The behavior of charged pion masses in the presence of a static uniform magnetic field is studied in the framework of the two-flavor NJL model, using a magnetic field-dependent (MFIR) regularization scheme. Analytical calculations are carried out employing the Ritus eigenfunction method, which allows us to properly take into account the presence of Schwinger phases in the quark propagators. Numerical results are obtained for definite model parameters, comparing the predictions of the model with present lattice QCD (LQCD) results.

**Charged Pions**

For charged pions, Schwinger phases do not cancel due to their different quark flavors, and therefore $J_{\gamma}$ is not translational invariant. However, it is still diagonal in the Ritus basis. The pion field is expanded as

$$\pi(x) = \sum_q \sum_p \rho_q^p \pi_q^p(\mathbf{x}) (\pi_q^p(x) - J_{\gamma}(x,x')) \delta M(\mathbf{r})$$

After bosonization and expansion of the meson field around the Mean Field (MF) values, the Euclidean action reads

$$S_{\text{bare}} = S_{\text{MF}} + S_{\text{pert}} + \ldots$$

The quadratic contribution is given by

$$S_{\text{pert}}^{\text{bare}} = \frac{1}{2} \sum_{x,x'} \int \delta M(x) \left[ \frac{1}{2} \delta^2 \pi(x',x') - J_{\gamma}(x,x') \right] \delta M(x')$$

$J_{\gamma}(x,x') = N_c \sum_q \sum_p \left[ S_{\text{pert}}^{\text{bare}} \gamma_5 S_{\text{pert}}^{\text{bare}} \right]_q^p \delta(x,x')$, where $S_{\text{pert}}^{\text{bare}}$ is the MFIR propagator.

Here $\delta(x,x') = \exp \left( \frac{m_0 B(x,x')}{x-x'} \right)$ is the Schwinger phase and

$$S_{\text{pert}}^{\text{bare}} = \int \delta^2 \pi(x',x') \left[ \left( m_0^2 + \frac{p^2}{2m_0} \right) \right]_q^p \delta(x,x')$$

where $B = \{B,0\}$, $\delta = (\delta_0, \delta_0', \delta_0''')$, $\rho_q = (\rho_q, \rho_0', \rho_0''')$, and $p = (p, p', p'')$.

**Parameterization – NJL vs LQCD condensate**

At the MF level, the condensates are given by

$$\langle \bar{q} q \rangle = - N_c \int d^4 x S_{\text{pert}}^{\text{bare}} = - \frac{N_c}{4 \pi^2} \int \frac{d^3 p}{p^0} \text{Re} \text{Tr} \ln B \text{Tr} \ln \left( \frac{1}{1 + \frac{m_0^2}{p^0}} \right)$$

The integral can be regulated in the MFIR scheme through a 3D cutoff. To compare with LQCD results we introduce the quantities

$$\Delta \Sigma(B) = \Delta \Sigma(B) = \left( \frac{\Delta \Sigma(B)}{\Delta \Sigma(B)} \right)$$

$$\Delta \Sigma(B) = - 2 m_0 \left( \frac{\langle \bar{q} q \rangle}{\langle \bar{q} q \rangle} \right)$$

We consider the following parameterizations which reproduce the empirical vacuum values $m_{\pi}(B=0) = 138$ MeV and $f_{\pi}(B=0) = 92.4$ MeV.

<table>
<thead>
<tr>
<th>Set</th>
<th>$M_0$ (MeV)</th>
<th>$m_0$ (MeV)</th>
<th>$\alpha$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>350.00</td>
<td>5.66</td>
<td>2.25</td>
</tr>
<tr>
<td>II</td>
<td>320.00</td>
<td>5.42</td>
<td>2.14</td>
</tr>
<tr>
<td>III</td>
<td>380.00</td>
<td>5.79</td>
<td>2.36</td>
</tr>
</tbody>
</table>

**Neutral Pion**

$\pi^0$ properties have been previously studied in the MFIR scheme expanding the propagator in Landau levels (LL) [1,2]. We use the Schwinger form instead. Since the Schwinger phases cancel out, $J_{\gamma}$ depends only on $x-x'$ (i.e. it is translational invariant). Momentum is conserved and $J_{\gamma}$ diagonalizable in the Fourier basis. The final expression coincides with the one found in [1,2].

**Conclusions**

- Within the NJL model, the charged pion two-point function was diagonalized in the Ritus basis, using the Schwinger form of the quark propagator and regularized in the MFIR scheme.
- The model parameterizations reproduce the quark condensates of LQCD.
- When $eB$ is enhanced, the $\pi^0$ mass slightly decreases, while $E_{\pi^0}$ steadily increases, remaining always larger than the one of a point-like pion.
- The results are rather independent of the parameterization.
- For a realistic charged pion mass, the results agree with LQCD [3] for low values of $eB$. At $0.15$ GeV/$c$, higher values show some discrepancy.
- For a heavy pion mass, although rescaled parameters show consistency with LQCD [4] for $E_{\pi^0}$, the errors are large to be conclusive. For $m_{\pi^0}$, the results disagree. The agreement is improved if $G(eB)$ is introduced in [2].

**Figure 1.** Condensate average vs $eB$ for Sets I, II, III and LQCD [5] (squares). Similar agreement is found for the condensate difference.

**Figure 2.** B-dependent neutral pion mass and charged pion "mass" as functions of $eB$ for Sets I, II and III. Also shown are the charged pion "mass" for point-like case (dotted) and the results from a LQCD calculation with physical pion mass [3] (squares).

More recent LQCD simulations [4] consider larger values of $eB$ but use a heavy pion mass of $m_{\pi}(B=0) = 415$ MeV. To compare we define the rescaled Set I, where $m_{\pi}$ is increased as in [2]. In that reference, the authors also consider a B-dependent coupling $G(eB)$ to reproduce LQCD results for the $\pi^0$ mass (this is denoted here as Set IV).

**Figure 3.** Normalized neutral pion mass and magnetic field-dependent charged pion mass as functions of $eB$ for Set I (solid), Set IV (dashed) of [2], the charged point-like case (dotted) and LQCD [4] (squares).

**REFERENCES**