Field theory amplitudes from the pure spinor superstring

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Introduction and motivation

- Since the discovery the pure spinor formalism many superstring amplitudes have been computed with manifest supersymmetry
  1. N-pts @ tree-level
  2. N-pts @ 1-loop (low energy limit)
  3. 4- and 5-pts @ 2-loops (5pt: low energy limit)
  4. 4-pt @ 3-loops (low energy limit)
- The $\alpha' \to 0$ limit gives rise to field theory amplitudes
- What can we say about FT amplitudes?

FT amplitudes from educated guesses

- The idea is to antecipate how the result of taking the $\alpha' \to 0$ limit will look like
- FT limit will be composed out of kinematics and propagators and loop momentum integrals
- The FT amplitude will be a BRST-invariant expression constructed out of these elements
- Strategy depends heavily on how much control we have over the string results
- Kinematics of string amplitudes given by pure spinor superspace expressions

SYM superfields in 10D (Witten '86)

- Covariant description of D=10 SYM theory with superfields

$$A_\alpha(x,\theta),\ A_m(x,\theta),\ W^\alpha(x,\theta),\ F_{mn}(x,\theta)$$

- Linearized EOMs

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m, \quad D_\alpha A_m = (\gamma_m W)_\alpha + \partial_m A_\alpha$$

$$D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}, \quad D_\alpha F_{mn} = \partial_{[m}(\gamma_{n]} W)_\alpha$$

- Well-known θ^α expansions:

$$\begin{aligned}
A_\alpha(x,\theta) &= \frac{1}{2} a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\chi\gamma_m\theta)(\gamma^m\theta)_\alpha + \cdots \\
A_m(x,\theta) &= a_m - (\chi\gamma_m\theta) + \cdots \\
W^\alpha(x,\theta) &= \chi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha f_{mn} + \cdots \\
F_{mn}(x,\theta) &= f_{mn} - 2(\partial_{[m}\chi\gamma_{n]}\theta) + \cdots
\end{aligned}$$

String tree-level amplitude

- N-point prescription (Berkovits '00):

$$\mathcal{A}_{\text{tree}} = \langle V_1 V_2 V_3 \int U_4 \ldots \int U_n \rangle$$

- where V and U are massless vertex operators

$$V = \lambda^\alpha A_\alpha(x,\theta), \quad QV = 0$$
$$U = \partial\theta^\alpha A_\alpha + A_m \Pi^m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} F_{mn}, \quad QU = \partial V$$

- λ^α is the pure spinor satisfying $(\lambda\gamma^m\lambda) = 0$
- $Q = \lambda^\alpha D_\alpha$ is the BRST charge
- The CFT computation of the tree-level correlator is usual
- Use OPEs to integrate out non-zero modes of $\partial\theta^\alpha, d_\alpha, \Pi^m$ and N^{mn}

Pure Spinor Superspace

- Surviving zero-modes of the pure spinor λ^α and θ^α integrated out with the prescription

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$$

- PSS defined as expressions containing $(\lambda^\alpha, \theta^\alpha)$ such that $\langle(\lambda^3\theta^5)\rangle = 1$
- Component expansions straightforward to compute

$$\langle V_1 V_2 V_3\rangle = (e^1\cdot e^2)(k^2\cdot e^3) + e^1_m(\chi_2\gamma^m\chi_3) + \mathrm{cyc}(1,2,3)$$

- PSS framework optimal to a BRST cohomology analysis of amplitudes

Pure Spinor Superspace Cohomology

- BRST charge $Q = \lambda^\alpha D_\alpha$ and SYM equations of motion are closely related

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m, \quad D_\alpha A_m = (\gamma_m W)_\alpha + \partial_m A_\alpha$$
$$D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathcal{F}_{mn}, \quad D_\alpha \mathcal{F}_{mn} = 2\partial_{[m}(\gamma_{n]} W)_\alpha$$

- String amplitudes are expressions in the BRST cohomology of pure spinor superspace

$$\mathcal{A}_3 = \langle V_1 V_2 V_3 \rangle \text{ and } Q(V_1 V_2 V_3) = 0, \quad V_1 V_2 V_3 \neq Q(\Omega_{123})$$

- Therefore the FT amplitudes must also be in the cohomology!
- The vertex V alone is not enough, need some OPE technology for pure spinor vertex operators

Multiparticle SYM superfields (CM, Schlotterer '14, '15)

- Recursive definition of multiparticle superfields

$$K_B \in \{A^B_\alpha,\ A^m_B,\ W^\alpha_B,\ \mathcal{F}^{mn}_B\}$$

inspired by OPE computations

$$U^1(z_1)U^2(z_2) \sim \frac{1}{z_{12}}\big[\partial\theta^\alpha A^{12}_\alpha + A^{12}_m \Pi^m + d_\alpha W^\alpha_{12} + \frac{1}{2}N^{mn}F^{12}_{mn}\big]$$

$$U^{12}(z_2)U^3(z_3) \sim \frac{1}{z_{23}}\big[\partial\theta^\alpha A^{123}_\alpha + A^{123}_m \Pi^m + d_\alpha W^\alpha_{123} + \frac{1}{2}N^{mn}F^{123}_{mn}\big]$$

- For example,

$$W^\alpha_{12} = \frac{1}{4}(\gamma^{mn}W^2)^\alpha F^1_{mn} + W^\alpha_2(k^2\cdot A^1) - (1\leftrightarrow 2)$$

$$W^\alpha_{123} = -(k^{12}\cdot A^3)W^\alpha_{12} + \frac{1}{4}(\gamma^{rs}W^3)^\alpha F^{12}_{rs} - (12\leftrightarrow 3)$$
$$+ \frac{1}{2}(k^1\cdot k^2)\big[W^\alpha_2(A^1\cdot A^3) - (1\leftrightarrow 2)\big]$$

Multiparticle SYM superfields

- The multiparticle superfields satisfy generalized SYM EOMs

$$
\begin{aligned}
D_\alpha W_1^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}^1 \\
D_\alpha W_{12}^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}^{12} \\
&\quad + (k^1 \cdot k^2)(A_\alpha^1 W_2^\beta - A_\alpha^2 W_1^\beta) \\
D_\alpha W_{123}^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}^{123} \\
&\quad + (k^1 \cdot k^2)\big[A_\alpha^1 W_{23}^\beta + A_\alpha^{13} W_2^\beta - (1 \leftrightarrow 2)\big] \\
&\quad + (k^{12} \cdot k^3)\big[A_\alpha^{12} W_3^\beta - (12 \leftrightarrow 3)\big] \,,
\end{aligned}
$$

- and similarly for the other superfields $A_\alpha^B, A_B^m, F_B^{mn}$
- Surprising and beautiful multiparticle structure hidden in SYM theory!

Multiparticle symmetries

- The superfields K_B satisfy generalized Lie symmetries

$$
\begin{aligned}
0 &= K_{12} + K_{21}, \\
0 &= K_{123} + K_{231} + K_{312}, \quad \text{(Jacobi identity)} \\
0 &= K_{1234} - K_{1243} + K_{3412} - K_{3421} \\
0 &= \text{general formula known}
\end{aligned}
$$

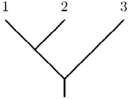
- These are the same symmetries obeyed by nested commutators
- Can make symmetries manifest in the notation

$$
K_{12} \to K_{[1,2]}, \quad K_{123} \to K_{[[1,2],3]}, \quad K_{1234} \to K_{[[[1,2],3],4]} \cdots
$$

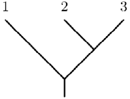
- Define new superfields

$$\mathcal{K}_B \in \{\mathcal{A}^B_\alpha, \mathcal{A}^m_B, \mathcal{W}^\alpha_B, \mathcal{F}^{mn}_B\}$$

from all binary trees (Catalan #) dressed with propagators and K_B



$\frac{K_{[[1,2],3]}}{s_{12}s_{123}}$



$\frac{K_{[1,[2,3]]}}{s_{23}s_{123}}$

$$\mathcal{K}_{123} \equiv \frac{K_{123}}{s_{12}s_{123}} + \frac{K_{321}}{s_{23}s_{123}}$$

- Satisfy simple EOMs for general multiparticle label, e.g.

$$\{D_\alpha, \mathcal{W}^\beta_B\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathcal{F}^B_{mn} + \sum_{XY=B}(\mathcal{A}^X_\alpha \mathcal{W}^\beta_Y - \mathcal{A}^Y_\alpha \mathcal{W}^\beta_X)$$

Towards FT tree amplitudes

- String OPE computations at tree-level can be written using only the multiparticle unintegrated vertex

$$V_B = \lambda^\alpha A^B_\alpha, \quad QV_1 = 0, \quad QV_{12} = s_{12} V_1 V_2, \ldots$$

- Corresponding Berends–Giele current $\mathcal{V}_B \equiv M_B$ satisfies

$$M_B \equiv \lambda^\alpha \mathcal{A}^B_\alpha, \quad QM_B = \sum_{XY=B} M_X M_Y$$

- BG current M_B has well-defined kinematic poles, eg $M_{12} = V_{12}/s_{12}$
- Tree amplitudes characterized by its poles too
- Can we guess the mapping between M_B and tree amplitudes?
- Is there a fundamental principle guiding this construction?

FT tree amplitudes from BRST invariance

- Guess FT amplitudes from BRST invariance!
- Cubic graph organization of tree-level amplitudes (BCJ '08)

$$A_{\rm YM}(1,2,\ldots,N) = \sum_i \frac{n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

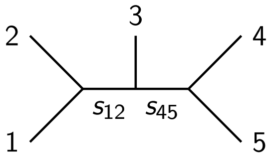
- Propagators contain only Mandelstam invariants
- BRST principle: assemble the kinematic building blocks M_B such that expression is in the BRST cohomology and contains correct kinematic poles

$$QA(1,2,3,\ldots,n) = 0, \quad A(1,2,3,\ldots,n) \neq Q(\mathit{something})$$

- Each term of Qn_i must have a factor of P_{α_i}

Tree-level FT amplitudes

$\mathcal{A}_5(1,2,3,4,5) \quad =$



$+ \quad \mathrm{cyclic}(12345)$

- Expressions in the BRST cohomology with correct kinematic poles

$$\mathcal{A}_5(1,2,3,4,5) = \langle M_{12} M_3 M_{45} \rangle + \mathrm{cyclic}(12345)$$

- Can also write as

$$\mathcal{A}_5(1,2,3,4,5) = \langle (M_{123} M_4 + M_{12} M_{34} + M_1 M_{234}) M_5 \rangle$$

- BRST closed object

$$E_{1234} \equiv M_{123} M_4 + M_{12} M_{34} + M_1 M_{234}, \qquad E_{1234} = Q M_{1234}$$

- Generalizes to N-pts!

PSS cohomology and FT amplitudes

(C.M., Schlotterer, Stieberger, Tsimpis, '10)

N-point color-ordered SYM tree amplitudes

$$A_n(1,2,\ldots,n) = \langle E_{123\ldots(n-1)} M_n \rangle$$

$$E_B \equiv \sum_{XY=B} M_X M_Y$$

- Recall Berends–Giele formula (Berends, Giele '88)

$$A_{\mathrm{YM}}(1,2,\ldots,n) = s_{123\ldots n-1} J^m(12\ldots n-1) J^m_n$$

- M_B plays the role of BG current J^m_B and E_B related to $s_B J^m_B$
- PSS formula is the supersymmetric generalization of BG method
- BG current J^m_B is nothing more than the SYM superfield $\mathcal{A}^m_B$! (in progress)

One-loop amplitudes, 5pts

- 5pt closed-string amplitude computed with the pure spinor formalism (Green, CM, Schlotterer '13)
- The OPE calculations give rise to three distinct kinematic building blocks: V_A, $T_{A,B,C}$ and $T^m_{A,B,C,D}$

$$T_{A,B,C} \equiv \frac{1}{3}(\lambda\gamma_m W_A)(\lambda\gamma_n W_B)F^{mn}_C + (C \leftrightarrow B, A)$$

$$T^m_{A,B,C,D} \equiv \big[T_{A,B,C}A^m_D + (D \leftrightarrow C, B, A)\big] + W^m_{A,B,C,D}$$

$$W^m_{A,B,C,D} \equiv \frac{1}{12}(\lambda\gamma_n W_A)(\lambda\gamma_p W_B)(W_C\gamma^{mnp}W_D) + (A, B|A, B, C, D)$$

- Multiparticle labels A, B, C, D

BRST variations of 1-loop building blocks

- Scalar building blocks

$$
\begin{aligned}
QT_{1,2,3} &= 0 \\
QT_{12,3,4} &= s_{12}(V_1 T_{2,3,4} - V_2 T_{1,3,4}) \\
QT_{12,34,5} &= s_{12}(V_1 T_{2,34,5} - V_2 T_{1,34,5}) + s_{34}(V_3 T_{12,4,5} - V_4 T_{12,3,5})
\end{aligned}
$$

- Vector building blocks

$$
\begin{aligned}
QT^m_{1,2,3,4} &= k_1^m V_1 T_{2,3,4} + (1 \leftrightarrow 2,3,4) \\
QT^m_{12,3,4,5} &= s_{12}(V_1 T^m_{2,3,4,5} - V_2 T^m_{1,3,4,5}) + k_{12}^m V_{12} T_{3,4,5} \\
&\quad + \left[k_3^m V_3 T_{12,4,5} + (3 \leftrightarrow 4,5)\right]
\end{aligned}
$$

- Nice algebraic structure (higher-point variations similar but omitted)

FT 1-loop amplitudes from BRST invariance

- Can we antecipate the FT integrands from BRST invariance like at tree-level?

$$A(1,2,3,\ldots,n) = \int \frac{d^D\ell}{(2\pi)^D} \langle A(1,2,3,\ldots,n|\ell)\rangle$$

- Can also use cubic graphs organization at loop-level (BCJ '10)

$$A(1,2,3,\ldots,n|\ell) = \sum_{\Gamma_i} \frac{N_i(\ell)}{\prod_k P_{k,i}(\ell)}$$

- In addition to Mandelstam propagators, also propagators with loop momentum ℓ^m
- BRST invariance principle: assemble the kinematic building blocks such that integrand is BRST closed

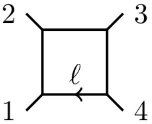
$$QA(1,2,3,\ldots,n|\ell) = 0$$

- Each term of $QN_i(\ell)$ must have a factor of $P_{k,i}(\ell)$

4-point 1-loop amplitude

- The 4-point 1-loop integrand is easy to write down

$$A(1,2,3,4|\ell) \quad =$$

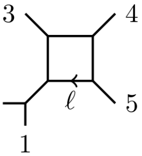


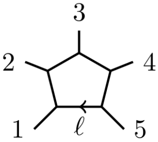
$$A(1,2,3,4|\ell) = \frac{V_1 T_{2,3,4}}{\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2}\,.$$

- not much choice of BRST invariants ...
- String 4pt amplitude and $V_1 T_{2,3,4}$ kinematics computed by Berkovits in 2004
- Later shown to reproduce standard $t_8 F^4$ (+ fermions) (CM '05)

5-point 1-loop amplitude

- 5-point 1-loop integrand from BRST invariance (CM, Schlotterer '14)

$A(1,2,3,4,5|\ell) \;=$ 

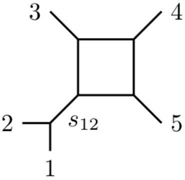
$+ \;\; \text{cyclic}(12345) \;\; +$ 

- Split integrand accordingly

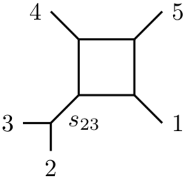
$$A(1,2,3,4,5|\ell) = A_{\text{box}}(1,2,3,4,5) + A_{\text{pent}}(1,2,3,4,5|\ell)$$

- Boxes independent of loop momentum, pentagon at most linear ($N-4$ powers in general)
- Mandelstam propagators in the boxes: this rings a Berends–Giele bell leading to the following ansatz

5pt boxes



$\leftrightarrow \quad \dfrac{V_{12}T_{3,4,5}}{s_{12}}$



$\leftrightarrow \quad \dfrac{V_{1}T_{23,4,5}}{s_{23}}$

- Leg number 1 treated differently due to fixed vertex position V_1 in 1-loop prescription

$$Q\frac{V_{12}}{s_{12}} = QM_{12} = M_1 M_2$$

$$Q\frac{T_{23,4,5}}{s_{23}} = QM_{23,4,5} = M_2 M_{3,4,5} - M_3 M_{2,4,5}$$

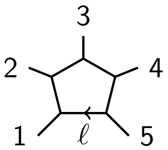
5pt boxes

$$
\begin{aligned}
A_{\text{box}}(1,2,3,4,5) = {} & \frac{V_{12}T_{3,4,5}}{(k_1+k_2)^2\ell^2(\ell-k_{12})^2(\ell-k_{123})^2(\ell-k_{1234})^2} \\
& + \frac{V_1 T_{23,4,5}}{(k_2+k_3)^2\ell^2(\ell-k_1)^2(\ell-k_{123})^2(\ell-k_{1234})^2} \\
& + \frac{V_1 T_{2,34,5}}{(k_3+k_4)^2\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{1234})^2} \\
& + \frac{V_1 T_{2,3,45}}{(k_4+k_5)^2\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2} \\
& + \frac{V_{51} T_{2,3,4}}{(k_1+k_5)^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2(\ell-k_{1234})^2}
\end{aligned}
$$

- BRST variations cancel the Mandelstams in the denominators, leaving boxes of loop momentum factors
- Need a pentagon whose BRST variation cancel those surviving boxes

5pt pentagon

- Pentagon: vector (ℓ^m) and scalar (no loop momentum)



$$A_{\text{pent}}(1,2,3,4,5|\ell) = \frac{N^{(5)}_{1|2,3,4,5}(\ell)}{\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell-k_{123})^2(\ell-k_{1234})^2}$$

- PSS numerator

$$2N^{(5)}_{1|2,3,4,5}(\ell) \equiv (\ell_m + \ell_m - k^1_m) V_1 T^m_{2,3,4,5} + \big[V_1 T_{23,4,5} + (2,3|2,3,4,5)\big]$$

5pt pentagon

- BRST principle: variation of numerator must cancel propagators

$$
\begin{aligned}
2QN^{(5)}_{1|2,3,4,5}(\ell) = {}& V_1 V_2 T_{3,4,5}\left[(\ell - k_{12})^2 - (\ell - k_1)^2\right] \\
&+ V_1 V_3 T_{2,4,5}\left[(\ell - k_{123})^2 - (\ell - k_{12})^2\right] \\
&+ V_1 V_4 T_{2,3,5}\left[(\ell - k_{1234})^2 - (\ell - k_{123})^2\right] \\
&+ V_1 V_5 T_{2,3,4}\left[\ell^2 - (\ell - k_{1234})^2\right]
\end{aligned}
$$

- Turns out to cancel BRST variation of the boxes, overall 5pt integrand is BRST closed!
- All BCJ identities satisfied, eg

$$
\langle N^{(4)}_{1|23,4,5}\rangle = \langle N^{(5)}_{1|2,3,4,5}(\ell) - N^{(5)}_{1|3,2,4,5}(\ell)\rangle
$$

Two-loop amplitudes, 5pts

- 5pt closed-string amplitude computed with the (non-minimal) pure spinor formalism led to kinematic building blocks: $T_{A,B|C,D}$ and $T^m_{A,B,C|D,E}$ (Gomez, CM, Schlotterer '15)
- Minimal pure spinor representation (CM, Schlotterer '15)

$$T_{A,B|C,D} \equiv \frac{1}{64}(\lambda\gamma_{mnpqr}\lambda)F^{mn}_A F^{pq}_B\left[F^{rs}_C(\lambda\gamma_s W_D) + F^{rs}_D(\lambda\gamma_s W_C)\right] + (A, B$$

$$T^m_{1,2,3|4,5} \equiv A^m_1 T_{2,3|4,5} + A^m_2 T_{1,3|4,5} + A^m_3 T_{1,2|4,5} + W^m_{1,2,3|4,5}$$

$$\begin{aligned} W^m_{3,4,5|1,2} \equiv\ & \frac{1}{48}(\lambda\gamma_{pq}\gamma^m W_{(1})F^{pq}_{2)}(\lambda\gamma_r W_5)(\lambda\gamma_s W_{(3})F^{rs}_{4)} \\ & - \frac{1}{128}(\lambda\gamma^m W_5)(\lambda\gamma_{pq}\gamma^r W_{(1})F^{pq}_{2)}(\lambda\gamma_{st}\gamma_r W_{(3})F^{st}_{4)} \\ & + \frac{1}{96}(W_3\gamma^{mst}W_4)(\lambda\gamma_{npqrs}\lambda)(\lambda\gamma_t W_5)F^{np}_1 F^{qr}_2 + (5 \leftrightarrow 3, 4) \end{aligned}$$

BRST variations of 2-loop building blocks

- Scalar building blocks

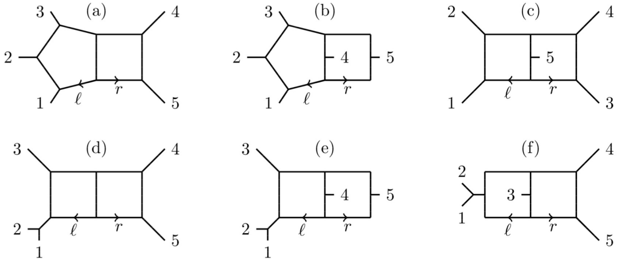
$$
\begin{aligned}
QT_{1,2|3,4} &= 0 \\
QT_{12,3|4,5} &= s_{12}(V_1 T_{2,3|4,5} - V_2 T_{1,3|4,5})
\end{aligned}
$$

- Vector building blocks

$$
QT^m_{1,2,3|4,5} = k_1^m V_1 T_{2,3|4,5} + k_2^m V_2 T_{1,3|4,5} + k_3^m V_3 T_{1,2|4,5}
$$

- Essentially the same algebraic structure as before!

2-loop 5pt topologies



- BCJ identities: master diagram (a) (Carrasco, Johansson '11)
- PSS representation

$$2N^{(a)}_{1,2,3|4,5}(\ell) \equiv (\ell_m + \ell_m - k^{123}_m)\, T^m_{1,2,3|4,5} + (T_{12,3|4,5} + T_{13,2|4,5} + T_{23,1|4,5})$$

2-loop 5-pt integrand

- BRST principle works exactly as before (CM, Schlotterer '15)
- Can assemble BRST-invariant 2-loop 5-pt integrand
- BCJ identities satisfied by construction, gravity amplitudes for free
- The solution for numerators look intuitive
- Hope to look for patterns allowing N-pt solution(?)