ARIZONA STATE UNIVERSITY

POLYTECHNIC CAMPUS

XIV Hadron Physics

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Florianópolis, Brazil

MAGNETIC CATALYSIS IN QCD IN A SUPERSTRONG MAGNETIC FIELD

XIV International Workshop on Hadron Physics

Lecture #1

Igor Shovkovy
Arizona State University

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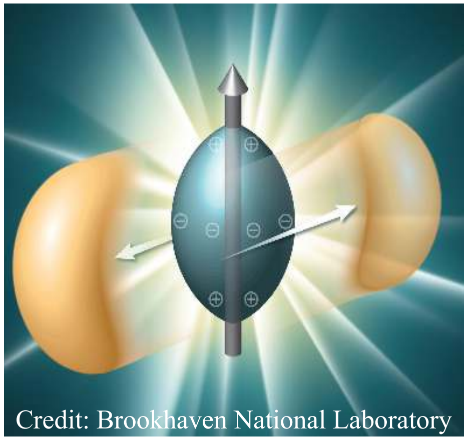
Magnetic Catalysis: Plan of Lectures

- Dirac fermions in magnetic field
- Dimensional reduction
- Magnetic catalysis: basics
- Magnetic catalysis in toy model
- Magnetic catalysis in QED
- Magnetic catalysis in QCD
- Anisotropic confinement
- Inverse catalysis
- Phase diagram

QCD in magnetic Fields

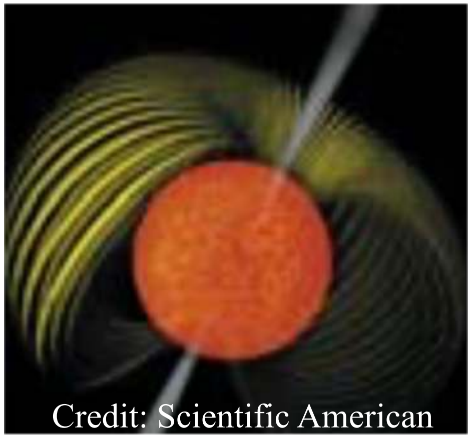
- Relativistic collisions of *heavy ions* produce quark-gluon plasma & strong magnetic fields

10^{18} - 10^{19} Gauss ($\sqrt{|eB|}\sim$ 100 MeV)



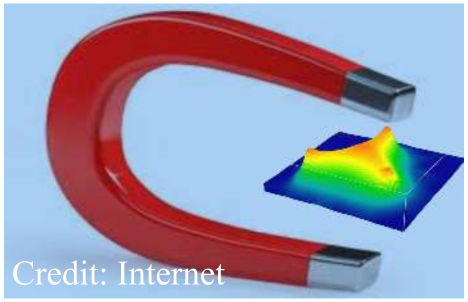
- Quark matter may form inside *magnetars*

10^{14} - 10^{16} Gauss ($\sqrt{|eB|}\sim$1 MeV to 10 MeV)

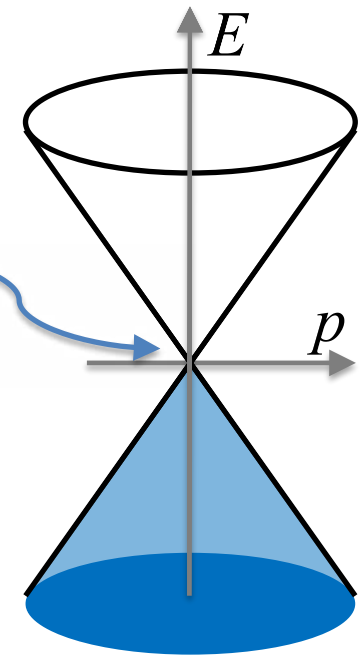


- Strong magnetic field is an instructive *theoretical tool* to study confined gauge theories such as QCD

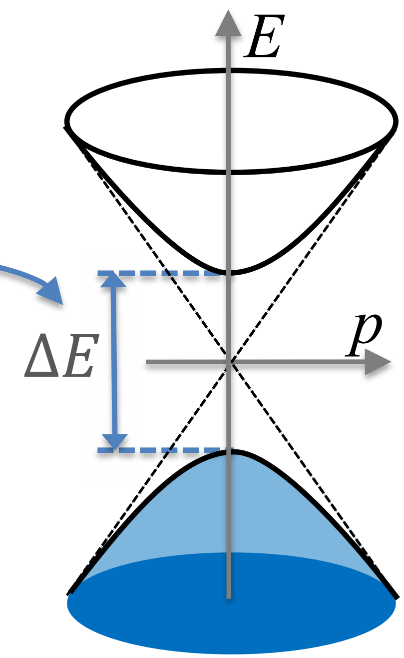
$\gtrsim 10^{19}$ Gauss ($\sqrt{|eB|}\gtrsim$100 MeV to 10 MeV)



DIRAC VACUUM



- At $m = 0$, the Dirac vacuum is a $\boxed{\text{semimetal}}$
  - No energy gap between the filled Dirac sea states and the empty positive-energy states ($E = \pm p$)
  - However, the density of states *vanishes* at E=0
  - A nonzero electric current could be produced by an arbitrarily small electric field



- At $m \neq 0$, the Dirac vacuum is an $\boxed{\text{insulator}}$
  - Energy gap $\Delta E = 2m$ between the antiparticle and particle states $\left(E = \pm\sqrt{p^2 + m^2}\right)$
  - the density of states @ E=0 *vanishes* (no states)
  - electric current is exponentially small, i.e., $e^{-\pi\, m^2/|eE|}$ (due to Schwinger pair creation)

DIRAC FERMIONS

- Lagrangian density for charged Dirac fermions (units with $c = 1$):

$$\mathcal{L} = \bar{\psi}\,(i\gamma^{\mu} D_{\mu} - m)\psi$$

where $D_{\mu} = \partial_{\mu} + ieA_{\mu}$, $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$ and $g^{\mu\nu} = (1,-1,-1,-1)$

- Consider the following two types of global transformations:

Electric charge conservation

$$\boxed{\psi \rightarrow e^{i\alpha}\psi} \quad \text{and} \quad \boxed{\psi \rightarrow e^{i\alpha\gamma^{5}}\psi}$$

Chiral charge "conservation" at m=0

where $\gamma^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$

The corresponding Noether's currents are

$$\boxed{j^{\mu} = \bar{\psi}\gamma^{\mu}\psi} \quad \text{and} \quad \boxed{j_{5}^{\mu} = \bar{\psi}\gamma^{\mu}\gamma^{5}\psi}$$

They satisfy the relations:

$$\boxed{\partial_{\mu} j^{\mu} = 0} \quad \text{and} \quad \boxed{\partial_{\mu} j_{5}^{\mu} = 2i\,m\,\bar{\psi}\gamma^{5}\psi}$$

Both transformations are **symmetries** when $m = 0$, but chiral symmetry is broken when $m \neq 0$. *[The chiral anomaly may complicate the situation]*

DIRAC FERMIONS AT B≠0

- Dirac equation for charged fermions:

$$\left(i\gamma^\mu D_\mu - m\right)\psi = 0$$

where $A_\mu = \left(A_0, -\vec{\boldsymbol{A}}\right)$ and the Landau gauge $\vec{\boldsymbol{A}} = (-By, 0{,}0)$ is used.

- Look for a solution in the form: $\psi = \left(i\gamma^\mu D_\mu + m\right)\phi$. Then,

$$\left[-\partial_0^2 + (\partial_x + ieBy)^2 + \partial_y^2 + \partial_z^2 + i\gamma^1\gamma^2 eB - m^2\right]\phi{=}0$$

- Normalized solutions for ϕ have the form

$$\phi_{k,\pm} \propto \frac{1 \pm i\mathrm{sgn}(eB)\gamma^1\gamma^2}{2}\varphi_k(y)e^{-i\omega t + ip_x x + ip_z z}$$

where φ_k are harmonic oscillator wave functions, i.e.,

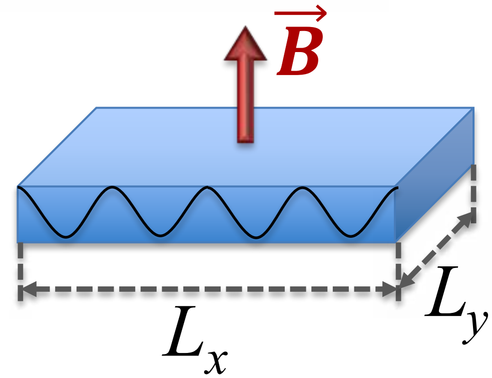
$$\varphi_k \propto H_k(\xi)e^{-\frac{\xi^2}{2}}, \quad \xi = \frac{y}{l} + p_x l\,\mathrm{sgn}(eB) \quad \text{and} \quad l = \frac{1}{\sqrt{|eB|}}$$

- The dispersion relation is given by

$$\omega = E_n^\pm = \pm\sqrt{2n|eB| + p_z^2 + m^2}$$

where $n = \underbrace{k + \frac{1}{2}}_{\text{orbital}} + \underbrace{\mathrm{sgn}(eB)s_z}_{\text{spin}}$ and $s_z = \pm\frac{1}{2}$ is an eigenvalue of $\frac{i}{2}\gamma^1\gamma^2$

Degeneracy of Landau levels



- The Landau level energies are independent of p_x

$$E_n^{\pm} = \pm\sqrt{2n|eB| + p_z^2 + m^2}$$

- This means that each level is highly degenerate
- Let's calculate the degeneracy by confining the system in a finite box of size $L_x \times L_y$ with periodic boundary conditions
- The wave function is a plane wave in the x direction: $\psi(x) \propto e^{ip_x x}$

$$\psi(0) = \psi(L_x) \quad \Longrightarrow \quad e^{ip_x L_x} = 1 \quad \Longrightarrow \quad p_x = \frac{2\pi n}{L_x}, \quad n = 1,2,\ldots,N_{\max}$$

- The value of p_x sets the center of the Landau orbit in y-direction:

$$y_c \approx p_x l^2 \quad \Longrightarrow \quad p_{x,\max} l^2 \lesssim L_y \quad \Longrightarrow \quad \frac{2\pi N_{\max}}{L_x}\frac{1}{|eB|} \approx L_y \quad \Longrightarrow \quad \frac{N_{\max}}{L_x L_y} \approx \frac{|eB|}{2\pi}$$

- The degeneracy is proportional to the field strength and the size (area) of the system in the spatial directions perpendicular to $\vec{\boldsymbol{B}}$

$$N_{\max} \approx \frac{|eB|}{2\pi} L_x L_y$$

Landau energy spectrum

- Landau energy levels at $m = 0$

$$E_n^{\pm} = \pm\sqrt{2n|eB| + p_z^2}$$

where $n = \underbrace{k + \frac{1}{2}}_{\text{orbital}} + \underbrace{\mathrm{sgn}(eB)s_z}_{\text{spin}}$

- Lowest Landau level is *spin polarized*

$$E_0^{\pm} = \pm p_z \qquad (k = 0,\ s_z = -\tfrac{1}{2})$$

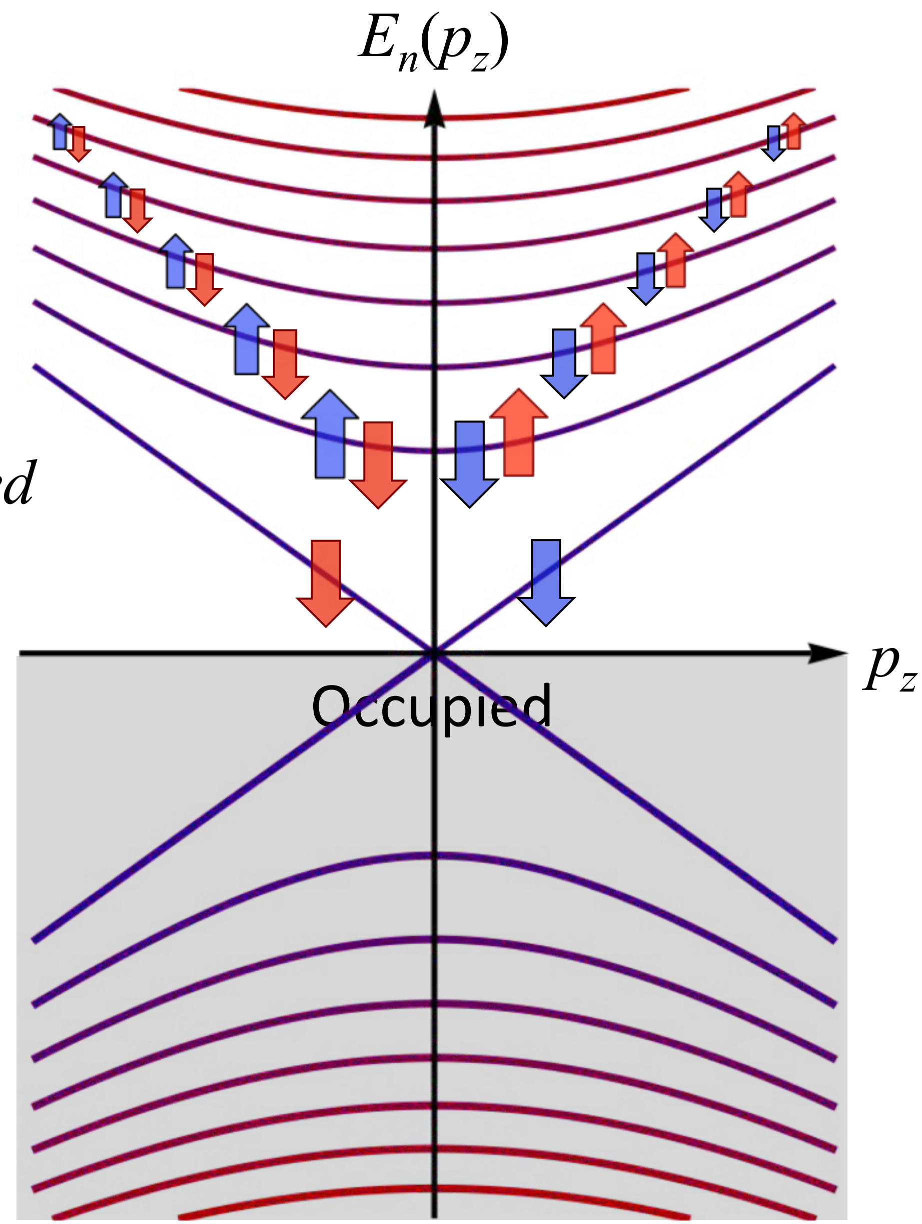
- Density of states at E=0:

$$\left.\frac{dn}{dE}\right|_{E=0} = \frac{|eB|}{2\pi}\frac{1}{2\pi} = \frac{|eB|}{4\pi^2}$$

- Higher Landau levels ($n \geq 1$) are twice as degenerate:

(i) $k = n$ & $s = -\frac{1}{2}$

(ii) $k = n - 1$ & $s = +\frac{1}{2}$



DIRAC PROPAGATOR AT B≠0

- By definition,

$$\mathbb{I} = \sum |k, p_z, s_z\rangle\langle k, p_z, s_z|$$

$$G(r,r') = i\left\langle r\middle|(i\gamma^\mu D_\mu - m)^{-1}\middle|r'\right\rangle$$

$$= i(i\gamma^\mu D_\mu + m)_r \left\langle r\middle|\left[(i\gamma^\mu D_\mu - m)(i\gamma^\nu D_\nu + m)\right]^{-1}\middle|r'\right\rangle$$

$$= i(i\gamma^\mu D_\mu + m)_r \left\langle r\middle|\left[-D^\mu D_\mu + i\gamma^1\gamma^2 eB - m^2\right]^{-1}\middle|r'\right\rangle$$

$$= i(i\gamma^\mu D_\mu + m)_r \sum\langle r|k, p_z, s_z\rangle\,(\omega^2 - E_n^2)^{-1}\langle k, p_z, s_z|r'\rangle$$

- Note that the explicit form of the wave functions is the same as before

$$\psi_{k,p_z,s_z}(r) = \langle r|k, p_z, s_z\rangle \propto H_k(\xi)e^{-\frac{\xi^2}{2}}e^{-i\omega t + ip_z z}\,U_{s_z}, \quad \text{where } \xi = \frac{y}{l} + p_x l$$

- The final expression for the propagator has the form

$$G(\omega, p_z; \vec{r}_\perp, \vec{r}_\perp') = e^{i\Phi(\vec{r}_\perp, \vec{r}_\perp')}\tilde{G}(\omega, p_z; \vec{r}_\perp - \vec{r}_\perp')$$

where $\Phi(\vec{r}_\perp, \vec{r}_\perp') = -\mathrm{e}\int_{\vec{r}_\perp'}^{\vec{r}_\perp} A_\nu dr^\nu$ is the *Schwinger phase* (!), and

$$\tilde{G}(\omega, p_z; \vec{r}_\perp - \vec{r}_\perp') = \int \frac{d^2\vec{p}_\perp}{(2\pi)^2} e^{i\vec{p}_\perp\cdot(\vec{r}_\perp - \vec{r}_\perp')}\,\tilde{G}(\omega, \vec{p})$$

Dirac Propagator at B≠0

- The Fourier transform of the *translation invariant* part reads

$$\tilde{G}(\omega,\vec{p}) = i e^{-\vec{p}_\perp^2 l^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(\omega,\vec{p})}{\omega^2 - E_n^2}$$

Laguerre polynomials

where

$$D_n(\omega,\vec{p}) = 2(\omega\gamma^0 - p_z\gamma^3 + m)[\mathcal{P}_+ L_n(2\vec{p}_\perp^2 l^2) - \mathcal{P}_- L_{n-1}(2\vec{p}_\perp^2 l^2)] + 4(\vec{p}_\perp \cdot \vec{\gamma}_\perp) L_{n-1}^1 (2\vec{p}_\perp^2 l^2)$$

and the following notation for the spin projectors is used

$$\mathcal{P}_\pm = \frac{1 \pm i\,\mathrm{sgn}(eB)\gamma^1\gamma^2}{2}$$

- Similarly, in momentum-coordinate space representation:

$$\tilde{G}(\omega,p_z;\vec{r}_\perp) = i\frac{e^{-\vec{r}_\perp^2/(4l^2)}}{2\pi l^2} \sum_{n=0}^{\infty} \frac{F_n(\omega,p_z;\vec{r}_\perp)}{\omega^2 - E_n^2}$$

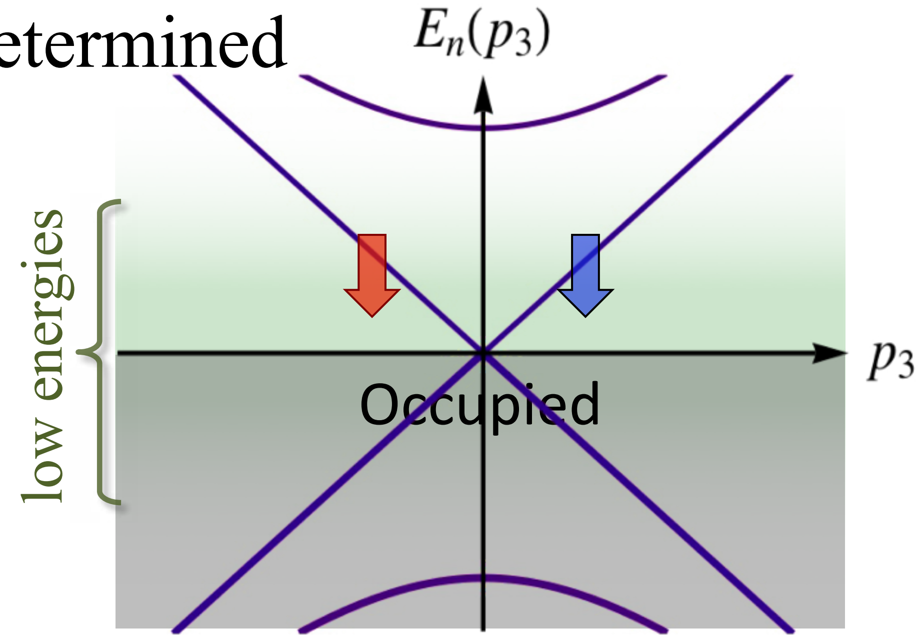
Laguerre polynomials

where $$F_n(\omega,p_z;\vec{r}_\perp) = 2(\omega\gamma^0 - p_z\gamma^3 + m)\left[\mathcal{P}_+ L_n\left(\frac{\vec{r}_\perp^2}{2l^2}\right) - \mathcal{P}_- L_{n-1}\left(\frac{\vec{r}_\perp^2}{2l^2}\right)\right] - \frac{i}{l^2}(\vec{r}_\perp \cdot \vec{\gamma}_\perp) L_{n-1}^1 \left(\frac{\vec{r}_\perp^2}{2l^2}\right)$$

Dimensional reduction

- The low-energy dynamics is determined by the lowest Landau level (n=0)

$$E_0^{\pm} = \pm p_z$$



- This is a (1+1)D spectrum!

- Propagator is also (1+1)D:

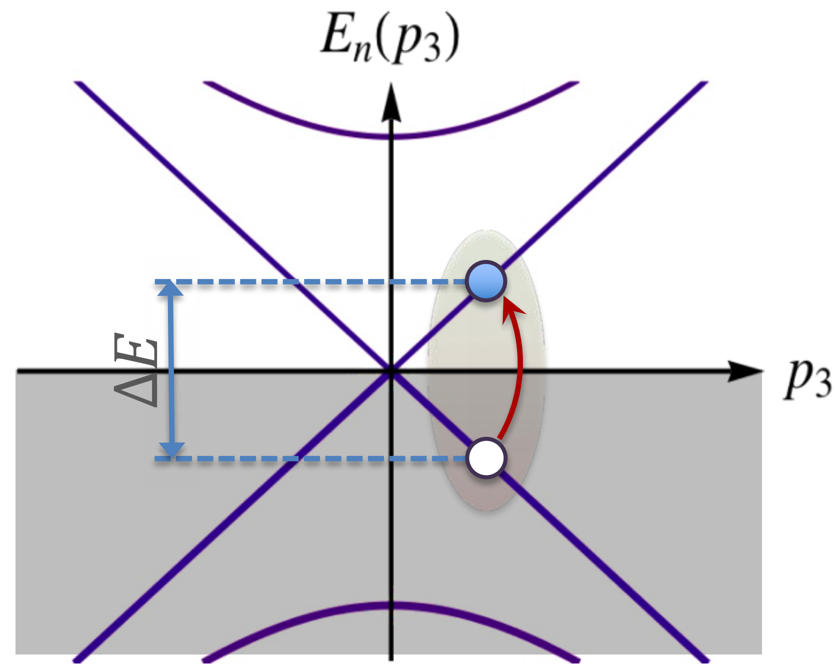
$$\tilde{G}_{LLL}(\omega, \vec{p}) = 2ie^{-\vec{p}_\perp^2 l^2} \frac{\omega\gamma^0 - p_z\gamma^3}{\omega^2 - p_z^2} \frac{1 + i\,\mathrm{sgn}(eB)\gamma^1\gamma^2}{2}$$

- In addition, there is a nonzero density of states at E=0:

$$\left.\frac{dn}{dE}\right|_{E=0} = \frac{1}{\delta E}\left(\frac{N_{\max}}{L_x L_y}\right)\left(\int_0^{\delta E} \frac{dp_z}{2\pi}\right) = \frac{|eB|}{4\pi^2}$$

Pairing instability

- Thought experiment:
  - Create a particle-antiparticle pair (energy price: ΔE)
  - The pair can form a bosonic bound state (energy gain: $-\epsilon_b$)
  - If $\epsilon_b > \Delta E$, copious formation of bound states is beneficial



  - Note, ΔE can be arbitrarily small when $m = 0$ (!)
  - The bound states of fermions are bosons
  - Bosons can (and will) occupy the lowest energy state ($\vec{P} = 0$), and thus form a Bose condensate $\langle \bar{\psi}\psi \rangle \neq 0$
  - Ground state (vacuum) changes its properties (e.g., chiral symmetry breaks down, an energy gap opens in spectrum)

Do bound states always form in 3D?

- Consider a 3D potential well in quantum mechanics
  [Landau-Lifshitz, Quantum Mechanics]

$$U(r) = \begin{cases} -g\dfrac{\pi^2\hbar^2}{8m_* a^2} & \text{for} \quad r \le a \\ 0 & \text{for} \quad r > a \end{cases}$$

- Bound states form only when the well is deep enough (namely, $g > 1$):

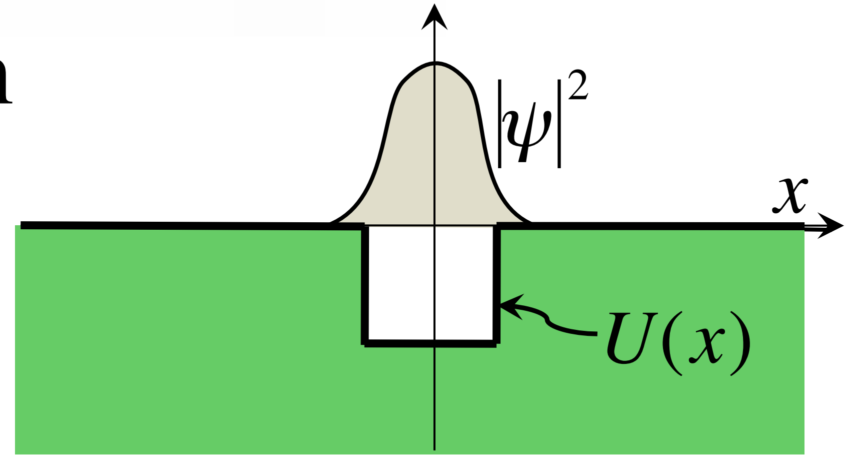
$$\left|E_{3D}\right| \approx \frac{\pi^4\hbar^2}{2^7 a^2 m_*}(g-1)^2\,, \quad \text{assuming} \quad 0 < g - 1 << 1$$

- There are no bound states when $g < 1$, i.e., when the well is not deep enough (in other words, when the coupling constant is not strong enough)

COMPARE: BOUND STATES IN 1D

- Bound states always form

$$\left|E_{1D}\right| \approx \frac{m_*}{2\hbar^2}\left(-\int_{-\infty}^{+\infty} U(x)\,dx\right)^2$$



- This is a perturbative result (!)

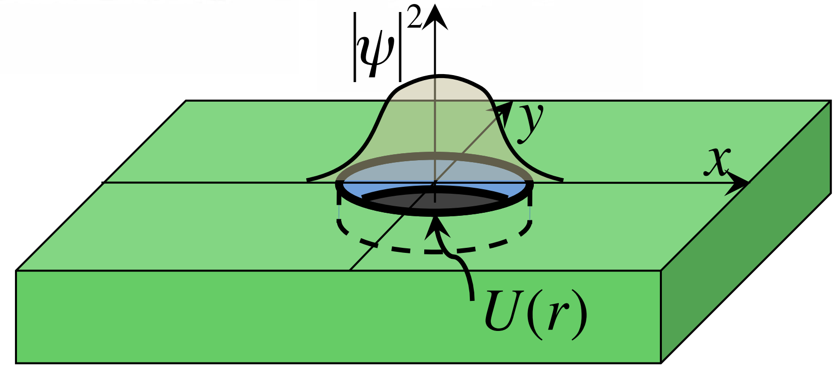
$$\left|E_{1D}\right| \propto g^2\,, \quad \text{when} \quad U(x) \to gU(x)$$

- Rigorous statement: at least one bound state exists if

$$\int (1+|x|)\,|U(x)|\,dx < \infty \quad \& \quad \int U(x)\,dx \le 0$$

[B. Simon, Annals Phys. 97 (1976) 279]

HOW ABOUT BOUND STATES IN 2D?



- Bound states always form

$$\left|E_{2D}\right| \approx \frac{\hbar^2}{a^2 m_*} \exp\left(- \frac{\hbar^2}{m_*} \left| \int_0^\infty r U(r) dr \right|^{-1} \right)$$

- This is a non-perturbative result

$$\left|E_{2D}\right| \propto \exp\left(- \frac{C}{g} \right), \quad \text{when} \quad U(x) \to g U(x)$$

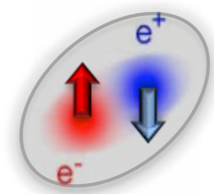
- Rigorous statement: at least one bound state exists if

$$\int |U(x)|^{1+\varepsilon} \, d^2x < \infty, \quad \int (1+x^2)^{\varepsilon} |U(x)| \, d^2x < \infty \quad \& \quad \int U(x) \, d^2x \le 0$$

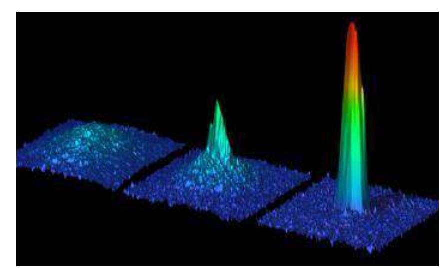
[B. Simon, Annals Phys. 97 (1976) 279]

UNIVERSAL MAGNETIC CATALYSIS

- Quantum field theory of charged fermions (m=0) at $\vec{\boldsymbol{B}} \neq \mathbf{0}$
  - Dimensional reduction (caused by a nonzero $\vec{\boldsymbol{B}}$)
  - Nonzero density of states ($\propto |eB|$) at E=0
  - Attraction between particles and antiparticles
- Universal outcome:



  - Copious particle-antiparticle pairing at low energies
  - Condensation of boson pairs that destabilizes the trivial Dirac vacuum



  - Spontaneous rearrangement of the ground state
  - Breakdown of chiral symmetry
  - Opening a nonzero gap in the Dirac spectrum

[Gusynin, Miransky, Shovkovy, Phys. Rev. Lett. **73**, 3499 (1994)]
[Miransky & Shovkovy, Physics Reports **576**, 1 (2015)]

- The mechanism is similar to superconductivity in metals due to Cooper pairing of electrons