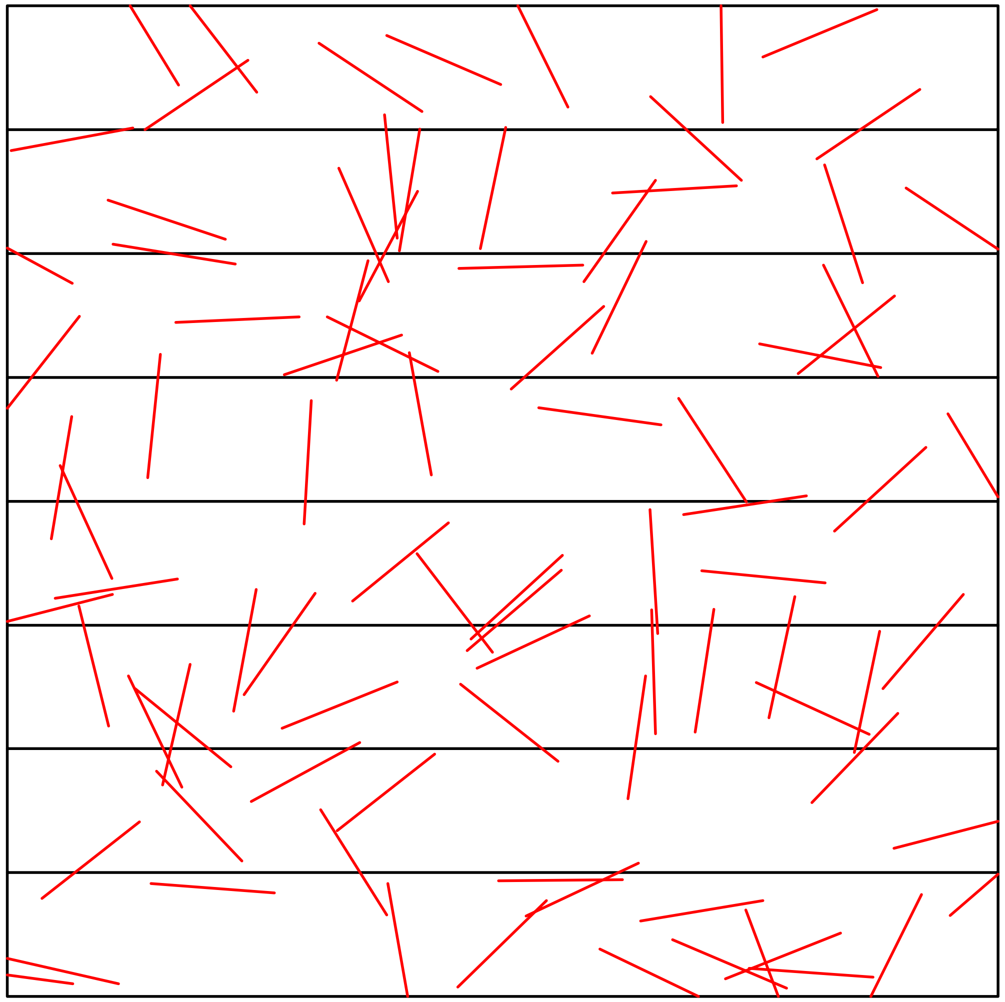
G-L Leclerc, Comte de Buffon, 1707-1788



2x80/47 = 3.404

Buffon's needle

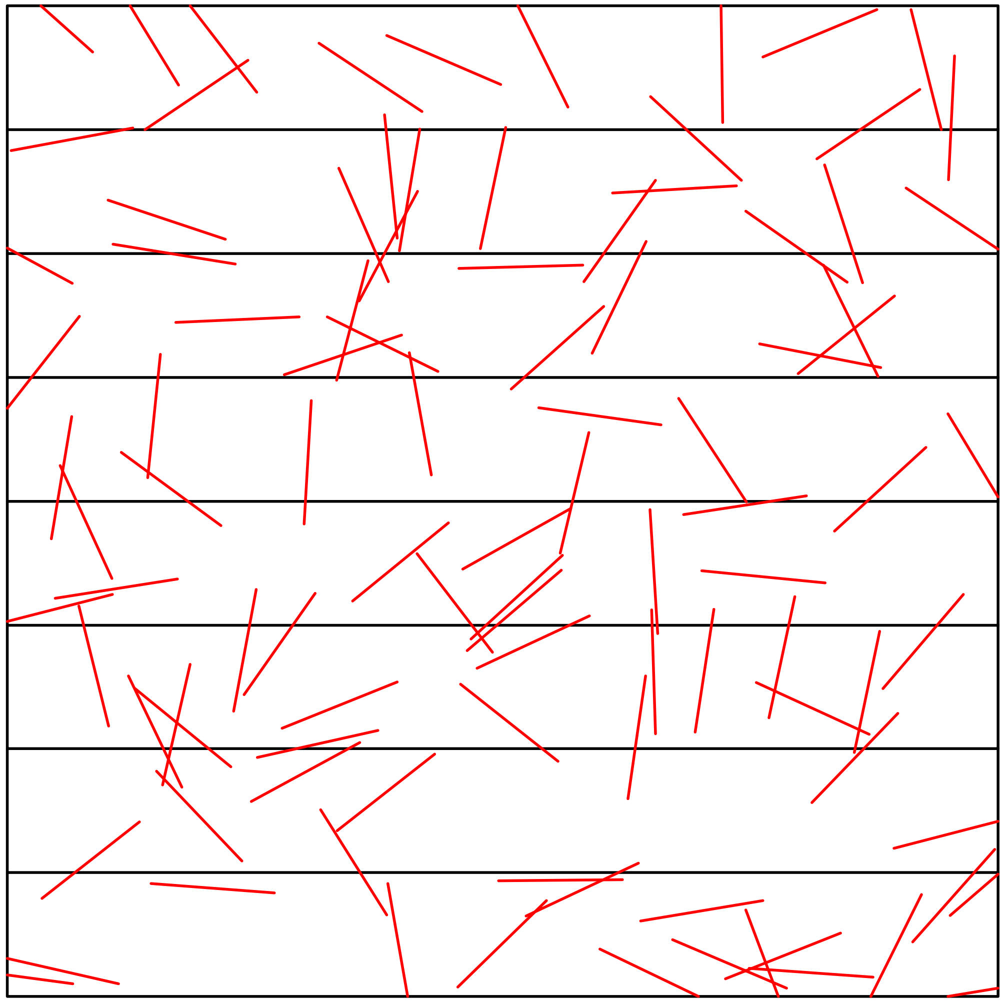
G-L Leclerc, Comte de Buffon, 1707-1788



2x90/55 = 3.273

Buffon's needle

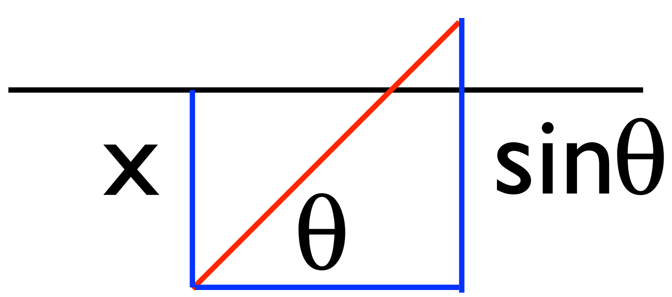
G-L Leclerc, Comte de Buffon, 1707-1788

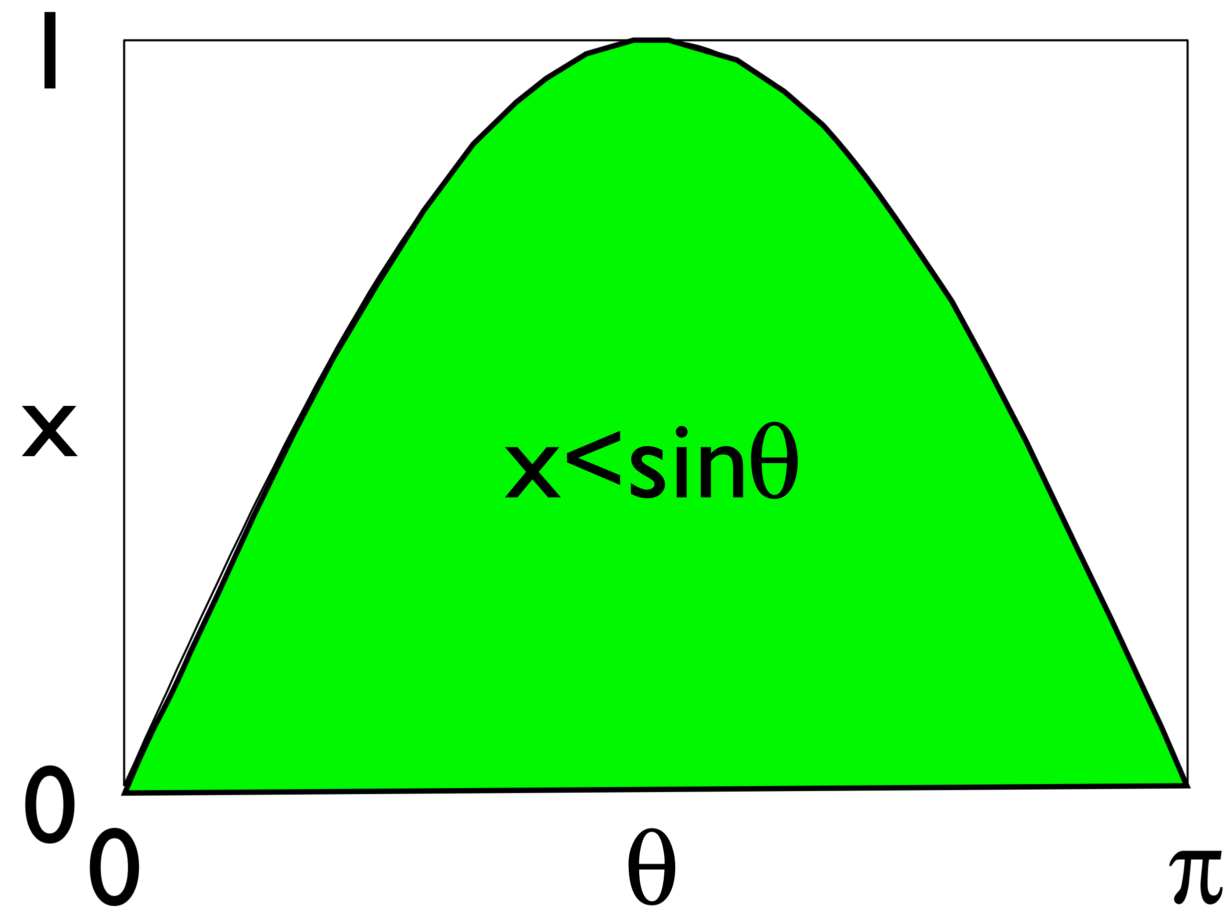


$$2 \times 100/63 = 3.175$$

Buffon's needle

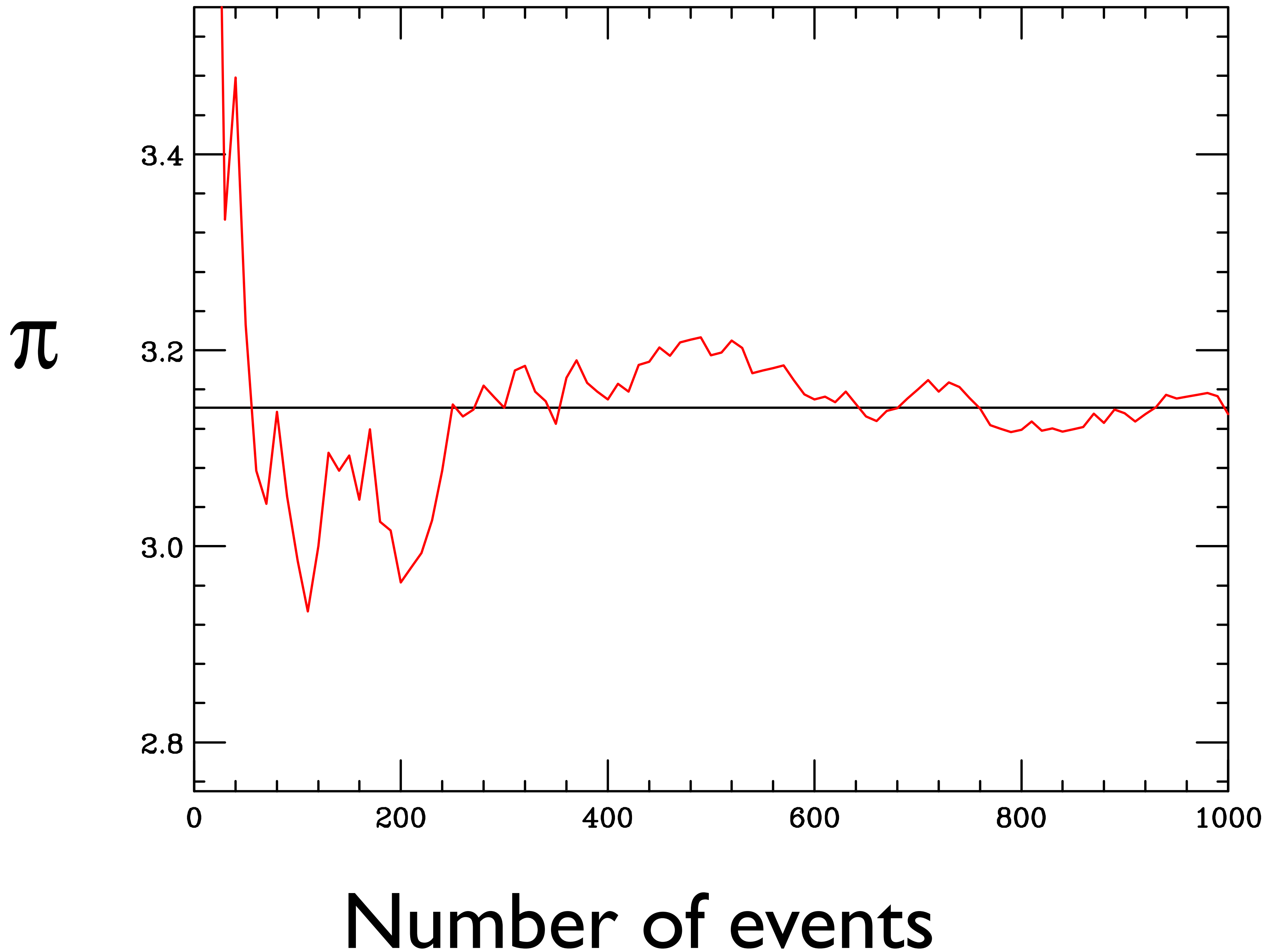
Events (needle drops) are represented by random points in (θ,x) phase space



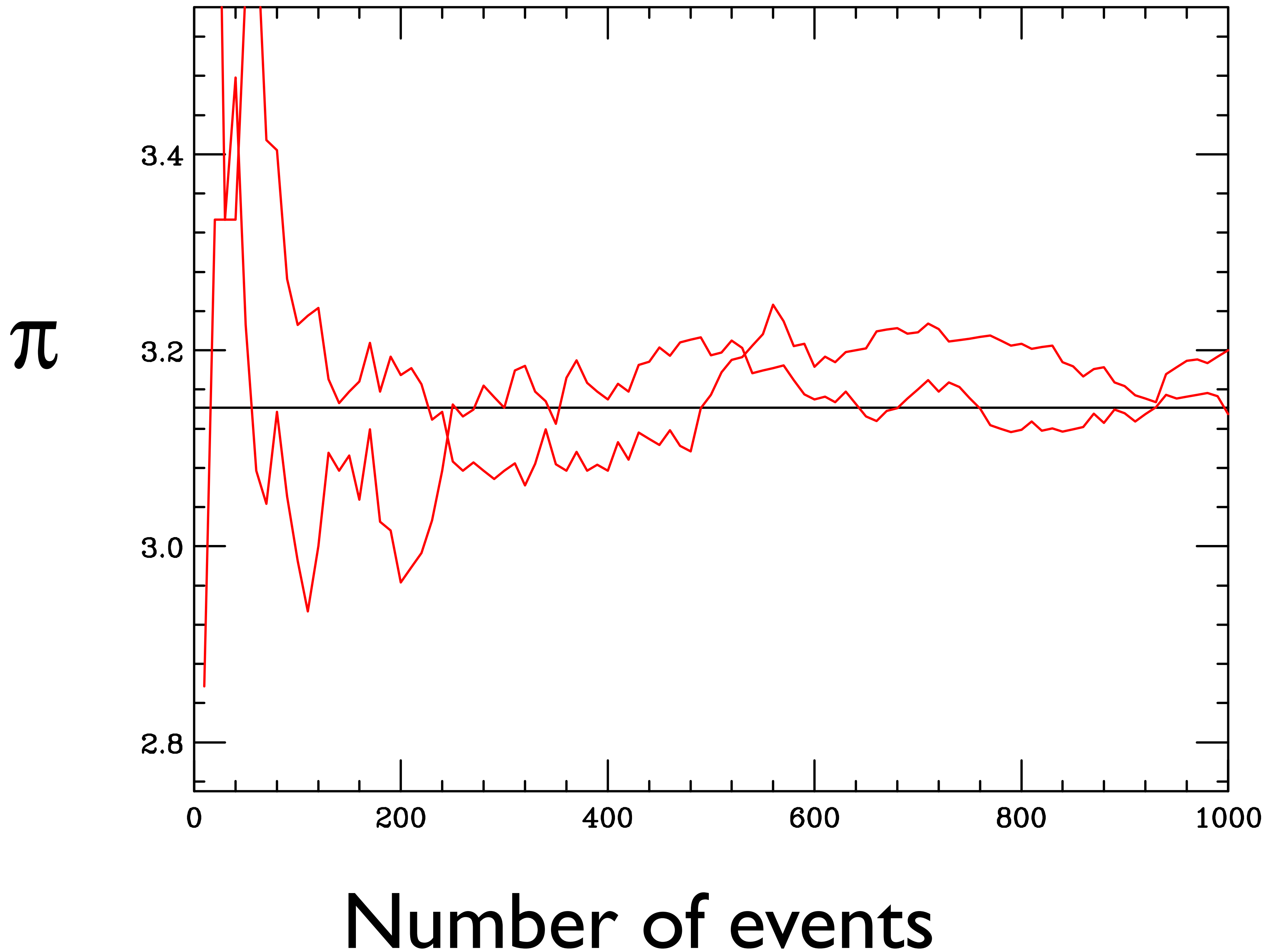


$P(x<\sin\theta)=2/\pi$

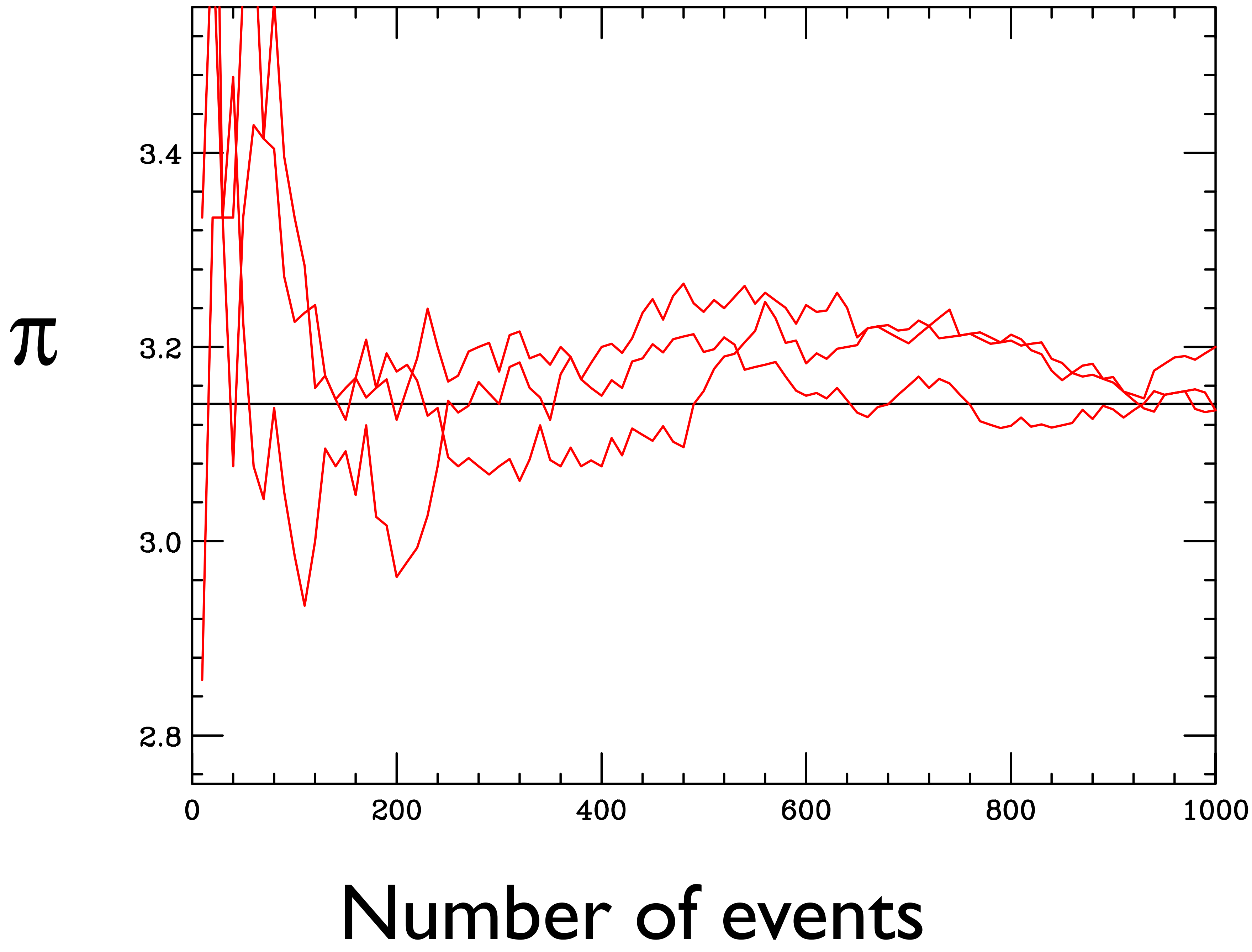
Buffon's needle



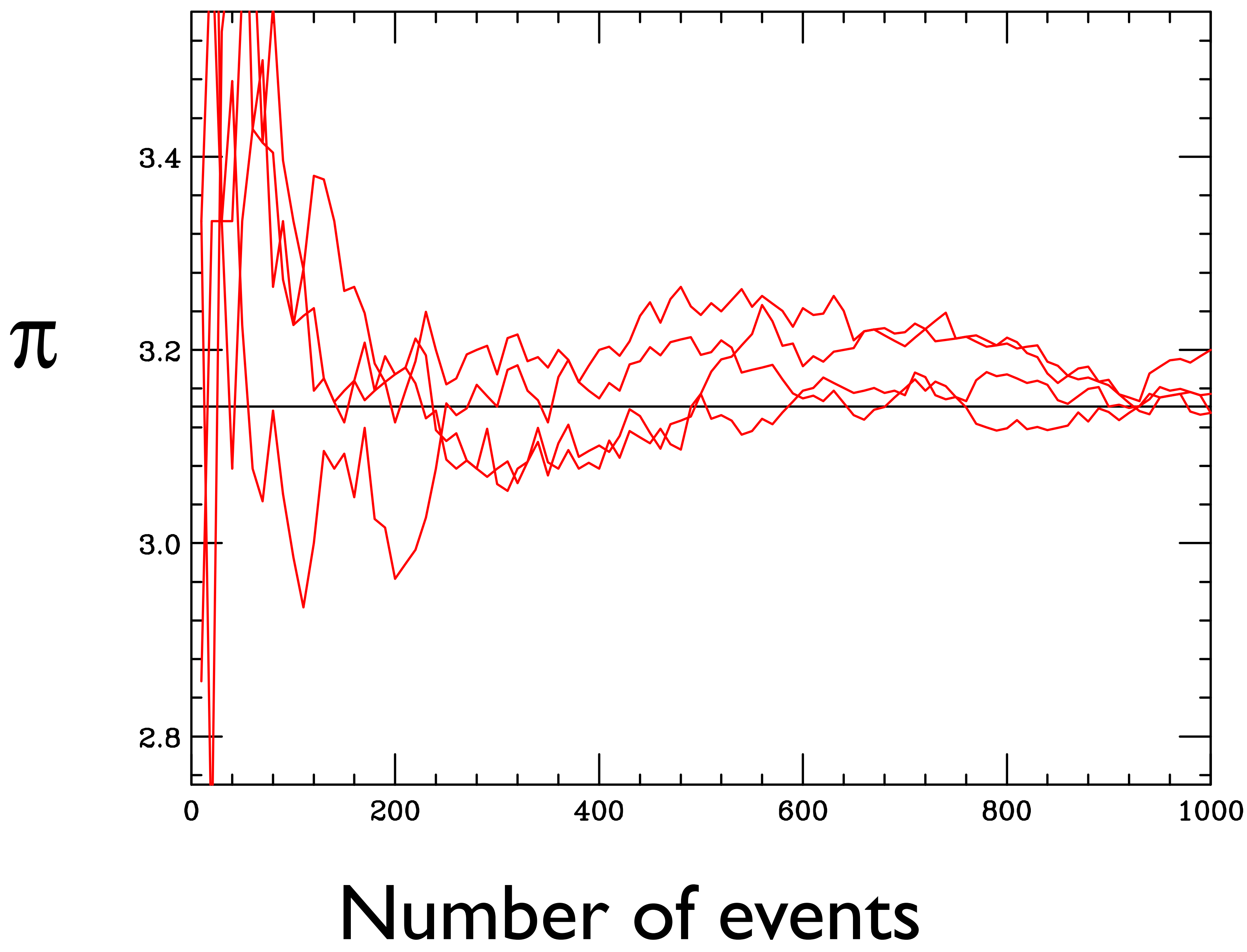
Buffon's needle



Buffon's needle



Buffon’s needle



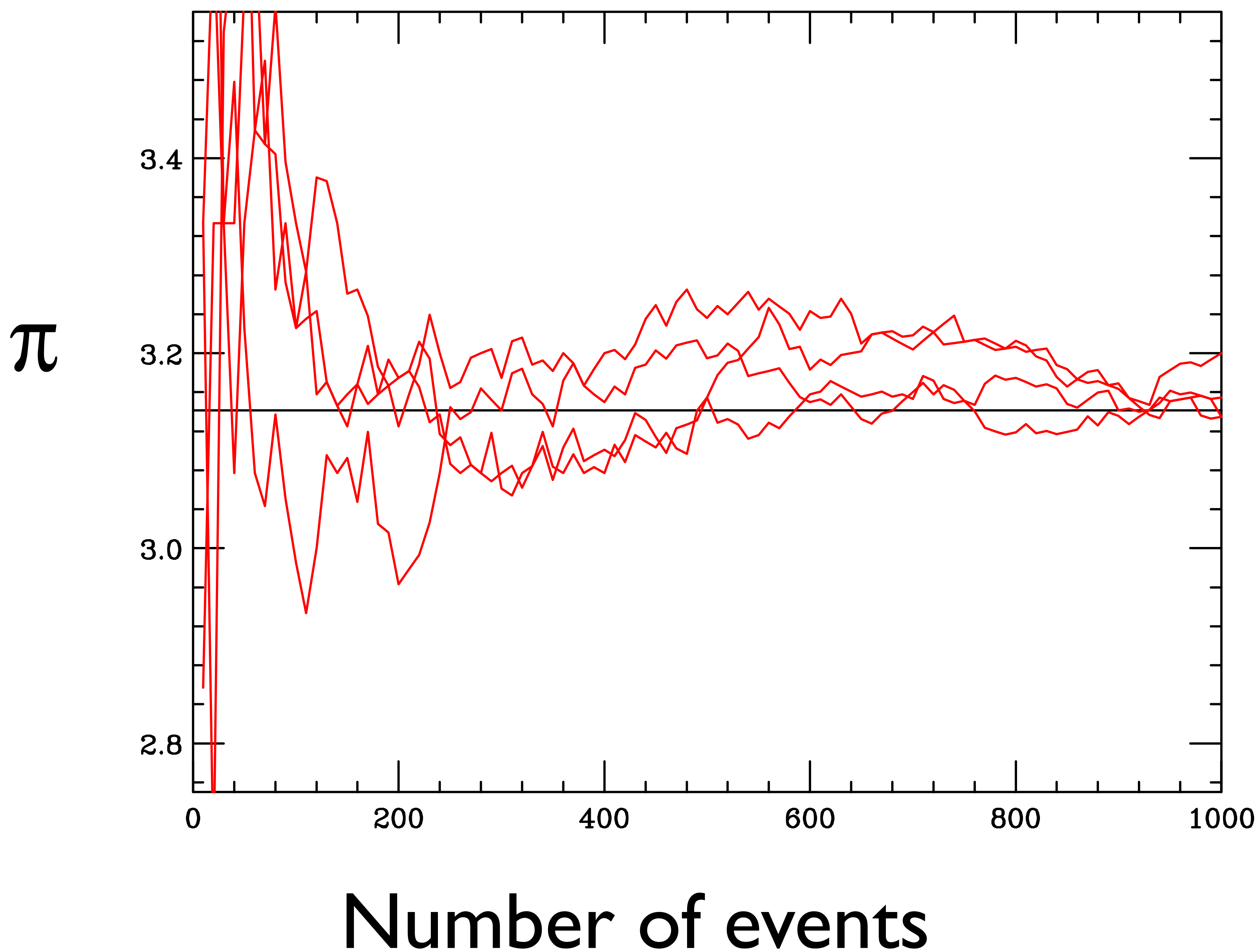
Some statistics

- Expected value of a discrete random variable = probability-weighted sum over possible outcomes = expected mean of a large number of independent trials
  - $E[X] = x_1p_1 + x_2p_2 + x_3p_3 + \ldots$
  - Here $x_1=1$ (needle on line, $p_1=2/\pi$) or else $x_1= 0$ (needle off line). Hence $E[X]=2/\pi$
- Variance = Mean square deviation
  - $Var[X] = E[(X-E[X])^2] = E[X^2]-(E[X])^2$
  - Here $Var[X] = p_1-p_1^2 = 2/\pi(1-2/\pi)$
- (RMS) Standard deviation $\sigma_X = \sqrt{Var[X]} = 0.481$ here

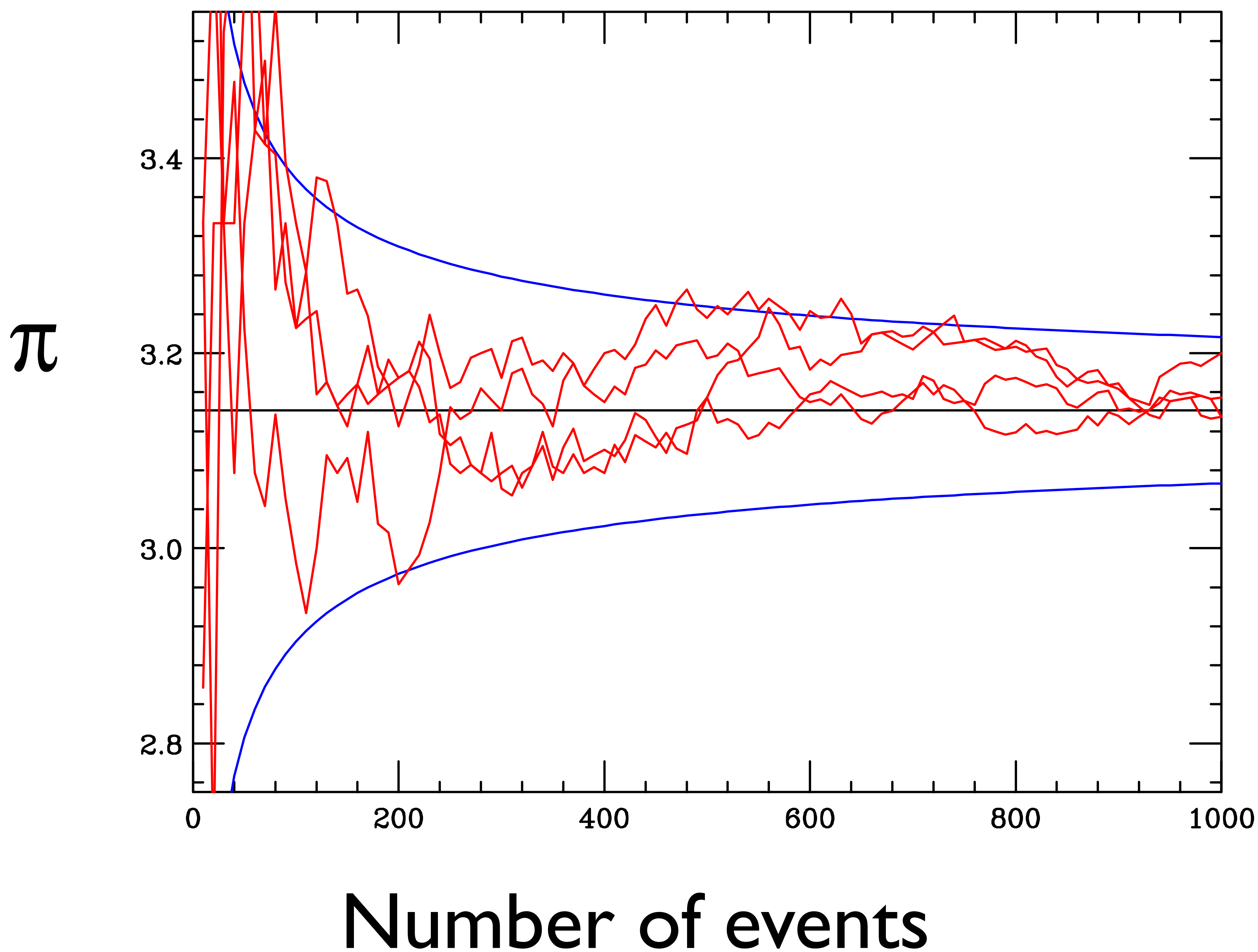
Statistics (cont'd)

- Variances for uncorrelated random variables are additive
  - $Var[X_1+X_2] = Var[X_1] + Var[X_2] +2(E[X_1X_2]-E[X_1][X_2])$
- For N identical independent trials, define $S_N=X_1+\ldots+X_N$, then
  - $E[S_N] = N\,E[X]$, $Var[S_N] = N\,Var[X]$, $\sigma_{SN} = \sqrt{N}\,\sigma_X$
- Here, for N needles, $S_N = 2N/\pi$, so $\sigma_\pi/\pi = \sigma_{SN}/S_N = \sqrt{N}\,\sigma_X/(2N/\pi)$, i.e. standard deviation in estimate of π is
  - $\sigma_\pi = \pi^2\sigma_X/(2\sqrt{N}) = 2.37/\sqrt{N}$

Buffon’s needle



Buffon's needle



Integrals as Averages

- Basis of all Monte Carlo methods: $$I = \int_{x_1}^{x_2} f(x)\, dx = (x_2 - x_1)\langle f(x)\rangle$$
- Draw N values from a uniform distribution: $$I \approx I_N \equiv (x_2 - x_1)\frac{1}{N}\sum_{i=1}^{N} f(x_i)$$
  weight
- Sum invariant under reordering: randomize
- Central limit theorem: $$I \approx I_N \pm \sqrt{V_N/N}$$

Variance $$V = (x_2 - x_1)\int_{x_1}^{x_2}\left[f(x)\right]^2 dx - \left[\int_{x_1}^{x_2} f(x)\, dx\right]^2$$

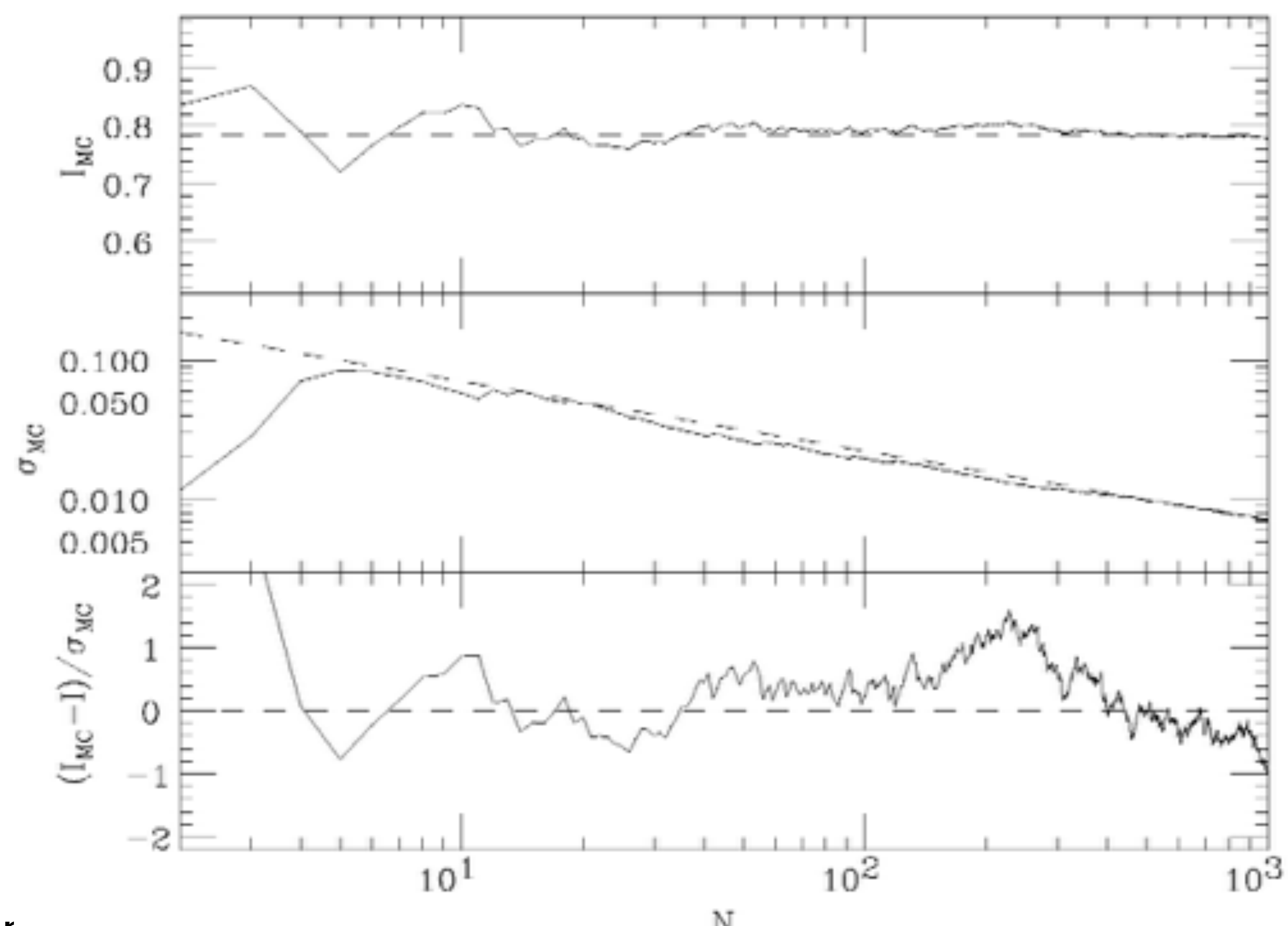
Convergence

- Monte Carlo integrals governed by Central Limit Theorem: error $\propto 1/\sqrt{N}$

c.f. trapezium rule $\propto 1/N^2$

Simpson's rule $\propto 1/N^4$

but only if derivatives exist and are finite: $\sqrt{1-x^2} \sim 1/N^{3/2}$



$$I = \frac{\pi}{4} = 0.785$$

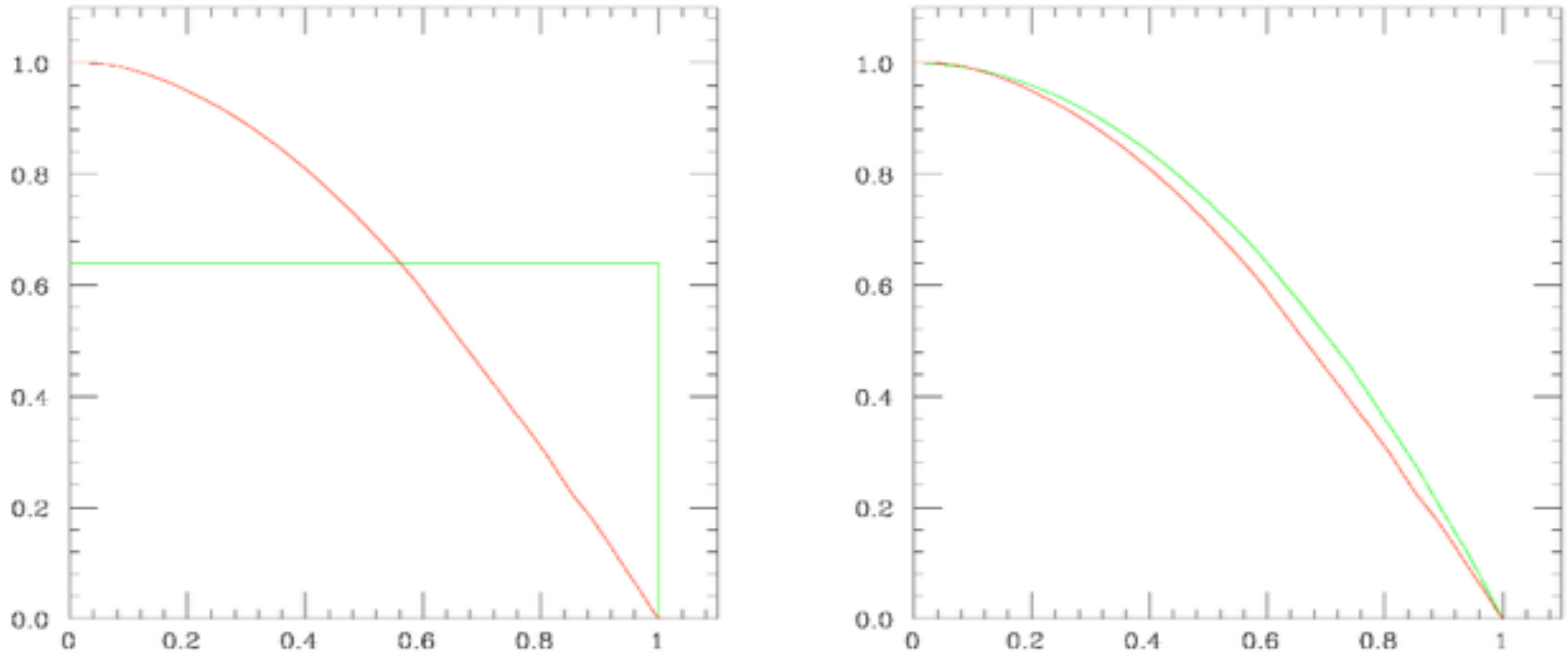
$$\sqrt{V} = 0.223$$

Importance Sampling

- Convergence improved by putting more samples in region where function is largest. ← "flattening" weight

  Corresponds to a Jacobian transformation.

- Hit-and-miss: accept points with probability = ratio (if < 1) ← unweighting



$$I = \int_0^1 dx \cos\tfrac{\pi}{2}x = 0.637 \pm 0.308/\sqrt{N}$$

$$I = \int_0^1 dx(1-x^2)\,\frac{\cos\frac{\pi}{2}x}{1-x^2} = \int d\rho\,\frac{\cos\frac{\pi}{2}x}{1-x^2}[x(\rho)] = 0.637 \pm 0.032/\sqrt{N}$$

Multi-dimensional Integration

- Formalism extends trivially to many dimensions
- Particle physics: very many dimensions,

  e.g. phase space = 3 dimensions per particles,

  LHC event ~ 250 hadrons.
- Monte Carlo error remains $\propto 1/\sqrt{N}$
- Trapezium rule $\propto 1/N^{2/d}$
- Simpson's rule $\propto 1/N^{4/d}$

Monte Carlo: Summary

Disadvantages of Monte Carlo:

- Slow convergence in few dimensions.

Advantages of Monte Carlo:

- Fast convergence in many dimensions.
- Arbitrarily complex integration regions (finite discontinuities not a problem).
- Few points needed to get first estimate ("feasibility limit").
- Every additional point improves accuracy ("growth rate").
- Easy error estimate.
- Hit-and-miss allows unweighted event generation, i.e. points distributed in phase space just like real events.

MC Efficiency = (Mean weight)/(Max weight)

Phase Space Generation

$$\sigma = \frac{1}{2s}\int |\mathcal{M}|^2 \, d\Pi_n(\sqrt{s})$$

$$\Gamma = \frac{1}{2M}\int |\mathcal{M}|^2 \, d\Pi_n(M)$$

- Phase space:

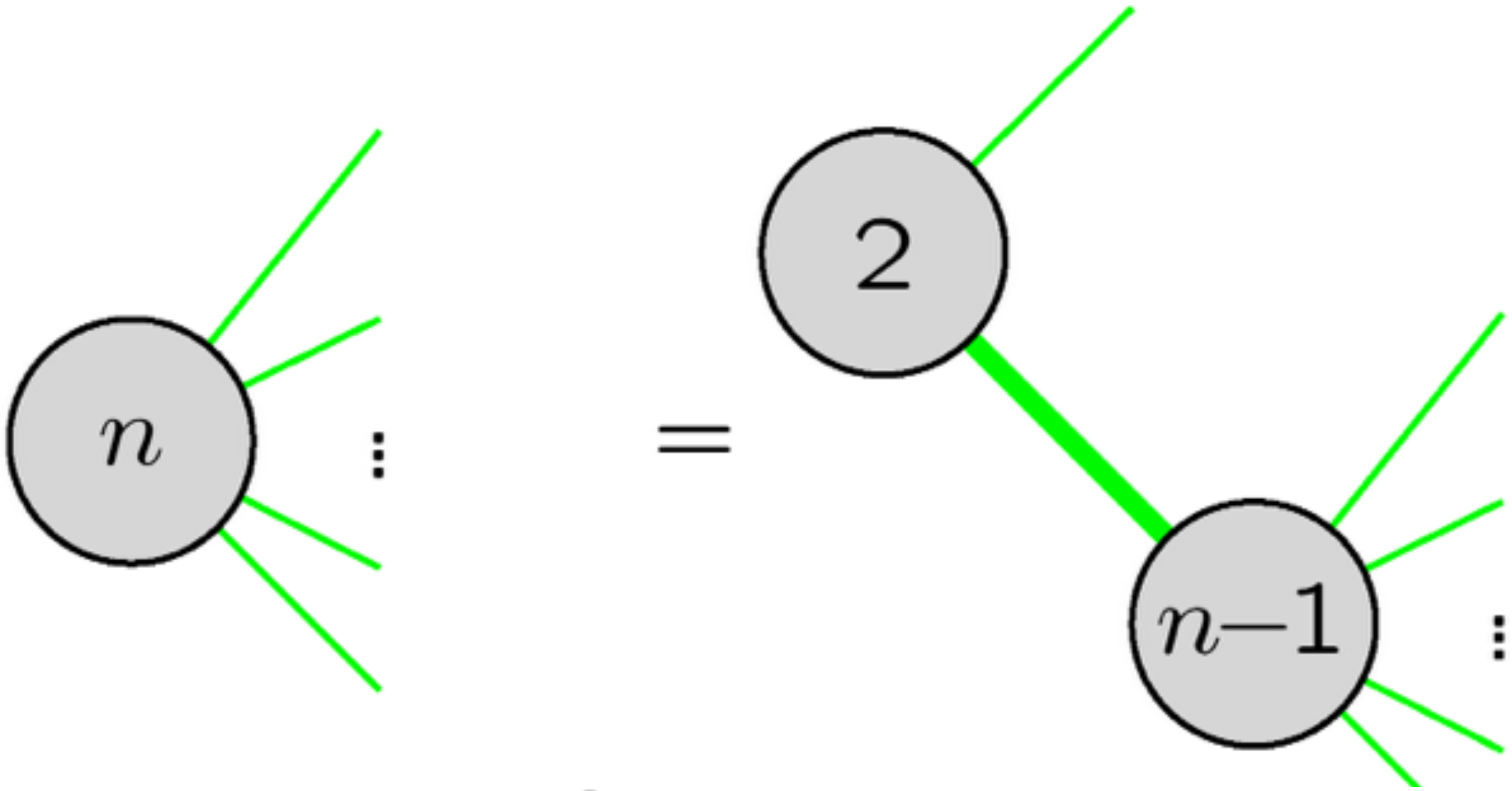
$$d\Pi_n(M) = \left[\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 (2E_i)}\right] (2\pi)^4 \delta^{(4)}\left(p_0 - \sum_{i=1}^{n} p_i\right)$$

- Two-body easy:

$$d\Pi_2(M) = \frac{1}{8\pi}\,\frac{2p}{M}\,\frac{d\Omega}{4\pi}$$

Phase Space Generation

- Other cases by recursive subdivision:



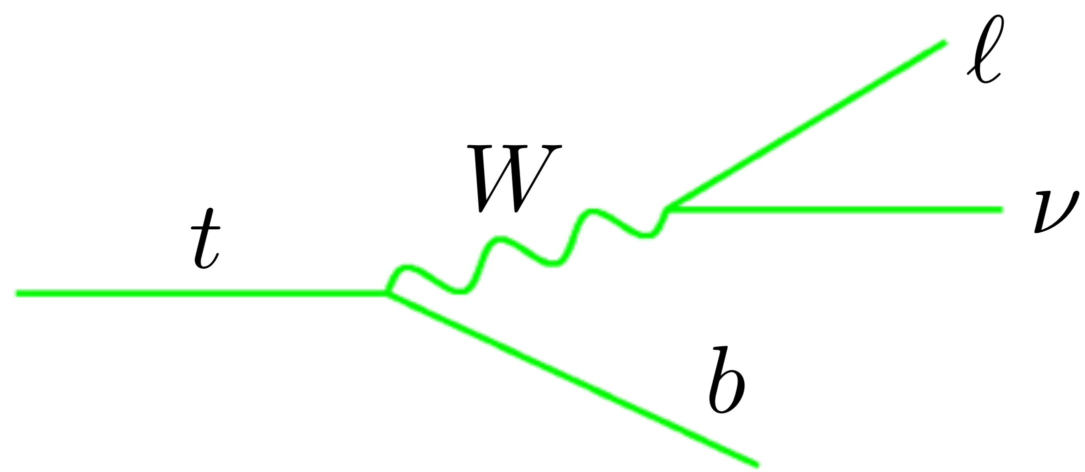
$$d\Pi_n(M) = \frac{1}{2\pi}\int_0^{(M-m)^2} dm_x^2 \, d\Pi_2(M) \, d\Pi_{n-1}(m_x)$$

- Or by 'democratic' algorithms: RAMBO, MAMBO
  Can be better, but matrix elements rarely flat.

Particle Decays

- Simplest example

  e.g. top quark decay:



$$|\mathcal{M}|^2 = \frac{1}{2}\left(\frac{8\pi\alpha}{\sin^2\theta_w}\right)^2 \frac{p_t\cdot p_\ell\, p_b\cdot p_\nu}{(m_W^2 - M_W^2)^2 + \Gamma_W^2 M_W^2}$$

$$\Gamma = \frac{1}{2M}\frac{1}{128\pi^3}\int |\mathcal{M}|^2 dm_W^2 \left(1 - \frac{m_W^2}{M^2}\right)\frac{d\Omega}{4\pi}\frac{d\Omega_W}{4\pi}$$

Breit-Wigner peak of W very strong - but can be removed by importance sampling:

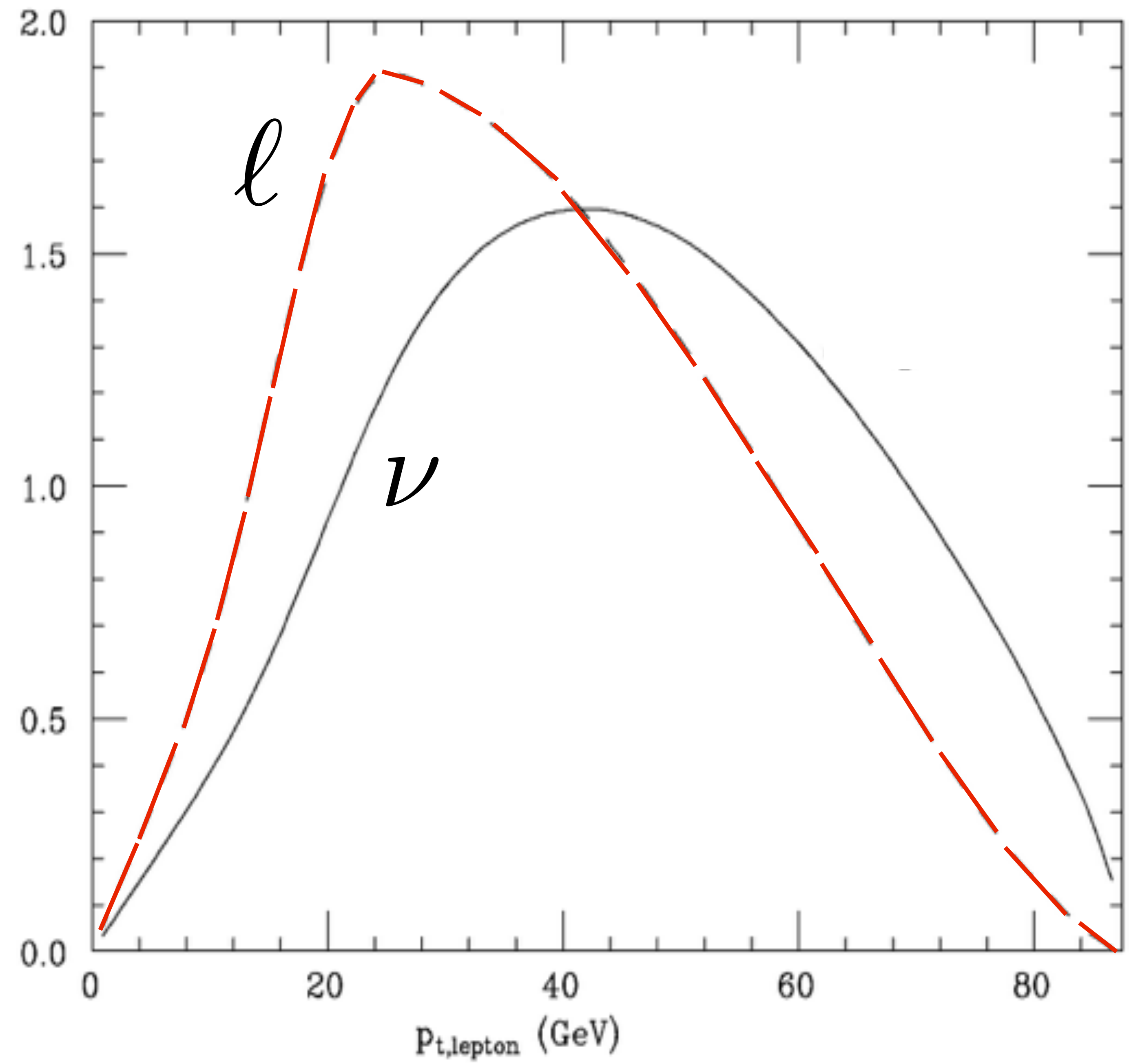
$$m_W^2 \to \arctan\left(\frac{m_W^2 - M_W^2}{\Gamma_W M_W}\right)$$ (prove it!)

Associated Distributions

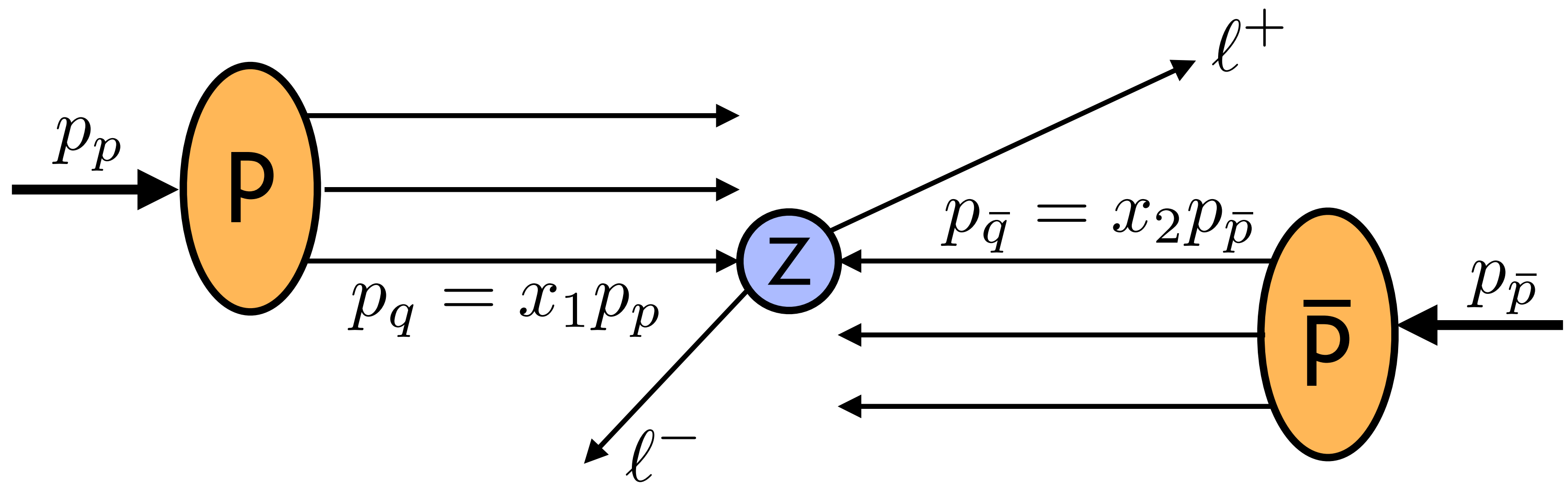
Big advantage of Monte Carlo integration:

- Simply histogram any associated quantities.
- Almost any other technique requires new integration for each observable.
- Can apply arbitrary cuts/ smearing.

e.g. lepton momentum in top decays:



Hadron-Hadron Cross Sections



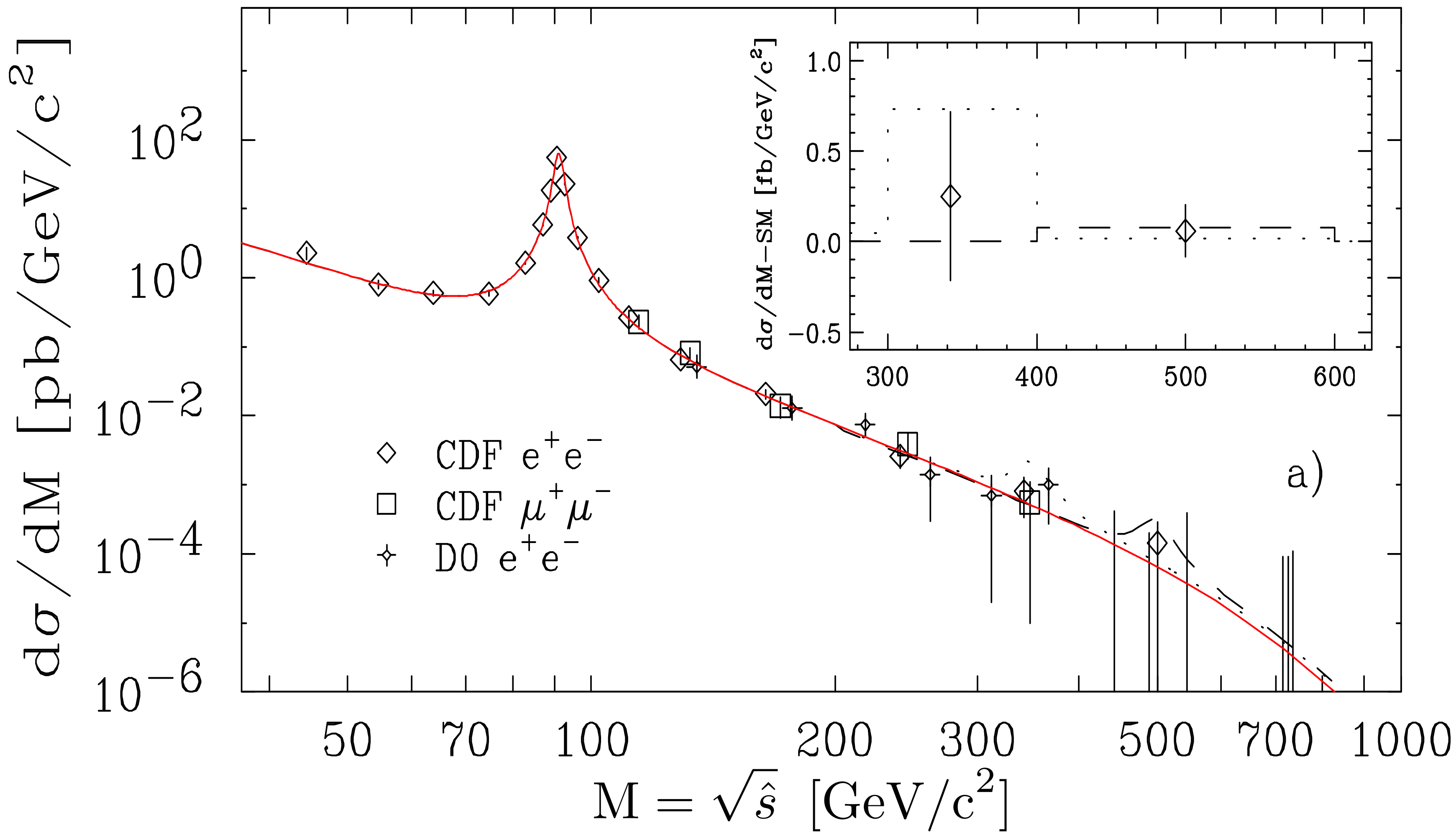
$$s = E_{\rm cm}^2 = 4 p_p p_{\bar{p}} \qquad \hat{s} = 4 p_q p_{\bar{q}} = x_1 x_2 s$$

- Consider e.g. $p\bar{p} \to Z^0 \to \ell^+ \ell^-$
- Integrations over incoming parton momentum distributions:

$$\sigma(s) = \int_0^1 dx_1 f(x_1) \int_0^1 dx_2 f(x_2)\, \hat{\sigma}(x_1 x_2 s)$$

- Hard process cross section $\hat{\sigma}(\hat{s})$ has strong peak, due to Z^0 resonance: needs importance sampling (like W in top decay)

$p\bar{p} \to \ell^+\ell^-$ cross section



$$\hat{\sigma}_{q\bar{q}\to Z^0\to\ell^+\ell^-} = \frac{4\pi\hat{s}}{3M_Z^2}\frac{\Gamma_\ell\Gamma_q}{(\hat{s}-M_Z^2)^2+\Gamma_Z^2 M_Z^2}$$

- "Background" is $q\bar{q} \to \gamma^* \to \ell^+\ell^-$

Parton-Level Monte Carlo Calculations

Now we have everything we need to make parton-level cross section calculations and distributions

Can be largely automated…

- MADGRAPH
- GRACE
- COMPHEP
- AMEGIC++
- ALPGEN

But…

- Fixed parton/jet multiplicity
- No control of large higher-order corrections
- Parton level

→ Need hadron level event generators

Summary of Lecture 1

- Monte Carlo is a very convenient numerical integration method.
- Well-suited to particle physics: difficult integrands, many dimensions.
- Integrand positive definite → event generator.
- Hard process: use parton-level generator.
- Next: parton showers and hadron-level event generation