Thermodynamic consistency of quasiparticle equation of state at finite baryon density

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Wei-Liang Qian (錢衛良)
University of Sao Paulo

In collaboration with
H.-H. Ma (馬鴻浩), K. Lin (林愷), Y. Hama (破魔陽二郎) and T. Kodama (小玉剛)
partly based on arXiv:1709.03586 (work in progress)
Abstract

- Quasiparticle model description
- Motivation
- Formalism and thermodynamic consistency
- Formal solution and Numerical results
- Concluding remarks and outlook
A brief review of quasiparticle description

- QCD predicts a phase transition between the quark-gluon dominated high energy region and the hadronic state in the low energy region.

- The two states are characterized by a dramatic difference in the number of degrees of freedom.

- A phenomenological approach for this phase transition which agree with the perturbative results at high energy and with the lattice QCD data at relatively low energies.
A brief review of quasiparticle picture

- Appropriate **degrees of freedom**
- View the system as composed of non-interacting **quasiparticle** in background mean field
- The interactions among basic constituents is incorporated through medium dependent **effective mass**

- Quasiparticles of gluon plasma with **temperature dependent mass** (Golovizinin and Satz 1993, Peshier et al 1994)
Systems of pure gauge theories for gluon plasma → inclusion of light dynamical quarks

The temperature dependence of particle mass are usually extracted from Lattice QCD calculations (Karsch et al 1988, Peshier et al 1996, etc) (for gluon plasma)

Thermal mass (Weldon 1982, Pisarski 1989, Frenkel and Taylor 1990, etc) via the pole of effective propagator

Debye screen mass through the behavior at small momentum (Klimov 1982, Pisarski 1989, Thoma 1995, etc)
A brief review of quasiparticle picture

- Various discussions on thermodynamic/statistical consistency following Gorenstein and Yang (Gorenstein and Yang 1995) and inclusively to the case of finite chemical potential

- Peshier et al 2000, 2002
- Biro et al 2003
- Gardim and Steffens 2003, 2007
- etc
Thermodynamic consistency for quasiparticle model (Gorenstein and Yang 1995)

Energy density obtained from ensemble average may not be the same as that from thermodynamic relation.
The solution is to visualize mass as an **intermediate field** which satisfies

\[
\left( \frac{\partial p}{\partial c_1} \right)_{T,\mu,c_2,...} = 0, \quad \left( \frac{\partial p}{\partial c_2} \right)_{T,\mu,c_1,...} = 0, \ldots
\]
The previous condition is achieved by introducing a specific temperature dependent bag constant

\[
\left( \frac{\partial p}{\partial c_1} \right)_{T,\mu,c_2,...} = 0, \quad \left( \frac{\partial p}{\partial c_2} \right)_{T,\mu,c_1,...} = 0, ...
\]

\[p(T, \mu, c_1, c_2, ...) = \mp T \frac{d}{2\pi^2} \int_0^\infty k^2 dk \ln \left[ 1 \mp \exp \left( -\frac{(\omega^* - \mu)}{T} \right) \right] - B^* \]

\[
\epsilon(T, \mu, c_1, c_2, ...) = \frac{d}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\exp[(\omega^* - \mu)/T] + 1} + B^* \]

\[
\frac{dB^*}{dm} = -m \frac{d}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\omega^*(k,T) \exp[\omega^*(k,T)/T]} - 1
\]
The study is extended to finite baryon density (Peshier et al 2000, Bluhm 2005, 2007)

The additional contribution due to the temperature/chemical potential dependence of particle mass should be canceled out by those of the bag constant.

The temperature/chemical potential dependence of the bag constant is through that of the screen mass.

\[
\frac{\partial B}{\partial \Pi_j^*} = \frac{\partial p_j(T, \mu_j; m_j^2)}{\partial m_j^2}
\]

\[
p(T, \mu; m_{0j}^2) = \sum_i p_i(T, \mu_i(\mu); m_i^2) - B(\Pi_j^*)
\]

\[
m_i^2 = m_{0i}^2 + \Pi_i^*,
\]
The screen mass depends on the temperature/chemical potential as well as the coupling, is given by the asymptotic values of the gauge independent hard-thermal/density-loop (HTL) self-energies.
On the other hand, thermodynamic quantities, such as entropy and particle number can be derived from the grand canonical partition function for a grand canonical ensemble.
The study is extended to finite baryon density (Peshier et al 2000, Bluhm 2005, 2007)

\[ \frac{\partial B}{\partial \Pi_j^*} = \frac{\partial p_j(T, \mu_j; m_j^2)}{\partial m_j^2} \]

\[ \sum_i \left[ \frac{\partial n_i}{\partial m_i^2} \frac{\partial \Pi_i^*}{\partial T} - \frac{\partial s_i}{\partial m_i^2} \frac{\partial \Pi_i^*}{\partial \mu} \right] = 0 \]

\[ \Pi_q^* = 2 \omega_{q0} (m_0 + \omega_{q0}), \quad \omega_{q0}^2 = \frac{N_c^2 - 1}{16N_c} \left[ T^2 + \frac{\mu_q^2}{\pi^2} \right] G^2 \]

\[ \Pi_g^* = \frac{1}{6} \left[ \left( N_c + \frac{1}{2} N_f \right) T^2 + \frac{N_c}{2\pi^2} \sum_q \mu_q^2 \right] G^2 \]

\[ s_i = \left. \frac{\partial p_i(T, \mu_i; m_i^2)}{\partial T} \right|_{m_i^2}, \quad n_i = \left. \frac{\partial p_i(T, \mu_i; m_i^2)}{\partial \mu_i} \right|_{m_i^2} \]

Therefore, Maxwell relation implies
Therefore, one obtains a **partial differential equation** for the coupling constant.
If one uses the running coupling in the asymptotic limit of large temperature and vanishing chemical potential as the boundary condition, one may obtain the value at finite chemical potential.
The arguments are simply based on the quasiparticle ansatz and thermodynamic consistency.
The study is extended to finite baryon density (Peshier et al 2000, Bluhm 2005, 2007)

The numerical results, inclusively those of the quark number susceptibility, are perfectly compared to those obtained by lattice QCD calculations.
Thermodynamic consistency in quasiparticle model in finite chemical potential

- The additional contribution due to the temperature/chemical potential dependence of particle mass should be **canceled out** by those of the bag constant.

\[
\frac{d \mathcal{B}}{d m} = \left. \frac{\partial \rho(T, \mu, m)}{\partial m} \right|_{T, \mu}
\]
 Thermodynamic consistency in quasiparticle model in finite chemical potential

- The condition of thermodynamic consistency is implemented assuming bag constant is a function of \text{screen mass}

\[
\frac{dB}{dm} = \left. \frac{\partial p(T, \mu, m)}{\partial m} \right|_{T,\mu}
\]
The condition of thermodynamic consistency is implemented assuming bag constant is a function of screen mass.

If one writes down the r.h.s. of the equation, one finds the explicit form of temperature and chemical potential dependence of the expression.
Thermodynamic consistency in quasiparticle model in finite chemical potential

- The condition of thermodynamic consistency is implemented assuming bag constant is a function of screen mass.

\[ \frac{dB}{dm} = \left. \frac{\partial p(T, \mu, m)}{\partial m} \right|_{T,\mu} \]

\[ -\frac{g}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\omega^*(k, T, \mu)} \exp\left[\frac{\omega^*(k, T, \mu) - \mu}{T}\right] + \frac{1}{m} \]

- The explicit form of temperature and chemical potential dependence of the r.h.s. implies that either bag constant is also an explicit function of temperature/chemical potential, or one cannot freely choose the form of particle mass as a function of those variables.
The condition of thermodynamic consistency is implemented assuming bag constant is a function of screen mass

\[
\frac{dB}{dm} = \left. \frac{\partial p(T, \mu, m)}{\partial m} \right|_{T, \mu}
\]

The former is not possible, because it implies extra terms for thermodynamical quantities which breaks the thermodynamic consistency, also it has been proven that it is not the case (Biro et al 2003)
The condition of thermodynamic consistency is implemented assuming bag constant is a function of screen mass.

Therefore, we conclude that there exist an extra constrain for the form of screen mass, again, owing to the thermodynamic consistency.
Gorenstein and Yang’s criterion + Maxwell relation for the bag constant leads to a partial differential equation which can be solved by the method of characteristics.
Thermodynamic consistency in quasiparticle model in finite chemical potential

\[
\frac{\partial m}{\partial \mu} = \frac{T}{(\omega^*(k, T, \mu) - \mu)} \frac{\partial m}{\partial T}.
\]

\[
w = \omega^* - \mu
\]

\[
w \frac{\partial w}{\partial \mu} - T \frac{\partial w}{\partial T} + w = 0.
\]

\[
\frac{d\mu}{w} = \frac{dT}{-T} = \frac{dw}{-w}.
\]

\[
d(\mu + w) = 0.
\]

\[
d \left[ \ln \left( \frac{w}{T} \right) \right] = d \left( \frac{w}{T} \right) = 0.
\]

\[
F \left( \frac{w}{T}, (w + \mu) \right) = 0.
\]

➢ Method of characteristics
Thermodynamic consistency in quasiparticle model in finite chemical potential

\[
\frac{\partial m}{\partial \mu} = \frac{T}{(\omega^* (k, T, \mu) - \mu)} \frac{\partial m}{\partial T}.
\]

\[ w = \omega^* - \mu \]

\[
w \frac{\partial w}{\partial \mu} - T \frac{\partial w}{\partial T} + w = 0.
\]

\[
d\mu \frac{w}{w} = dT \frac{w}{-T} = dw \frac{w}{-w}.
\]

\[ d(\mu + w) = 0. \]

\[ d \left[ \ln \left( \frac{w}{T} \right) \right] = d \left( \frac{w}{T} \right) = 0. \]

\[ F \left( \frac{w}{T}, (w + \mu) \right) = 0. \]

- (Unfortunately) the above analytic solution only serves for the case where one ignores antiparticles.
When considering antiparticles, the solution is a surface in temperature-chemical potential coordinates which consists of characteristic curves for given omega star.
Thermodynamic consistency in quasiparticle model in finite chemical potential

When considering antiparticles, the solution is a surface in temperature-chemical potential coordinates which consists of characteristic curves for given omega star.

While the boundary condition is given by lattice data at vanishing chemical potential.
Present work: Energy density and particle number are expressed in terms of ensemble average which is consistent with those obtained by grand canonical partition function (Gorenstein et al 1995, 2010)

\[ Q_G = \langle \exp[-\alpha \hat{N} - \beta \hat{H}_{\text{eff}}] \rangle, \]

\[ \hat{H}_{\text{eff}} = \hat{H}_{\text{id}} + E_0 + E_1. \]

\[ \hat{H}_{\text{id}} = \sum_j \sum_k \omega(k) a_{k,j}^\dagger a_{k,j}, \]

There is additional term E1 which is linear in temperature
Present work: Energy density and particle number are expressed in terms of ensemble average which is consistent with those obtained by grand canonical partition function (Gorenstein et al 1995, 2010)

\[ Q_G = \langle \exp[-\alpha \hat{N} - \beta \hat{H}_{\text{eff}}]\rangle, \]

\[ \varepsilon = \frac{\langle E \rangle}{V} = -\frac{1}{V} \frac{\partial \ln Q_G}{\partial \beta} = \varepsilon_{id} + \frac{E_0}{V} + \frac{E_1}{V} + \frac{1}{V} \langle \beta \frac{\partial E_1}{\partial \beta} \rangle = \varepsilon_{id} + B. \]

\[ \varepsilon_{id} = \frac{g}{2\pi^2} \int_0^{\infty} \frac{k^2 dk \omega^*(k, T, \mu)}{\exp[(\omega^*(k, T, \mu) - \mu)/T]} + \text{c.t.,} \]

\[ \lim_{V \to \infty} \frac{E_0}{V} = B \]
Present work: Energy density and particle number are expressed in terms of ensemble average which is consistent with those obtained by grand canonical partition function (Gorenstein et al 1995, 2010)

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\]

\[
p = \frac{1}{\beta} \frac{\partial \ln Q_G}{\partial V} = \frac{1}{\beta} \frac{\ln Q_G}{V} = p_{id} - B - \frac{E_1}{V}.
\]

There is **one more free parameter** in the expression of pressure
Present work: Energy density and particle number are expressed in terms of ensemble average which is consistent with those obtained by grand canonical partition function (Gorenstein et al 1995, 2010)

\[ Q_G = \langle \exp[-\alpha \hat{N} - \beta \hat{H}_{\text{eff}}] \rangle, \]

\[ \varepsilon = \frac{\langle E \rangle}{V} = -\frac{1}{V} \frac{\partial \ln Q_G}{\partial \beta} = \varepsilon_{id} + \frac{E_0}{V} + \frac{E_1}{V} + \frac{1}{V} \langle \beta \frac{\partial E_1}{\partial \beta} \rangle = \varepsilon_{id} + B. \]

\[ p = \frac{1}{\beta} \frac{\partial \ln Q_G}{\partial V} = \frac{1}{\beta} \frac{\ln Q_G}{V} = p_{id} - B - \frac{E_1}{V}. \]

\[ n = \frac{\langle N \rangle}{V} = -\frac{1}{V} \frac{\partial \ln Q_G}{\partial \alpha} = n_{id}, \]
The quasiparticle model employed in the present study

- We note that the derived quantities must satisfy the “first law of thermodynamics”, therefore any other quantity can be unambiguously derived from thermodynamic potential.
The quasiparticle model employed in the present study

- We note that the derived quantities must satisfy the “first law of thermodynamics”, therefore any other quantity can be unambiguously derived from thermodynamic potential.

- In fact, one can readily verify that

\[ dq = -\langle N \rangle d\alpha - \langle E \rangle d\beta - \beta pdV. \]

- By matching

\[ \beta = \frac{1}{k_B T}, \]
\[ \alpha = -\frac{\mu}{k_B T}, \]
\[ q + \alpha N + \beta E = \frac{S}{k_B}. \]

- One has

\[ d\langle E \rangle = TdS - pdV + \mu d\langle N \rangle. \]
Indeed, one can readily verify that the quantities obtained by ensemble average indeed furnish the “first law of thermodynamics”, and subsequently one can derive the thermodynamic potential and thus any other quantity without any ambiguity.
Thermodynamical consistency check: we compare our results with an approach which assumes the form of screen mass inspired by HTL results.

\[
\begin{align*}
    m_g^2 &= \frac{3}{2} \omega_p^2 \\
    m_q^2 &= (m_{q0} + m_f)^2 + m_f^2 \\
    \omega_p^2 &= a_q g^2 \frac{n_q}{T} + \sum_q a_q^2 g^2 \frac{n_q}{T}, \\
    m_f^2 &= b_q g^2 \frac{n_q}{T}, \\
    \alpha_s(T) &\equiv \frac{g^2}{4\pi} = \frac{6\pi}{(33 - 2n_f) \ln(T/\Lambda_T)} \left(1 - \frac{3(153 - 19n_f) \ln(2\ln(T/\Lambda_T))}{(33 - 2n_f)^2 \ln(T/\Lambda_T)}\right).
\end{align*}
\]
Thermodynamical consistency check: we compare our results with an approach which assumes the form of screen mass inspired by HTL results.

\[ m^2_g = \frac{3}{2} \omega_p^2 \]
\[ m^2_q = (m_{q0} + m_f)^2 + m_f^2 \]
\[ \omega_p^2 = a_q^2 g^2 n_q \frac{n_q}{T} + \sum_q a_q^2 g^2 n_q \frac{n_q}{T}, \]
\[ m_f^2 = b_q^2 g^2 n_q \frac{n_q}{T}, \]
\[ \alpha_s(T) \equiv \frac{g^2}{4\pi} = \frac{6\pi}{(33 - 2n_f) \ln(T/\Lambda_T)} \left(1 - \frac{3(153 - 19n_f) \ln(2 \ln(T/\Lambda_T))}{(33 - 2n_f)^2 \ln(T/\Lambda_T)} \right). \]
➢ Thermodynamical consistency check: we compare our results with an approach which assumes the form of screen mass inspired by HTL results

➢ We compare the calculated bag constant by using different paths in the temperature-chemical potential plane
We compare the calculated bag constant with a more straightforward calculation which assumes the form of screening mass inspired by HTL results.

### Numerical Results

<table>
<thead>
<tr>
<th>T</th>
<th>μ</th>
<th>the present model</th>
<th>discrepancy</th>
<th>preassumed form for $m(T,\mu)$</th>
<th>discrepancy</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>path 1</td>
<td>path 2</td>
<td>path 1</td>
<td>path 2</td>
</tr>
<tr>
<td>0.1 GeV</td>
<td>0.1 GeV</td>
<td>$-6.63 \times 10^{-3}$</td>
<td>$-6.63 \times 10^{-3}$</td>
<td>0.043%</td>
<td>$1.52 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.15 GeV</td>
<td>0.1 GeV</td>
<td>$-3.34 \times 10^{-3}$</td>
<td>$-3.34 \times 10^{-3}$</td>
<td>0.047%</td>
<td>$9.27 \times 10^{-3}$</td>
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<tr>
<td>0.20 GeV</td>
<td>0.1 GeV</td>
<td>$2.49 \times 10^{-2}$</td>
<td>$2.49 \times 10^{-2}$</td>
<td>0.010%</td>
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<tr>
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<td>$-6.83 \times 10^{-3}$</td>
<td>0.072%</td>
<td>$2.15 \times 10^{-2}$</td>
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<tr>
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<td>$-2.65 \times 10^{-3}$</td>
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<td>$-1.44 \times 10^{-3}$</td>
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<td>$2.92 \times 10^{-2}$</td>
<td>0.066%</td>
<td>$1.07 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Preliminary results
Numerical Results

Preliminary results

We compare our results with a more straightforward calculation which assumes the form of screen mass inspired by HTL results.

We compare the calculated bag constant by using different paths in the temperature-chemical potential plane.
Next, we try to see whether the present scheme, with less controllable free parameters, can still accommodate the lattice QCD results, especially, those on chemical potential dependence.
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We adjust the temperature (at zero chemical potential) dependent quark masses to fix the quark number susceptibility.

Then use the temperature dependent gluon screen mass and E1 to reproduce pressure and energy density at zero baryon density.
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We adjust the temperature (at zero chemical potential) dependent quark masses to fix the quark number susceptibility.

Then use the temperature dependent gluon screen mass and E1 to reproduce pressure and energy density at zero baryon density.
Next, we try to see whether the present scheme, with less controllable free parameters, can accommodate the lattice QCD results on chemical potential dependence. We adjust the temperature (and therefore chemical potential) dependent quark mass to fix the quark number susceptibility and then use the temperature dependent gluon screen mass to reproduce pressure at zero baryon density.

Numerical Results

Preliminary results

\[ p(T^4) \]

\[ (\varepsilon-3p)/T^4 \]

\[ \varepsilon/T^4 \]

\[ s/T^3 \]
Numerical Results

- Results of Peshier et al. can be obtained by considering another possible solution where particle mass is only a function of temperature and chemical potential.

Preliminary results
We review the thermodynamic consistency in quasiparticle model with finite chemical potential and suggest that there is an possible solution where the screen mass is also a function of momentum.

The treatment is essentially based on the quasiparticle ansatz and arguments concerning thermodynamical consistency.

Our results shows that the quasiparticle ansatz is consistent with the lattice QCD data.

Future plan: inclusion of critical end point, particle number fluctuations via hydro…
谢谢!

thanks!