Magnetic catalysis in QCD in a superstrong magnetic field

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Magnetic Catalysis: Plan of Lectures

- Dirac fermions in magnetic field
- Dimensional reduction
- Magnetic catalysis: basics
- Magnetic catalysis in toy model
- Magnetic catalysis in QED
- Magnetic catalysis in QCD
- Anisotropic confinement
- Inverse catalysis
- Phase diagram
QCD in Magnetic Fields

• Relativistic collisions of *heavy ions* produce quark-gluon plasma & strong magnetic fields

\[ 10^{18} - 10^{19} \text{ Gauss} \left( \sqrt{|eB|} \sim 100 \text{ MeV} \right) \]

• Quark matter may form inside *magnetars*

\[ 10^{14} - 10^{16} \text{ Gauss} \left( \sqrt{|eB|} \sim 1 \text{ MeV to 10 MeV} \right) \]

• Strong magnetic field is an instructive *theoretical tool* to study confined gauge theories such as QCD

\[ \gtrsim 10^{19} \text{ Gauss} \left( \sqrt{|eB|} \gtrsim 100 \text{ MeV to 10 MeV} \right) \]
• At $m = 0$, the Dirac vacuum is a **semimetal**
  
  – No energy gap between the filled Dirac sea states and the empty positive-energy states ($E = \pm p$)
  
  – However, the density of states *vanishes* at $E=0$
  
  – A nonzero electric current could be produced by an arbitrarily small electric field

• At $m \neq 0$, the Dirac vacuum is an **insulator**
  
  – Energy gap $\Delta E = 2m$ between the antiparticle and particle states ($E = \pm \sqrt{p^2 + m^2}$)
  
  – The density of states @ $E=0$ *vanishes* (no states)
  
  – Electric current is exponentially small, i.e., $e^{-\pi m^2/|eE|}$ (due to Schwinger pair creation)
• Lagrangian density for charged Dirac fermions (units with $c = 1$):

$$\mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu - m \right) \psi$$

where $D_\mu = \partial_\mu + ieA_\mu$, $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$ and $g^{\mu\nu} = (1, -1, -1, -1)$

• Consider the following two types of global transformations:

$$\psi \rightarrow e^{i\alpha} \psi$$

and

$$\psi \rightarrow e^{i\alpha\gamma^5} \psi$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

The corresponding Noether’s currents are

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

and

$$j_5^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$$

They satisfy the relations:

$$\partial_\mu j^\mu = 0$$

and

$$\partial_\mu j_5^\mu = 2i \, m \, \bar{\psi} \gamma^5 \psi$$

Both transformations are symmetries when $m = 0$, but chiral symmetry is broken when $m \neq 0$.  

[The chiral anomaly may complicate the situation]
**Dirac Fermions at $B \neq 0$**

- Dirac equation for charged fermions:

$$\left( i \gamma^\mu D_\mu - m \right) \psi = 0$$

where $A_\mu = (A_0, -\vec{A})$ and the Landau gauge $\vec{A} = (-By, 0,0)$ is used.

- Look for a solution in the form: $\psi = (i \gamma^\mu D_\mu + m) \phi$. Then,

$$\left[ -\partial_0^2 + (\partial_x + ieBy)^2 + \partial_y^2 + \partial_z^2 + i \gamma^1 \gamma^2 eB - m^2 \right] \phi = 0$$

- Normalized solutions for $\phi$ have the form

$$\phi_{k,\pm} \propto \frac{1 \pm i \text{sgn}(eB) \gamma^1 \gamma^2}{2} \varphi_k(y) e^{-i\omega t + ip_xx + ip_zz}$$

where $\varphi_k$ are harmonic oscillator wave functions, i.e.,

$$\varphi_k \propto H_k(\xi) e^{-\frac{\xi^2}{2}}, \quad \xi = \frac{y}{l} + p_x l \text{sgn}(eB) \quad \text{and} \quad l = \frac{1}{\sqrt{|eB|}}$$

- The dispersion relation is given by

$$\omega = E_n^{\pm} = \pm \sqrt{2n|eB| + p_z^2 + m^2}$$

where $n = k + \frac{1}{2} + \text{sgn}(eB)s_z$ and $s_z = \pm \frac{1}{2}$ is an eigenvalue of $\frac{i}{2} \gamma^1 \gamma^2$.
**Degeneracy of Landau Levels**

- The Landau level energies are independent of \( p_x \)
  \[
  E_n^\pm = \pm \sqrt{2n|eB| + p_z^2 + m^2}
  \]
- This means that each level is highly degenerate
- Let’s calculate the degeneracy by confining the system in a finite box of size \( L_x \times L_y \) with periodic boundary conditions
- The wave function is a plane wave in the \( x \) direction: \( \psi(x) \propto e^{ip_xx} \)
  \[
  \psi(0) = \psi(L_x) \implies e^{ip_xL_x} = 1 \implies p_x = \frac{2\pi n}{L_x}, \quad n = 1, 2, ..., N_{\text{max}}
  \]
- The value of \( p_x \) sets the center of the Landau orbit in \( y \)-direction:
  \[
  y_c \approx p_x l^2 \implies p_{x,\text{max}} l^2 \lesssim L_y \implies \frac{2\pi N_{\text{max}}}{L_x} \frac{1}{|eB|} \approx L_y \implies \frac{N_{\text{max}}}{L_x L_y} \approx \frac{|eB|}{2\pi}
  \]
- The degeneracy is proportional to the field strength and the size (area) of the system in the spatial directions perpendicular to \( \vec{B} \)
  \[
  N_{\text{max}} \approx \frac{|eB|}{2\pi} L_x L_y
  \]
**Landau Energy Spectrum**

- Landau energy levels at $m = 0$
  
  $$E_n^\pm = \pm \sqrt{2n|eB| + p_z^2}$$

  where $n = k + \frac{1}{2} + \text{sgn}(eB)s_z$

  - Lowest Landau level is *spin polarized*
    
    $$E_0^\pm = \pm p_z \quad (k = 0, \ s_z = -\frac{1}{2})$$

- Density of states at $E=0$:
  
  $$\left| \frac{dN}{dE} \right|_{E=0} = \frac{|eB|}{2\pi} \frac{1}{2\pi} = \frac{|eB|}{4\pi^2}$$

- Higher Landau levels ($n \geq 1$) are twice as degenerate:
  
  (i) $k = n \quad \& \quad s = -\frac{1}{2}$

  (ii) $k = n - 1 \quad \& \quad s = +\frac{1}{2}$
**DIRAC PROPAGATOR AT B≠0**

- By definition,

\[
G(r, r') = i \left\langle r \left| (i\gamma^\mu D_\mu - m)^{-1} \right| r' \right\rangle
\]

\[
= i (i\gamma^\mu D_\mu + m)_r \left\langle r \left| \left( (i\gamma^\mu D_\mu - m)(i\gamma^v D_v + m) \right)^{-1} \right| r' \right\rangle
\]

\[
= i (i\gamma^\mu D_\mu + m)_r \left\langle r \left| \left( -D^\mu D_\mu + i\gamma^1\gamma^2 eB - m^2 \right)^{-1} \right| r' \right\rangle
\]

\[
= i (i\gamma^\mu D_\mu + m)_r \sum \left\langle r \left| k, p_z, s_z \right\rangle (\omega^2 - E_n^2)^{-1} \left\langle k, p_z, s_z \right| r' \right\rangle
\]

- Note that the explicit form of the wave functions is the same as before

\[
\psi_{k,p_z,s_z}(r) = \left\langle r \left| k, p_z, s_z \right\rangle \propto H_k(\xi) e^{-\frac{\xi^2}{2}} e^{-i\omega t + ip_z z} U_{s_z}, \quad \text{where} \quad \xi = \frac{\gamma}{l} + p_x l
\]

- The final expression for the propagator has the form

\[
G(\omega, p_z; \vec{r}_1, \vec{r}_1') = e^{i\Phi(\vec{r}_1, \vec{r}_1')} \tilde{G}(\omega, p_z; \vec{r}_1 - \vec{r}_1')
\]

where \(\Phi(\vec{r}_1, \vec{r}_1') = -e \int_{\vec{r}_1}^{\vec{r}_1'} A_\nu dr^\nu\) is the Schwinger phase (!), and

\[
\tilde{G}(\omega, p_z; \vec{r}_1 - \vec{r}_1') = \int \frac{d^2 \vec{p}_1}{(2\pi)^2} e^{i\vec{p}_1 \cdot (\vec{r}_1 - \vec{r}_1')} \tilde{G}(\omega, \vec{p})
\]

\[= \sum |k, p_z, s_z\rangle \langle k, p_z, s_z| \]
The Fourier transform of the translation invariant part reads

\[ \tilde{G}(\omega, \vec{p}) = i e^{-\vec{p}_\perp l^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(\omega, \vec{p})}{\omega^2 - E_n^2} \]

where

\[ D_n(\omega, \vec{p}) = 2(\omega \gamma^0 - p_z \gamma^3 + m) [\mathcal{P}_+ L_n(2\vec{p}_\perp l^2) - \mathcal{P}_- L_{n-1}(2\vec{p}_\perp l^2)] \]

\[ + 4(\vec{p}_\perp \cdot \vec{\gamma}_\perp) L^1_{n-1} (2\vec{p}_\perp l^2) \]

and the following notation for the spin projectors is used

\[ \mathcal{P}_\pm = \frac{1 \pm \text{sgn}(eB) \gamma^1 \gamma^2}{2} \]

Similarly, in momentum-coordinate space representation:

\[ \tilde{G}(\omega, p_z; \vec{r}_\perp) = i \frac{e^{-\vec{r}_\perp^2/(4l^2)}}{2\pi l^2} \sum_{n=0}^{\infty} \frac{F_n(\omega, p_z; \vec{r}_\perp)}{\omega^2 - E_n^2} \]

where

\[ F_n(\omega, p_z; \vec{r}_\perp) = 2(\omega \gamma^0 - p_z \gamma^3 + m) \left[ \mathcal{P}_+ L_n \left( \frac{\vec{r}_\perp^2}{2l^2} \right) - \mathcal{P}_- L_{n-1} \left( \frac{\vec{r}_\perp^2}{2l^2} \right) \right] \]

\[ - \frac{i}{l^2} (\vec{r}_\perp \cdot \vec{\gamma}_\perp) L^1_{n-1} \left( \frac{\vec{r}_\perp^2}{2l^2} \right) \]
**DIMENSIONAL REDUCTION**

- The low-energy dynamics is determined by the lowest Landau level \((n=0)\)

\[
E_0^\pm = \pm p_z
\]

- This is a \((1+1)D\) spectrum!

- Propagator is also \((1+1)D\):

\[
\tilde{G}_{LLL}(\omega, \vec{p}) = 2ie^{-\frac{\vec{p}_1^2}{2}} \frac{\omega\gamma^0 - p_z\gamma^3}{\omega^2 - p_z^2} \frac{1 + isgn(eB)\gamma^1\gamma^2}{2}
\]

- In addition, there is a nonzero density of states at \(E=0\):

\[
\left. \frac{dn}{dE} \right|_{E=0} = \frac{1}{\delta E} \left( \frac{N_{\text{max}}}{L_x L_y} \right) \left( \int_0^{\delta E} \frac{dp_z}{2\pi} \right) = \frac{|eB|}{4\pi^2}
\]
• Thought experiment:
  – Create a particle-antiparticle pair (energy price: $\Delta E$)
  – The pair can form a bosonic bound state (energy gain: $-\epsilon_b$)
  – If $\epsilon_b > \Delta E$, copious formation of bound states is beneficial
  – Note, $\Delta E$ can be arbitrarily small when $m = 0$ (!)
  – The bound states of fermions are bosons
  – Bosons can (and will) occupy the lowest energy state ($\vec{P} = 0$), and thus form a Bose condensate $\langle \bar{\psi} \psi \rangle \neq 0$
  – Ground state (vacuum) changes its properties (e.g., chiral symmetry breaks down, an energy gap opens in spectrum)
DO BOUND STATES ALWAYS FORM IN 3D?

• Consider a 3D potential well in quantum mechanics
  [Landau-Lifshitz, Quantum Mechanics]

\[
U(r) = \begin{cases} 
-g \frac{\pi^2 \hbar^2}{8 m^* a^2} & \text{for } r \leq a \\
0 & \text{for } r > a
\end{cases}
\]

• Bound states form only when the well is deep enough (namely, \( g > 1 \)):

\[
|E_{3D}| \approx \frac{\pi^4 \hbar^2}{2^7 a^2 m^*} (g - 1)^2, \quad \text{assuming} \quad 0 < g - 1 \ll 1
\]

• There are no bound states when \( g < 1 \), i.e., when the well is not deep enough (in other words, when the coupling constant is not strong enough)
**Compare: Bound States in 1D**

- Bound states always form-bound states always form

\[ |E_{1D}| \approx \frac{m^*_\text{eff}}{2\hbar^2} \left( -\int_{-\infty}^{+\infty} U(x) dx \right)^2 \]

- This is a perturbative result (!)

\[ |E_{1D}| \propto g^2 \text{, when } U(x) \rightarrow gU(x) \]

- Rigorous statement: at least one bound state exists if

\[ \int (1 + |x|)|U(x)| \, dx < \infty \quad \& \quad \int U(x) \, dx \leq 0 \]

[B. Simon, Annals Phys. 97 (1976) 279]
**How about bound states in 2D?**

- Bound states always form
  \[ |E_{2D}| \approx \frac{\hbar^2}{a^2 m_*} \exp \left( -\frac{\hbar^2}{m_*} \int_0^\infty r U(r) dr \right)^{-1} \]

- This is a non-perturbative result
  \[ |E_{2D}| \propto \exp \left( -\frac{C}{g} \right), \text{ when } U(x) \to gU(x) \]

- Rigorous statement: at least one bound state exists if
  \[
  \int |U(x)|^{1+\epsilon} d^2x < \infty, \quad \int (1 + x^2)^\epsilon |U(x)| d^2x < \infty \quad \& \quad \int U(x) d^2x \leq 0
  \]

[B. Simon, Annals Phys. 97 (1976) 279]
Universal magnetic catalysis

• Quantum field theory of charged fermions \( (m=0) \) at \( \vec{B} \neq 0 \)
  – Dimensional reduction (caused by a nonzero \( \vec{B} \))
  – Nonzero density of states \( (\propto |eB|) \) at \( E=0 \)
  – Attraction between particles and antiparticles

• Universal outcome:
  – Copious particle-antiparticle pairing at low energies
  – Condensation of boson pairs that destabilizes the trivial Dirac vacuum
  – Spontaneous rearrangement of the ground state
  – Breakdown of chiral symmetry
  – Opening a nonzero gap in the Dirac spectrum

[Gusynin, Miransky, Shovkovy, Phys. Rev. Lett. 73, 3499 (1994)]
[Miransky & Shovkovy, Physics Reports 576, 1 (2015)]

• The mechanism is similar to superconductivity in metals due to Cooper pairing of electrons