BCS Pairing Gap in the Infrared Limit of the Similarity Renormalization Group

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Effective interactions have been used to compute the pairing gap for nuclear and neutron matter in several schemes. In this work we analyze the impact of phase-shift equivalent interactions within the BCS theory on the $^1S_0$-channel pairing gap for a translational invariant many-fermion system such as nuclear and neutron matter. We solve the BCS pairing gap equation on a finite momentum grid for a toy model separable Gaussian potential in the $^1S_0$-channel explicitly evolved through the Similarity Renormalization Group (SRG) transformation and show that in the on-shell and continuum limits the pairing gap vanishes. For finite size systems the momentum is quantized and the on-shell limit is realized for SRG cutoffs comparable to the momentum resolution. In this case the pairing gap can be computed directly from the scattering phase-shifts by an energy-shift formula. While the momentum grid is usually used as an auxiliary way of solving the BCS pairing gap equation, we show that it actually encodes some relevant physical information, suggesting that in fact finite grids may represent the finite size of the system.

I. INTRODUCTION

The microscopic origin of pairing in nuclei was first driven by the analogy to the BCS theory of superconductivity [1]. Since then, the nature of pairing correlations has provided a lot of insight in Nuclear Physics (for a review see e.g. Ref. [2] and references therein). A renaissance of the subject was experienced by the production of heavy $N \sim Z$ nuclei which are achieved by Radiactive Ion Beams [3].

Effective interactions have been used to compute the pairing gap for nuclear and neutron matter in several schemes [2]. There are claims in the literature that what determines the pairing gap are the phase-shifts [4] and most often the BCS approach is based on having the scattering phase-shift as the basic input of the calculation. On the other hand, there is an arbitrariness in this procedure, as there are infinitely many interactions leading to the identical phase-shift.

For instance, in the case of the pairing gap in the $^1S_0$ channel, most $V_{\rm low-k}$ calculations provide a BCS gap which has maximum at Fermi momentum $p_F \sim 0.8\ {\rm fm}^{-1}$ and strength about 3 MeV [5, 6]. In medium T-matrix has been used to provide an improvement on the standard BCS theory [7] yielding a 30% reduction in the $^1S_0$ pairing gap. *Ab initio* calculations, however, may provide completely different results [8–10]. Thus, it is disconcerning that the BCS gap is so different and so much scheme dependent. In this paper we analyze these ambiguities.

The BCS state provides a paring gap given by

$$\Delta(\mathbf{k}) = -\frac{1}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{V(\mathbf{k},\mathbf{p})\Delta(\mathbf{p})}{E(\mathbf{p})} \ , \tag{1}$$

where $E(\mathbf{p})^2 = ((\mathbf{p}^2 - p_F^2)/(2M))^2 + \Delta(\mathbf{p})^2$ (M is the nucleon mass). After the partial-wave decomposition [2],

$$V^S(\mathbf{p}',\mathbf{p}) = \frac{4\pi^2}{M}\sum_{JMLL'} \mathcal{Y}^{JM}_{LS}(\hat{p}')V^{JS}_{LL'}(p',p)\mathcal{Y}^{JM\,\dagger}_{L'S}(\hat{p}) \tag{2}$$

$$\Delta^S(\mathbf{p}) = \sum_{JML}\mathcal{Y}^{JM}_{LS}(\hat{p})\Delta^{JS}_L(p) \ , \tag{3}$$

we have,

$$\Delta^J_L(p) = -\frac{1}{\pi}\int_0^\infty k^2 dk \sum_{L'}\frac{V^J_{L,L'}(k,p)\Delta^J_{L'}(p)}{ME(p)} \ , \tag{4}$$

which is the generalized gap equation in all channels. These equations are solved iteratively until convergence is achieved (for several strategies see e.g. [11]). The pairing gap in a given channel is defined as $\Delta_F = \Delta(p)|_{p=p_F}$.

We aim to analyze numerically the behaviour of the pairing gap as a function of the SRG cutoff towards the infrared limit. These are demanding calculations, particularly with interactions having a strong short distance repulsive core which provide long momentum tails. For our illustration purposes, in the present study we consider the toy model gaussian separable potential in the $^1S_0$ channel discussed in our previous works [12–15] because the long momentum tails are suppressed from the start, considerably reducing the computational effort for the SRG numerical treatment in the infrared limit.

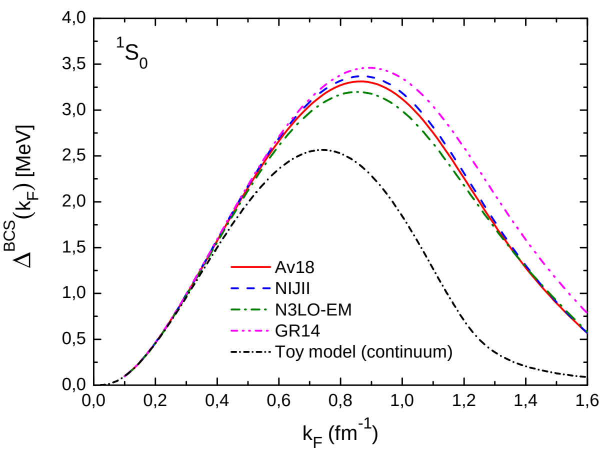

FIG. 1. BCS pairing gap for the $^1S_0$-state in MeV of the toy model compared to realistic potentials AV18 [16], NijII [17], N3LO-EM [18] and GR14 [19], as a function of the Fermi momentum k_F (in fm^{-1}).

In Fig. 1 we show the the pairing gap obtained with the toy model potential. As one can see, the toy model potential yields a sufficiently reasonable behaviour compared to the results obtained with the realistic potentials AV18 [16], NijII [17], N3LO-EM [18] and GR14 [19].

Most analyses based on the BCS approach usually end here. However, it is important to note that the only physical input information in these calculations is contained in the phase-shift but *not* in the potentials. As it is well known, this generates some off-shell ambiguity which cannot be directly related to measurable physical information. This off-shellness corresponds to the non-dagonal matrix elements of the potential $V(p',p)$. In fact, one can make a unitary transformation $V \to UVU^\dagger$ such that phase-shifts remain invariant. In the following sections we will show that the BCS pairing gap can in principle depend strongly on this unitary transformation and hence on the off-shellness.

II. BCS PAIRING GAP EQUATIONS ON A FINITE MOMENTUM GRID

The BCS equations can be solved numerically on an N-dimensional momentum grid, $p_1 < \dots p_N$ [20] by implementing a high-momentum ultraviolet (UV) cutoff, $p_{\max} = \Lambda$, and an infrared (IR) momentum cutoff $p_{\min} = \Delta p$. The integration rule becomes

$$\int_{\Delta p}^{\Lambda} dp f(p) \to \sum_{n=1}^{N} w_n f(p_n)\,. \tag{5}$$

The completeness relation in discretized momentum-space reads:

$$1 = \frac{2}{\pi}\sum_{n=1}^{N} w_n p_n^2 |p_n\rangle\langle p_n|\,. \tag{6}$$

By inserting this into Eq. (4) and defining the matrix-element of the potential as $V(p_n,p_m) \equiv \langle p_n|V|p_m\rangle$, we obtain the BCS pairing gap equation on the finite momentum grid

$$\Delta(p_n) = -\frac{2}{\pi}\sum_{k=1}^{N} w_k p_k^2 \frac{V(p_n,p_k)\Delta(p_k)}{2ME(p_n)}\,, \tag{7}$$

where $2ME(p_n) = \sqrt{(p_n^2 - p_F^2)^2 + 4M^2\Delta(p_n)^2}$. Of course, on the grid the Fermi momentum must also belong to the grid ($p_F = p_m$).

While the momentum grid is usually regarded as an auxiliary element for solving the BCS gap equation, we will show that it actually encodes some relevant physical information, suggesting that in fact finite grids may represent the finite size of the system. Moreover, we will show that using the inherent arbitrariness of the off-shellness in the potential one may get a large variety of results. As a matter of fact, we will present a scheme which is free of any off-shell ambiguities, and for this scheme the continuum limit is shown to produce a vanishing BCS gap for an infinitely large system.

III. PHASE EQUIVALENT INTERACTIONS AND THE ON-SHELL LIMIT

Quite generally, for a given hamiltonian we can always perform a unitary transformation $H \to UHU^\dagger$ keeping the phase-shift invariant. On a finite momentum grid the definition of the phase-shift must be specified, since on the one hand one replaces the scattering boundary conditions with standing waves boundary conditions and on the other hand one wants to preserve the invariance under unitary transformations on the grid.

The unitary transformation U can be quite general, and for our study we will generate them by means of the so-called similarity renormalization group (SRG), proposed by Glazek and Wilson [21, 22] and independently by Wegner [23] who showed how high- and low-momentum degrees of freedom can decouple while keeping scattering equivalence.

The general SRG equation is given by [24],

$$\frac{dH_s}{ds} = [[G_s, H_s], H_s]\,, \tag{8}$$

and supplemented with a generator G_s and an initial condition at $s = 0$, H_0. This correspond to a one-parameter operator evolution dynamics and, as it is customary, we will often switch to the SRG cutoff $\lambda = s^{-1/4}$ which has momentum dimensions. The generator G_s can be chosen according to certain requirements, and three popular choices are the kinetic energy T [22] (Wilson-Glazek generator), the diagonal part of the hamiltonian Diag(H) [23] (Wegner generator) or a block-diagonal (BD) generator $PH_sP + QH_sQ$ where $P + Q = 1$ are orthogonal projectors $P^2 = P$, $Q^2 = Q$, $QP = PQ = 0$, for states below and above a given momentum scale [25].

On the finite momentum grid the SRG equations become a set of non-linear coupled differential equations. For the Wegner generator, which will be taken here for definiteness, the equations take a quite simple form

$$\frac{dH_s(p_n,p_k)}{ds} = \frac{2}{\pi}\sum_m H_s(p_n,p_m) w_m p_m^2 H_s(p_m,p_k) \times [H_s(p_n,p_n) - 2H_s(p_m,p_m) + H_s(p_k,p_k)] \ . \quad (9)$$

The BCS pairing gap equation for the SRG-evolved hamiltonian can be written as

$$\Delta_\lambda(p_n) = -\sum_{k=1}^{N} \frac{[H_\lambda(p_n,p_k) - p_n^2\delta_{nk}]\Delta_\lambda(p_k)}{2ME_\lambda(p_n)} \ , \qquad (10)$$

Clearly, the BCS pairing gap becomes a function of the SRG cutoff λ.

The limit $\lambda \to 0$ corresponds to an infrared fixed-point of the SRG evolution, at which the Hamiltonian becomes a diagonal matrix [15],

$$\lim_{\lambda\to 0} H_\lambda(p_n,p_m) = P_n^2 \ , \qquad (11)$$

where P_n^2 the $n-$th ordered eigenvalue of the Hamiltonian. In this limit the potential also becomes diagonal, and hence all off-shellness is eliminated. Thus, in the SRG infrared limit $\lambda \to 0$ we get an on-shell interaction. An important result derived in Ref. [14] is the energy-shift formula,

$$\delta^{\rm ES}(p_n) = -\pi \lim_{\lambda\to 0} \frac{H^{G,\lambda}_{n,n} - p_n^2}{2w_n p_n} = -\pi \frac{P_n^2 - p_n^2}{2w_n p_n} \ , \quad (12)$$

which provides phase-shifts that remain constant along the SRG-trajectory, i.e.

$$\delta^{\rm ES}_\lambda(p_n) = \delta^{\rm ES}_\infty(p_n) = \delta^{\rm ES}_0(p_n) \ . \qquad (13)$$

Furthermore, in the SRG infrared limit $\lambda \to 0$ the BCS pairing gap equation on the grid is determined by the energy-shift at the Fermi surface for on-shell interactions,

$$\Delta^{ES}_\lambda(p_n) = \lim_{\lambda\to 0}\Delta_\lambda(p_n) = w_n \frac{p_n \delta^{\rm ES}(p_n)}{\pi M} \ . \qquad (14)$$

It is important to note that the integration weights w_n appear explicitly in the formula, and in the continuum limit $N \to \infty$ they vanish as $w_n = \mathcal{O}(1/N)$. Therefore, if we denote by $\Delta p_F \equiv w_F$ the integration weight corresponding to the Fermi momentum, in the continuum limit the BCS pairing gap becomes

$$\Delta_F \sim \Delta p_F \frac{p_F \delta(p_F)}{\pi M} \ . \qquad (15)$$

whenever $\Delta_F > 0$ and zero otherwise. This is our main result. One should also note that while the shape is rather universal, the strength is related to Δp_F which ultimately depends on the system size R and geometry, and for large systems $\Delta p_F = \mathcal{O}(1/R)$.

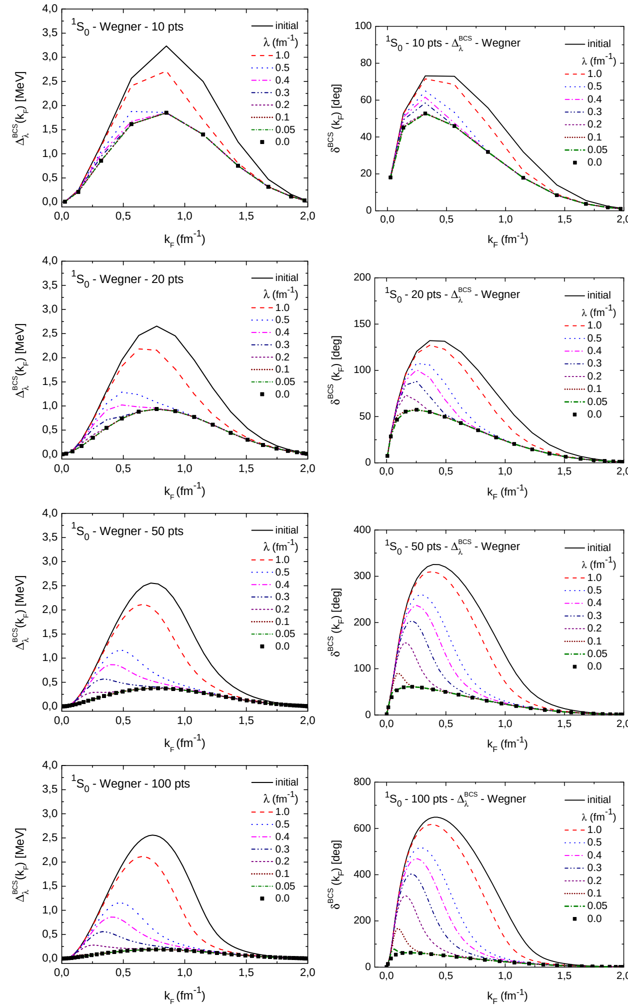

FIG. 2. SRG evolution of the BCS pairing gap for the toy model potential in the $^1S_0$ channel and the corresponding BCS phase-shifts as a function of the Fermi momentum k_F.

In Fig. 2 we show the evolution of the pairing gaps $\Delta_\lambda(p_n)$ obtained by solving the BCS equation on the grid as a function of the SRG cutoff λ for the range $\lambda = 1, \ldots, 0.05$ fm^{-1} and for several choices on the number of points $N = 10, \ldots, 100$. Alternatively, we illustrate the scaling behavior of the BCS pairing gap by defining the "BCS phase-shift" as

$$\delta^{\rm BCS}_\lambda(p_n) = \frac{\Delta^{\rm BCS}_\lambda(p_n)\pi M}{w_n p_n} \ , \qquad (16)$$

which, as expected, converges to the phase-shift obtained from the energy-shift formula, in the limit $\lambda \to 0$. We remind that, as pointed out and illustrated in Ref. [13], the phase-shifts calculated through the solution of the Lippmann-Schwinger equation *does not* fulfill the phase-invariance on the finite momnetum grid, but only in the continuum limit, i.e. for $N \to \infty$. On the other hand, the phase-shifts remain invariant if we consider the energy-shift definition [13].

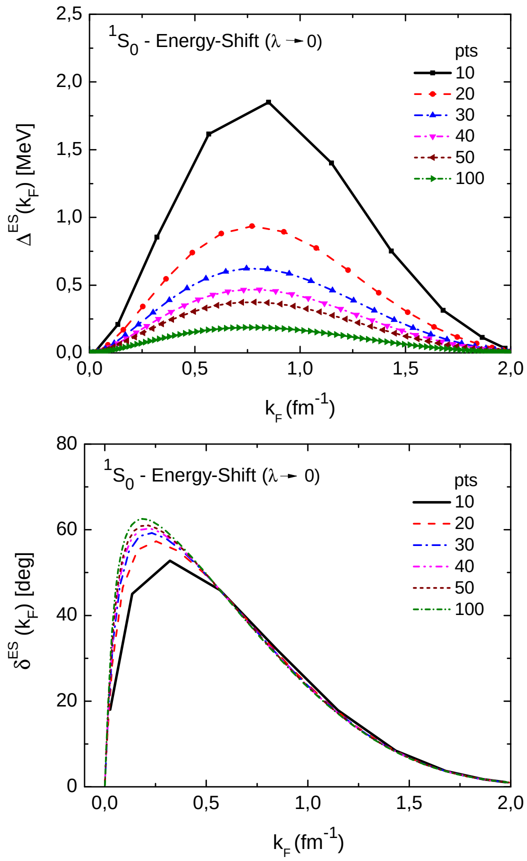

FIG. 3. BCS pairing gap for the toy model potential in the $^1S_0$ - channel and the corresponding phase-shifts in the infrared limit $\lambda \to 0$, obtained from the energy-shift formula.

In Fig. 3 we show the pairing gap $\Delta^{\rm ES}(p_n)$ and the corresponding phase-shifts $\delta^{\rm ES}(p_n)$ in the infrared limit $\lambda \to 0$, obtained from the energy-shift formula. As one can see, when we take the SRG infrared limit the pairing gap is compatible with zero in the strict continuum limit.

IV. FINAL REMARKS

In the present work we have explored the freedom on reducing the off-shellness of the NN interaction through the SRG evolution towards the infrared limit $\lambda \to 0$ as a way to analyze the BCS pairing gap. Remarkably, we find that there is an on-shell regime where the BCS pairing gap can be directly determined by the NN phase-shifts in the $^1S_0$ channel. We have verified by explicit numerical calculations that in the infrared ($\lambda \to 0$) and continuum ($N \to \infty$) limits the pairing gap vanishes, suggesting that finite momentum grids may represent the finite size of the system.

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