# Symmetries in P and S wave for $B \to K^* \mu\mu$

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LHCb workshop Edinburgh - May 2015

# Outline of the talk

- The angular distribution: P+ S wave. S-wave observables.
- Symmetries and counting.
- Structure of the relations in P and S wave sector.
- Phenomenological implications.
- Conclusions.

# Angular Distribution

The differential decay rate of $B \to K\pi\ell^+\ell^-$ receives contributions from:

- P-wave decay $B \to K^*(\to K\pi)\ell^+\ell^-$
- S-wave decay $B \to K_0^*(\to K\pi)\ell^+\ell^-$ with $K_0^*$ a broad scalar resonance

$$\frac{d^5\Gamma}{dq^2\, dm_{K\pi}^2\, d\cos\theta_K\, d\cos\theta_\ell\, d\phi} = W_P + W_S,$$

$\Rightarrow$ $W_P$ contains the pure P-wave contribution.

$\Rightarrow$ $W_S$ contains the pure S-wave exchange and S-P interference.

# Angular Distribution (2)

**Structure:**

- $B \to K^* \ell^+ \ell^-$ described by 6 complex amplitudes $A^{L,R}_{\|,\perp,0}$ ($m_\ell = 0$, $A_S = 0$)
- $B \to K_0^* \ell^+ \ell^-$ described by 2 complex amplitudes $A'^{L,R}_0$

These amplitudes are multiplied by a $BW_i(m^2_{K\pi})$ with $i = K^*, K_0^*$ describing the propagation of the $K^*$ and $K_0^*$ meson.

A bit more on the structure. $W_P$ is well known and contains $J_i(q^2, m_{K\pi})$ while

$$W_S = \frac{1}{4\pi}\Big[\tilde{J}^c_{1a} + \tilde{J}^c_{1b}\cos\theta_K + (\tilde{J}^c_{2a} + \tilde{J}^c_{2b}\cos\theta_K)\cos 2\theta_\ell + \tilde{J}_4 \sin\theta_K \sin 2\theta_\ell \cos\phi$$
$$+\tilde{J}_5 \sin\theta_K \sin\theta_\ell \cos\phi + \tilde{J}_7 \sin\theta_K \sin\theta_\ell \sin\phi + \tilde{J}_8 \sin\theta_K \sin 2\theta_\ell \sin\phi\Big] .$$

we assume lepton masses to zero (for simplicity).

# S-wave observables

The S-wave observables are defined as

$$A_S = \frac{8}{3}\frac{\tilde{J}^c_{1b} + \bar{\tilde{J}}^c_{1b}}{\Gamma'_{\rm full} + \bar{\Gamma}'_{\rm full}}, \qquad A_S^{\rm CP} = \frac{8}{3}\frac{\tilde{J}^c_{1b} - \bar{\tilde{J}}^c_{1b}}{\Gamma'_{\rm full} + \bar{\Gamma}'_{\rm full}},$$

$$A_S^i = \frac{4}{3}\frac{\tilde{J}_i + \bar{\tilde{J}}_i}{\Gamma'_{\rm full} + \bar{\Gamma}'_{\rm full}}, \qquad A_S^{i\,\rm CP} = \frac{4}{3}\frac{\tilde{J}_i - \bar{\tilde{J}}_i}{\Gamma'_{\rm full} + \bar{\Gamma}'_{\rm full}},$$

$i = 4, 5, 7, 8$. The total differential decay width $\Gamma'_{\rm full}$ and $F_S$ are

$$\Gamma'_{\rm full} = \Gamma'_{K^*} + \Gamma'_{K_0^*}, \qquad F_S = \frac{\Gamma'_{K_0^*} + \bar{\Gamma}'_{K_0^*}}{\Gamma'_{\rm full} + \bar{\Gamma}'_{\rm full}}$$

One can reexpress $W_S$ in terms of this set of observables.

In the following and for simplicity neglect CPV contributions (bar observables) exact expressions can be found in [L. Hofer, J.M'15].

# Symmetries of the angular distribution:

The angular distribution of $B \to K^*(\to K\pi)\mu^+\mu^-$ exhibit symmetries:

$\Rightarrow$ *Transformation of its amplitudes $A^{L,R}_{\perp,\|,0}$ leaving invariant the distribution.*

- Number of independent observables given a set of amplitudes and generators of symmetry.
  - **Minimal number of observables** $\to$ a complete basis.
  - Provide **consistency relations** among observables.
- Not any combination of transversity amplitudes $A^{L,R}_{\perp,\|,0}$ defines an observable. For instance,

$$A_T^1 = -2\frac{\mathrm{Re}(A_\| A_\perp^*)}{|A_\perp|^2 + |A_\||^2}$$

  is ”gauge dependent” and cannot be extracted from the distribution.
- “Amplitude analysis approach” uses/has all symmetries embedded.

# Counting

The **counting of degrees of freedom** for the P-wave sector ALONE shows:

$$n^{P-wave}_{Observables} = n_J - n_{relations} = 2n_A - n^P_{symmetries}$$

| Case | Coeff. $J_i$ | Amplitudes | Rel. | Symmetries | Observables |
|---|---|---|---|---|---|
| $m_\ell = 0, A_S = 0$ | 11 | 6 | **3** | 4 | **8** |
| $m_\ell = 0$ | 11 | 7 | 2 | 5 | **9** |
| $m_\ell > 0, A_S = 0$ | 11 | 7 | 1 | 4 | **10** |
| $m_\ell > 0$ | 12 | 8 | 0 | 4 | **12** |

All symmetries (massive and scalars) were found explicitly later on.

[JM, Mescia, Ramon, Virto'12]

ADDING the S-wave sector:

$$n^{S-wave}_{Observables} = 2n_A - n^{P+S}_{symmetries} - n^P_{Observables} = \mathbf{4}$$

| Case | Coeff. $J_i$, $\tilde{J}_i$ | Amplitudes | Rel. | Symmetries | Observables |
|---|---|---|---|---|---|
| $m_\ell = 0, A_S = 0$ | 11+**8** | 6+**2** | **3**+**4** | 4 | **8**+**4** |

Natural language to make explicit all those symmetries is to introduce:

$$n_{\|} = \begin{pmatrix} A_{\|}^{L} BW_P \\ A_{\|}^{R*} BW_P^* \end{pmatrix}, \; n_{\perp} = \begin{pmatrix} A_{\perp}^{L} BW_P \\ -A_{\perp}^{R*} BW_P^* \end{pmatrix}, \; n_0 = \begin{pmatrix} A_0^{L} BW_P \\ A_0^{R*} BW_P^* \end{pmatrix}, \; n_S = \begin{pmatrix} A_0'^{L} BW_S \\ A_0'^{R*} BW_S^* \end{pmatrix}.$$

All P-wave coefficients can be expressed in terms of them:

$$\mathbf{J_{1s} = 3J_{2s}} \;=\; \frac{3}{4}\left(|n_{\perp}|^2 + |n_{\|}|^2\right), \quad \mathbf{J_{1c} = -J_{2c}} = |n_0|^2 \quad J_3 = \frac{1}{2}\left(|n_{\perp}|^2 - |n_{\|}|^2\right),$$

$$J_4 \;=\; \frac{1}{\sqrt{2}} \mathrm{Re}(n_0^{\dagger}\, n_{\|}), \qquad J_5 = \sqrt{2}\, \mathrm{Re}(n_0^{\dagger}\, n_{\perp}), \quad J_{6s} = 2\, \mathrm{Re}(n_{\perp}^{\dagger}\, n_{\|}),$$

$$J_7 \;=\; -\sqrt{2}\, \mathrm{Im}(n_0^{\dagger}\, n_{\|}), \quad J_8 = -\frac{1}{\sqrt{2}} \mathrm{Im}(n_0^{\dagger}\, n_{\perp}) \quad J_9 = -\mathrm{Im}(n_{\perp}^{\dagger}\, n_{\|})$$

Modulus and Real and Imaginary bilinear scalar products of $n_i$

Also S-wave can also be expressed:

$$\tilde{\mathbf{J}}^{\mathbf{c}}_{\mathbf{1a}} = -\tilde{\mathbf{J}}^{\mathbf{c}}_{\mathbf{2a}} = \frac{3}{8}|n_S|^2, \quad \tilde{\mathbf{J}}^{\mathbf{c}}_{\mathbf{1b}} = -\tilde{\mathbf{J}}^{\mathbf{c}}_{\mathbf{2b}} = \frac{3}{4}\sqrt{3}\mathrm{Re}(n_S^\dagger\, n_0),$$
$$\tilde{J}_4 = \frac{3}{4}\sqrt{\frac{3}{2}}\mathrm{Re}(n_S^\dagger\, n_\parallel), \quad \tilde{J}_5 = \frac{3}{2}\sqrt{\frac{3}{2}}\mathrm{Re}(n_S^\dagger\, n_\perp),$$
$$\tilde{J}_7 = \frac{3}{2}\sqrt{\frac{3}{2}}\mathrm{Im}(n_\parallel^\dagger\, n_S), \quad \tilde{J}_8 = \frac{3}{4}\sqrt{\frac{3}{2}}\mathrm{Im}(n_\perp^\dagger\, n_S).$$

- Exactly same symmetries for P and S wave sector.
- Only 4 out of these 8 $\tilde{J}_i$ are independent.
- We have found **2** (P-wave) + **2** (S-wave) relations out of **7**.

$\Rightarrow$ Three relations lacking!

# Symmetries

A **symmetry** of the angular distribution will be a unitary transformation

$$n_i' = U n_i = \begin{bmatrix} e^{i\phi_L} & 0 \\ 0 & e^{-i\phi_R} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cosh i\tilde{\theta} & -\sinh i\tilde{\theta} \\ -\sinh i\tilde{\theta} & \cosh i\tilde{\theta} \end{bmatrix} n_i \,.$$

$U$ defines the **four symmetries** of the massless angular distribution:

- two global phase transformations ($\phi_L$ and $\phi_R$),
- a rotation $\theta$ among the real and imaginary components of the amplitudes independently.
- another rotation $\tilde{\theta}$ that mixes real and imaginary components of the transversity amplitudes.

# How to obtain the lacking 3 relations?

There are basically two possible procedures:

1) Since $n_{||}$ and $n_\perp$ span the space of complex 2-component vectors:

$$n_i = a_i n_{||} + b_i n_\perp, \quad i = 0, S.$$

Contracting with $n_{||}$ and $n_\perp$ we get a system of linear equations $\rightarrow a_i, b_i$.

Using $n_0, n_S$ in terms of $n_{||}, n_\perp$ to calculate $|n_0|^2, |n_S|^2, n_0^\dagger n_S$, one finds

$$\begin{aligned}
|n_0|^2 &= a_0(n_0^\dagger n_{||}) + b_0(n_0^\dagger n_\perp), \quad (\textbf{1st relation : P - wave}) \\
|n_S|^2 &= a_S(n_S^\dagger n_{||}) + b_S(n_S^\dagger n_\perp), \quad (\textbf{1st relation : S - wave}) \\
n_0^\dagger n_S &= a_S(n_0^\dagger n_{||}) + b_S(n_0^\dagger n_\perp) \quad (\textbf{2nd relation : S - wave})
\end{aligned}$$

These are the 3 extra relations to be expressed in terms of $J_i, \tilde{J}_i$.

2) Solve the system of $A_i$ in terms of $J_i, \tilde{J}_i$ and these three same equations appear.

# Structure of the relations

The three relations has a interestingly similar structure:

• P-wave in a compact form (not including CPV observables for simplicity)

$$P_2 = +\frac{1}{2}\left[(P_4' P_5' + \delta_1) + \frac{1}{\beta}\sqrt{(-1 + P_1 + P_4'^2)(-1 - P_1 + \beta^2 P_5'^2) + \delta_2 + \delta_3 P_1 + \delta_4 P_1{}^2}\right]$$

where $\delta_{1,2,3} = f(P'_{4,5}, P_3, P'_{6,8})$ and $\delta_4 = 0$ (if no CPV)

• First S-wave (quadratic and same structure) is identical to the P-wave relation with replacements:

$$P_4' \to \frac{2}{\alpha}A_S^4, \quad P_5' \to \frac{1}{\alpha}A_S^5, \quad P_6' \to -\frac{1}{\alpha}A_S^7, \quad P_8' \to -\frac{2}{\alpha}A_S^8 \qquad \alpha = \sqrt{3F_T F_S(1 - F_S)}$$

where all ratios are $\leq \sqrt{2}$.

• Second S-wave is linear, but combined with the previous ones can be written as the P-wave relation with the replacements:

$$P_4' \to \mathcal{P}_4' = 2\omega A_S^4 + \rho P_4' \quad P_5' \to \mathcal{P}_5' = \omega A_S^5 + \rho P_5'$$
$$P_6' \to \mathcal{P}_6' = -\omega A_S^7 + \rho P_6' \quad P_8' \to \mathcal{P}_8' = -2\omega A_S^8 + \rho P_8'$$

where $\omega, \rho$ are functions of $F_L, F_S, A_S$.

# Phenomenological implications: P-wave relation

⇒ The first **exact** relation (no scalars) establishes a consistency relation between the 6 independent $P_i$ + a seventh redundant $P_3$ (including the $P_i^{CP}$).

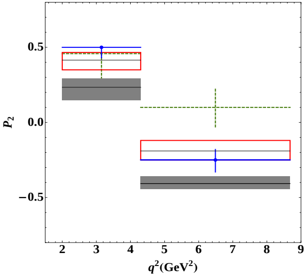
Under the hypothesis: **No New Physics in weak phases of Wilson coefficients. Then**

$$P_2^{derived} = \frac{1}{2}\left[P_4' P_5' + \frac{1}{\beta}\sqrt{(-1 + P_1 + P_4'^2)(-1 - P_1 + \beta^2 P_5'^2)}\right]$$

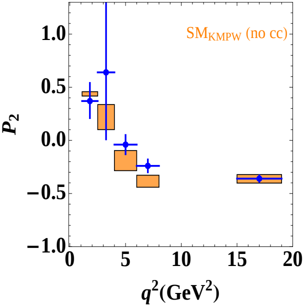
[N. Serra, J.M.'14]

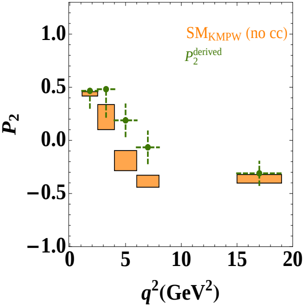
In 3fb$^{-1}$ slight shift up of $P_2$ and down of $P_5'$ as required.

2013 data:




2015 data:



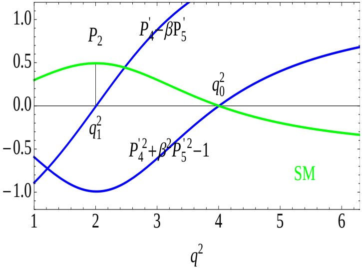



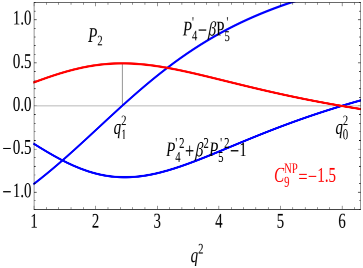

1 fb$^{-1}$ data: Difference of **2**$\sigma$ on third bin between $P_2^{measured}$ and $P_2^{derived}$

3 fb$^{-1}$ data move to increase agreement to 1.3$\sigma$ [4,6] and 1.1$\sigma$ [6,8] between $P_2^{measured}$ and $P_2^{derived}$

# Symmetries and P-wave

**Relation between $P'_{4,5}$ at $q_0^2$ and at $q_1^2$ of $P_2$**




- **At the zero of $P_2$ called $q_0^2$**

$$P_4'^2(q_0^2) + \beta^2 P_5'^2(q_0^2) = 1 + \eta(q_0^2)$$

where $\eta(q_0^2) \to 0$ if $P_1 \to 0$ (NO RHC)

**Example**: $q_0^{2SM} \simeq 4$ GeV$^2$. In the SM I would get in a 1 GeV$^2$ bin around the zero:

$$\langle P_4' \rangle_{[3.5,4.5]} = 0.714, \ \langle P_5' \rangle_{[3.5,4.5]} = -0.674$$

$$\Rightarrow [P_4'^2 + P_5'^2]_{[3.5,4.5]} = 0.96$$

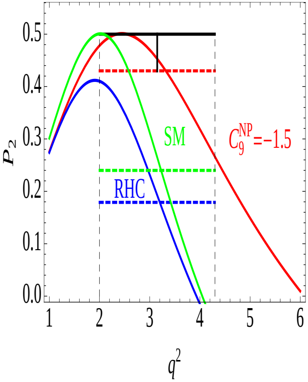
If you miss the right position of the zero you get:
$[P_4'^2 + P_5'^2]_{[2.5,3.5]} = 0.37$ and
$[P_4'^2 + P_5'^2]_{[4.5,5.5]} = 1.37$!!!

- **At the maximum of $P_2$ called $q_1^2$**

$$P_4'(q_1^2) = \left[\beta P_5' \sqrt{\frac{1-P_1}{1+P_1}}\right]_{q_1^2}$$

# A closer look at the 2nd bin of $P_2 = A_T^{\rm re}/2$

[L.Hofer, JM'15]




$$P_2^{exp}{}_{[2,4.3]} = 0.5 \pm 0.07$$
$$P_2^{SM**}{}_{[2,4.3]} = 0.24^{+0.11}_{-0.14}$$
$$P_2^{C_9^{NP}=-1.5}{}_{[2,4.3]} = 0.43$$

** KMPW in BZ: $0.16 \pm 0.12$.

This bin is as interesting/important as the anomaly bins of $P_5'$. It contains two type of important infos:

- **Position of maximum:**

$$q_1^{2LO} = \frac{2 m_b M_B C_7^{eff}}{C_{10} - {\rm Re} C_9^{eff}(q_1^2)}$$

assuming $C_7^{eff\prime} = C_9' = C_{10}' = 0$.
It shifts with the zero $q_0^2$.

- **Value of $P_2^{max}$**. Two possibilities:
  - SM or NP in $C_{7,9,10} \Rightarrow P_2^{max} = 1/2$
  - RHC NP in $C'_{7,9,10} \Rightarrow P_2^{max} < 1/2$

1 fb$^{-1}$ data pointed towards shift in $q_1^2$ and $P_2$ in the bin [2,4.3] was $+0.5 \pm 0.07$, little space for RHC. 3 fb$^{-1}$ data too large error in this bin to discern.

# Phenomenological implications: S-wave relations

- Two S-wave relations $\Rightarrow A_S, A_S^{4,5,7,8}, F_S$ **are NOT independent**

Basis: $\left\{\frac{d\Gamma}{dq^2}, F_L, P_1, P_2, P_4', P_5', P_6', P_8', F_S, A_S, A_S^4, A_S^5\right\}$

- The requirement of real solutions when solving 1st S-wave relation for $A_S^{4,5}$ $\Rightarrow$ constraints on the allowed ranges of values:

$$|A_S^4| \leq \frac{1}{2}\sqrt{3F_T F_S(1-F_S)(1-P_1)},$$

$$|A_S^5| \leq \sqrt{3F_T F_S(1-F_S)(1+P_1)},$$

$$|A_S^7| \leq \sqrt{3F_T F_S(1-F_S)(1-P_1)},$$

$$|A_S^8| \leq \frac{1}{2}\sqrt{3F_T F_S(1-F_S)(1+P_1)}.$$

$$|A_S| \leq 2\sqrt{3F_S(1-F_S)(1-F_T)}$$

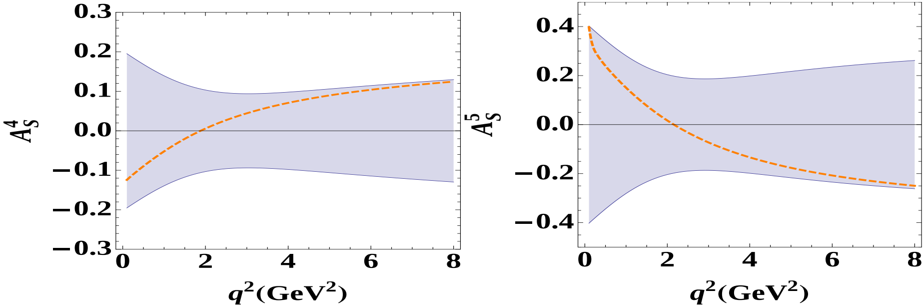
[S. Descotes, T. Hurth, JM, J. Virto'13]

In agreement with previous Cauchy-Schwartz inequalities.

# Phenomenological implications: Constraints on $A_S^i$

*Symmetry relations impose correlations among $A_S^i$ beyond individual bounds.*

**Bounds on $A_S^{4,5}$ assuming $F_S \simeq 6\%$ in SM**




As an illustration:

Assume a measurement of $A_S^{5,7,8}$ gives:

$$A_S^5 = \alpha P_5', \; A_S^7 = -\alpha P_6' \simeq 0 \text{ and } A_S^8 = -\frac{\alpha}{2} P_8' \simeq 0 \qquad (\alpha = \sqrt{3 F_T F_S (1 - F_S)})$$

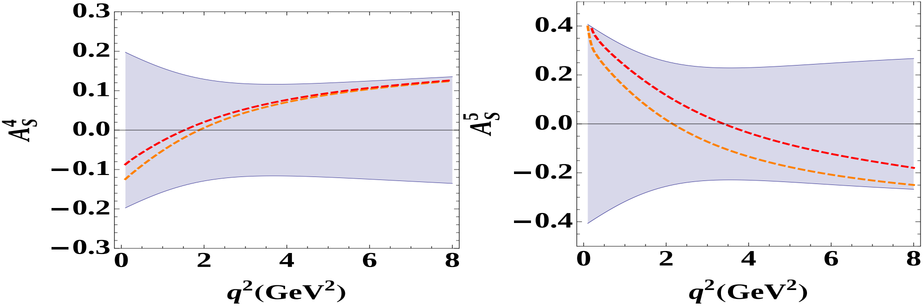
$\Rightarrow$ **symmetry relation** fixes completely $A_S^4$ to be $\frac{\alpha}{2} P_4'$.

Similarly if we measure $A_S^{4,7,8}$ symmetry fixes $A_S^5$.

# Phenomenological implications: Constraints on $A_S^i$

*Symmetry relations impose correlations among $A_S^i$ beyond individual bounds.*

**Bounds on $A_S^{4,5}$ assuming $F_S \simeq 6\%$ in presence of NP** $C_9^{NP} = -1.5$

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$\Rightarrow$ **symmetry relation** fixes completely $A_S^4$ to be $\frac{\alpha}{2} P_4'$.

Similarly if we measure $A_S^{4,7,8}$ symmetry fixes $A_S^5$.

# Constraints on the $A_S^{(i)}$ at maximum and zero of $P_2$

The requirement of *real solutions* when solving the quadratic S-wave relation at the position $q_1^2$ of the maximum of $P_2$ fixes completely the ratios $\mathbf{r_S^{4,5}} = \mathbf{A_S^4}/\mathbf{A_S^5}$ and $\mathbf{r_S^{7,8}} = \mathbf{A_S^7}/\mathbf{A_S^8}$ at $q_1^2$:

$$2\mathbf{A_S^4}(q_1^2) = \left[\mathbf{A_S^5}\sqrt{\frac{1-P_1}{1+P_1}}\right]_{q_1^2} \quad \text{and} \quad \mathbf{A_S^7}(q_1^2) = \left[2\mathbf{A_S^8}\sqrt{\frac{1-P_1}{1+P_1}}\right]_{q_1^2} .$$

Using the symmetry relations at the zero $q_0^2$ of $P_2$ ($q_0^{2SM} = 4\ \text{GeV}^2$):

$$\mathbf{A_S}(q_0^2) = \left[\frac{2\sqrt{F_L}(2\mathbf{A_S^4}(1+P_1)P_4' + \mathbf{A_S^5}(1-P_1)P_5'}{\sqrt{F_T}(1-P_1^2)}\right]_{q_0^2}$$

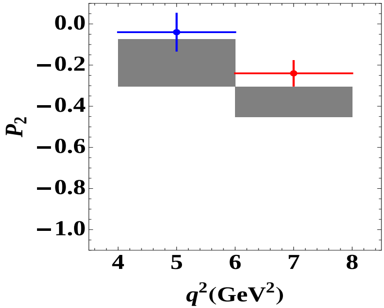
In the absence of RHC ($P_1 \simeq 0$) the relation simplifies.

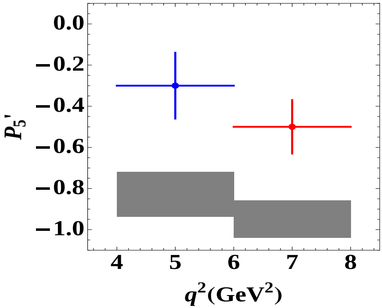
# Conclusions

- The angular distribution of $B \to K\pi\ell\ell$ exhibit a set of 4 **symmetries** common to the P and S-wave sector.
- The symmetries imply **3 non trivial relations** among the $J_i$ and $\tilde{J}_i$.
  - **P-wave**: Assuming no NP in weak phases we establish a relation between $P_2$ and $P'_{4,5}$, $P_1$. Also relations at the zero and maximum of $P_2$ are found that make explicit the impact of the anomaly of $P'_5$ in $P_2$.
  - **S-wave**: Only 4 out of the 6 S-wave observables are independent. We establish bounds but also strong correlations among them.
- $P_2$ contains not only the information of the zero of $A_{FB}$ but the **position and value of its maximum** is sensitive to NP.
  - NP in SM-like operators shift the position keeping its value to $1/2$.
  - RHC reduces the peak below $1/2$.

# Back-up slides

# Phenomenological implications(I)




Consistency test on data compare $\mathbf{P_2^{exp}}$ with $\mathbf{P_2 = f(P_1^{exp}, P_{4,5}'^{exp})}$ (assume: no new weak phases):

$$P_2 = \frac{1}{2}\left(P_4' P_5' + \frac{1}{\beta}\sqrt{(-1 + P_1 + P_4'^2)(-1 - P_1 + \beta^2 P_5'^2)}\right)$$

- If $P_2 = -\epsilon$ and $P_4' = 1 + \delta$ ($P_1 < -2\delta$) then $\mathbf{P_5' \leq -2\epsilon/(1+\delta)}$

2013: $\langle P_2 \rangle_{[4.3,8.68]} \sim -0.25$ and $\langle P_5' \rangle_{[4.3,8.68]} \sim -0.19$ approx. $\epsilon = -0.25$, $\langle P_5' \rangle_{[4.3,8.68]} \leq -0.42$

2015: $\langle P_2 \rangle_{[6,8]} \sim -0.24$ and $\langle P_5' \rangle_{[6,8]} \sim -0.5$ approx. $\epsilon = -0.24$, $\langle P_5' \rangle_{[6,8]} \leq -0.4$

**Now $P_2$ and $P_5'$ bins have the expected order! (in both [4,6] and [6,8] bins)**

[Becirevic, Taygudanov'12]

$$\tilde{J}_{1a}^c = -\tilde{J}_{2a}^c = \frac{3}{8}(|A_0'^L|^2 + |A_0'^R|^2)|BW_S|^2$$

$$\tilde{J}_{1b}^c = -\tilde{J}_{2b}^c = \frac{3}{4}\sqrt{3}\mathrm{Re}\left[(A_0'^L A_0^{L*} + A_0'^R A_0^{R*})BW_S BW_P^*\right],$$

$$\tilde{J}_4 = \frac{3}{4}\sqrt{\frac{3}{2}}\mathrm{Re}\left[(A_0'^L A_\parallel^{L*} + A_0'^R A_\parallel^{R*})BW_S BW_P^*\right],$$

$$\tilde{J}_5 = \frac{3}{2}\sqrt{\frac{3}{2}}\mathrm{Re}\left[(A_0'^L A_\perp^{L*} - A_0'^R A_\perp^{R*})BW_S BW_P^*\right],$$

$$\tilde{J}_7 = \frac{3}{2}\sqrt{\frac{3}{2}}\mathrm{Im}\left[(A_0'^L A_\parallel^{L*} - A_0'^R A_\parallel^{R*})BW_S BW_P^*\right],$$

$$\tilde{J}_8 = \frac{3}{4}\sqrt{\frac{3}{2}}\mathrm{Im}\left[(A_0'^L A_\perp^{L*} + A_0'^R A_\perp^{R*})BW_S BW_P^*\right].$$

# Second S-wave relation parameters

The parameters entering the second S-wave relation are provided here:

$$\omega = \frac{3}{z_2} \qquad \rho = -4\frac{z_1}{z_2}$$

where

$$z_1 = \sqrt{F_T F_L}(-1 + F_S)$$

and

$$z_2 = \sqrt{F_T}\sqrt{(1 - F_S)(12A_S + 16F_L(1 - F_S) + 27F_S)}$$