Determination of α_s from the QCD static energy

Xavier Garcia i Tormo
Universität Bern

Based on:

A. Bazavov, N. Brambilla, XGT, P. Petreczky, J. Soto and A. Vairo, Phys. Rev. D **86**, 114031 (2012) [arXiv:1205.6155 [hep-ph]];

Phys. Rev. D **90**, 074038 (2014) [arXiv:1407.8437 [hep-ph]]

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QCD static energy $E_0(r)$

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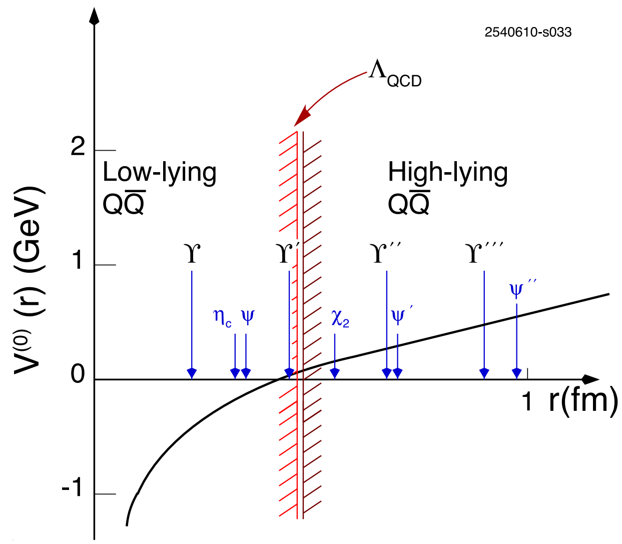

From N. Brambilla *et al.*, Eur. Phys. J. **C71** (2011) 1534

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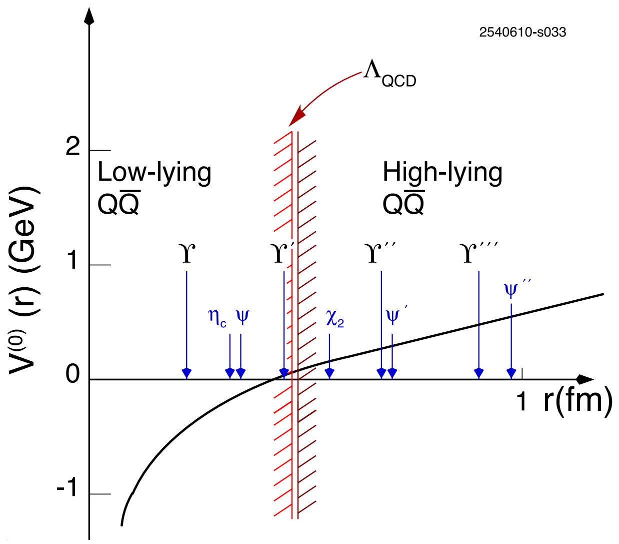

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Short-distance part $\longleftrightarrow$ Long-distance part

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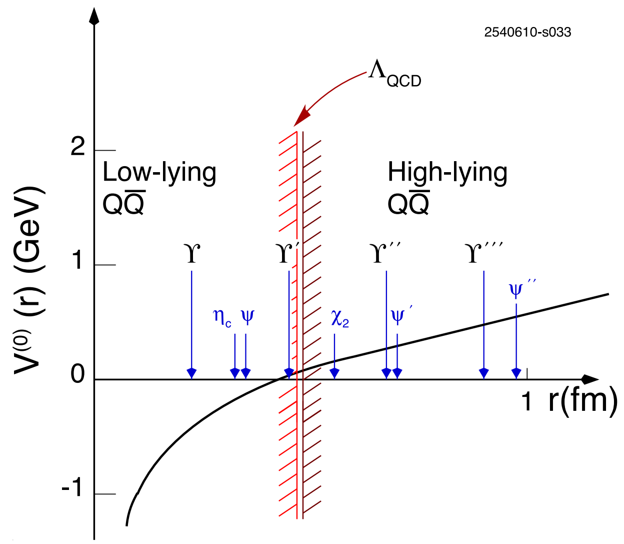

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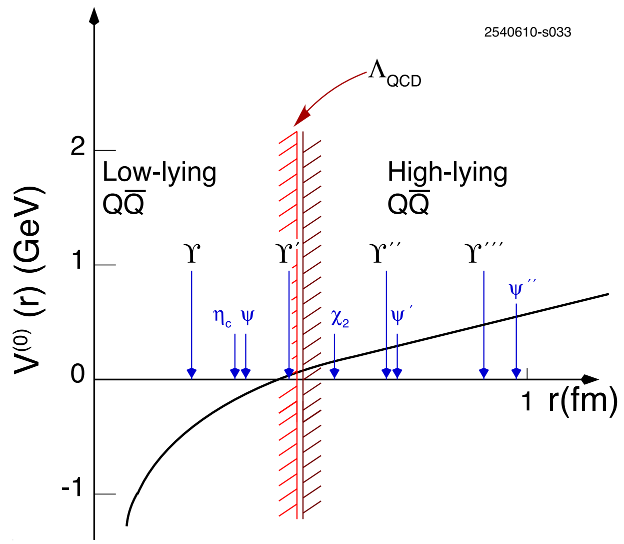

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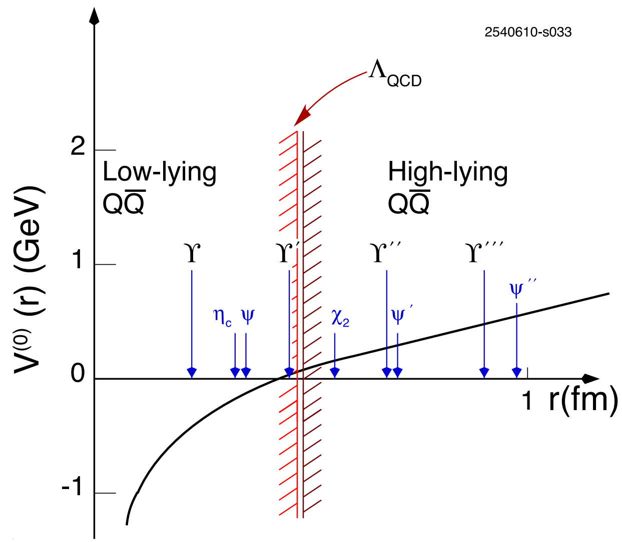

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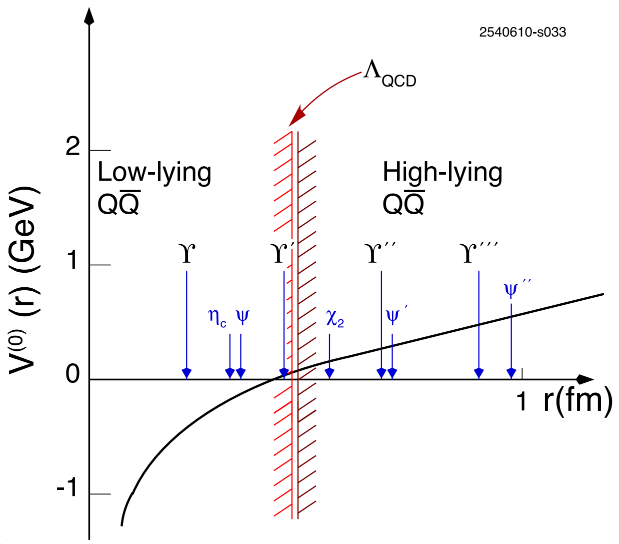

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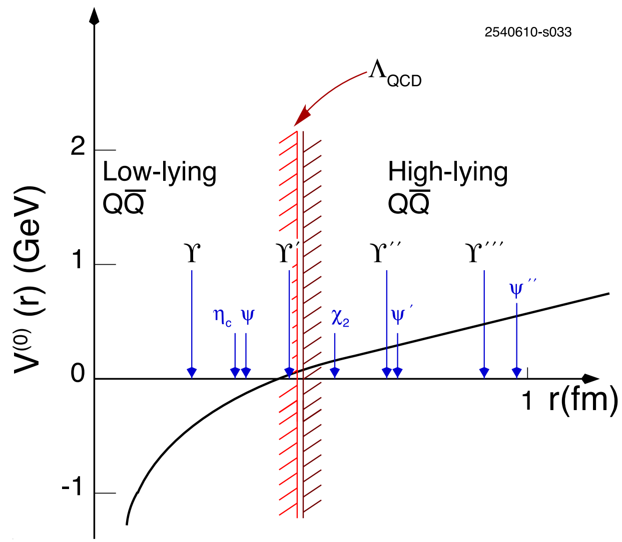

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Perturbation theory Encodes confining dynamics

Compare perturbative and lattice results for the static energy at short distances to extract α_s

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$$E_0(r) \sim -C_F \frac{\alpha_s(1/r)}{r}$$

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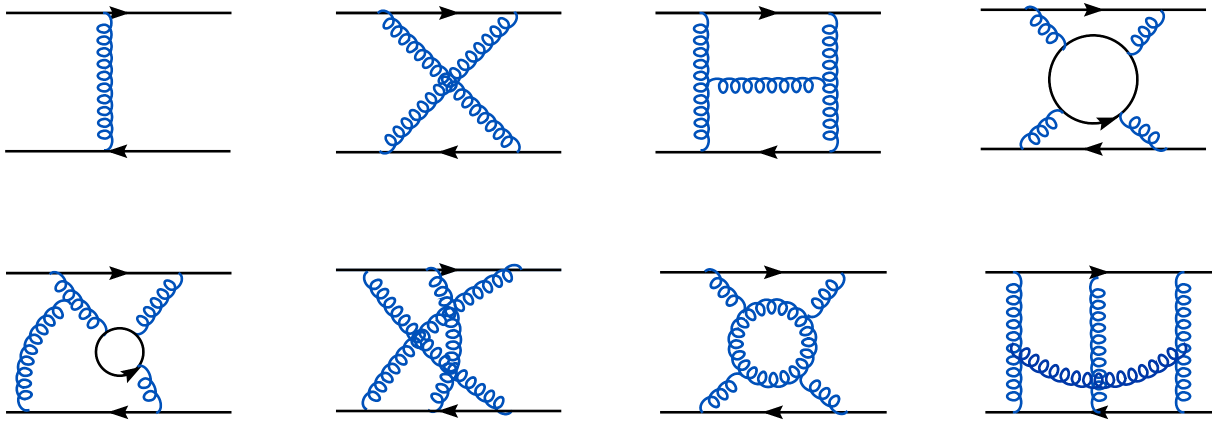

(Picture from A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys.Rev.Lett. **104** (2010) 112002 [arXiv:0911.4742 [hep-ph]])

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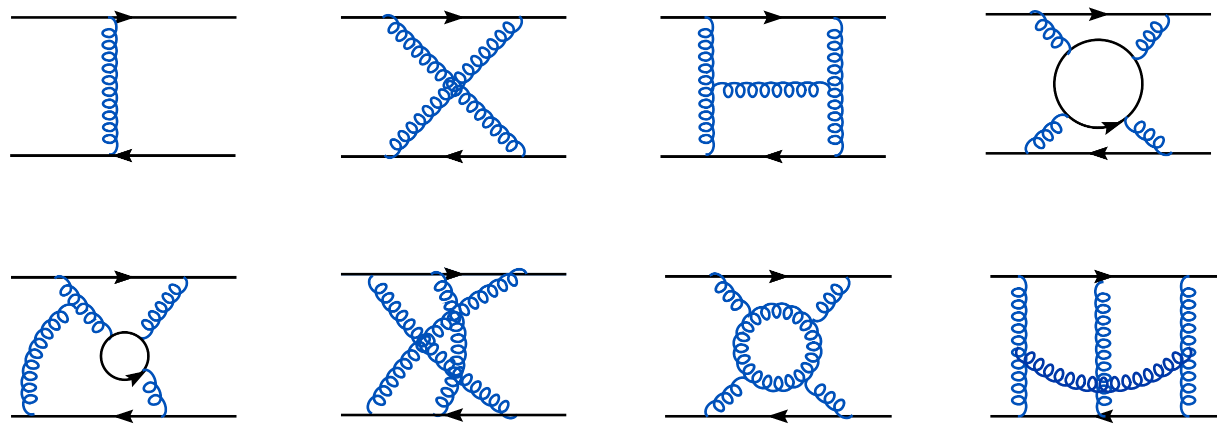

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Virtual emissions that change the color state of the pair (*Ultrasoft gluons*) Appelquist Dine Muzinich'78

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$E_0(r)$ calculated on the lattice in $n_f = 2 + 1$ flavor QCD

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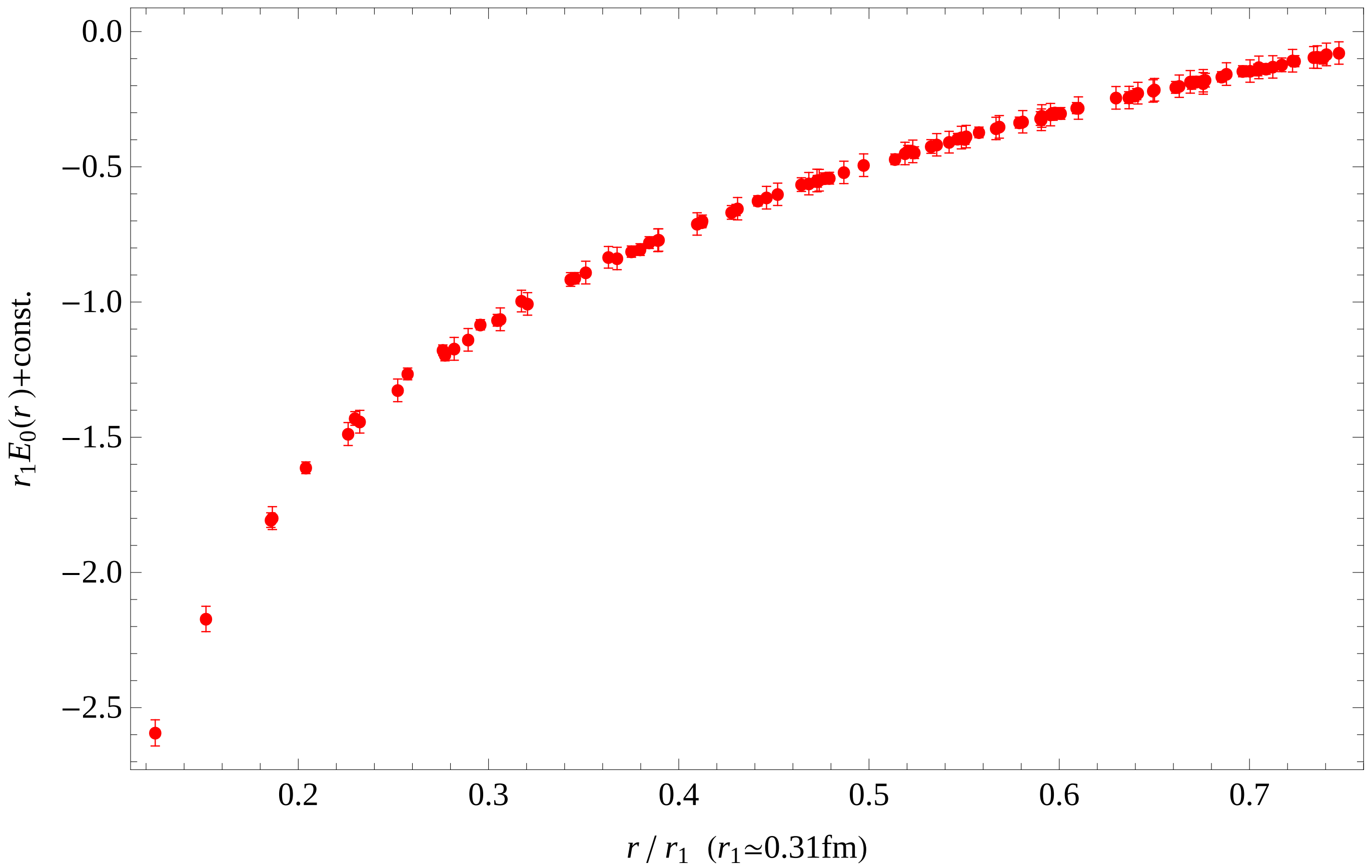

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- Normalization (mass) $\rightarrow$ affected by renormalon
- Slope (relevant for $\alpha_{\rm s}$ extraction) $\rightarrow$ renormalon free

Perturbative expression best suited for the comparison. Use pert. expression for the force

$$E_0 \sim -\frac{C_F}{r}\alpha_E(r,\nu) + RS(\rho)$$

Beneke'98; Hoang *el al.*'99; Pineda'01

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$\alpha_E(r,\nu)$: series in $\alpha_{\rm s}(\nu)$, contain $\ln(r\nu)$ terms
$RS(\rho)$: series in $\alpha_{\rm s}(\rho)$, affected by uncertainties in computation of renormalon

$$E_0 \sim -\frac{C_F}{r}\alpha_E(r,\nu) + RS(\rho) \rightarrow$$

$$\rightarrow F(r,\nu) = \frac{dE_0}{dr} = \frac{C_F}{r^2}\left(\alpha_E(r,\nu) - r\alpha'_E(r,\nu)\right)$$

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No logs, not affected by renormalon uncertainties

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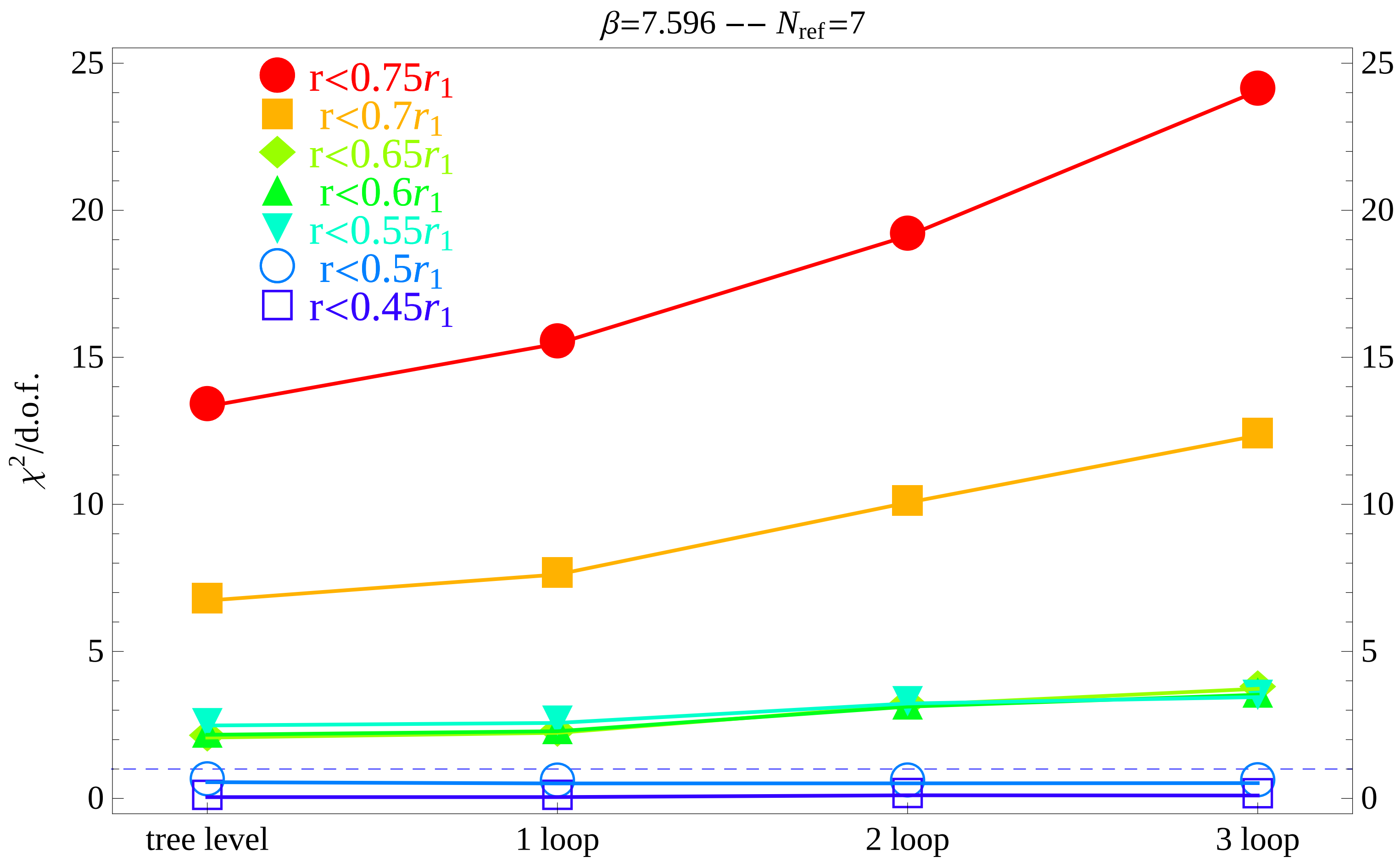
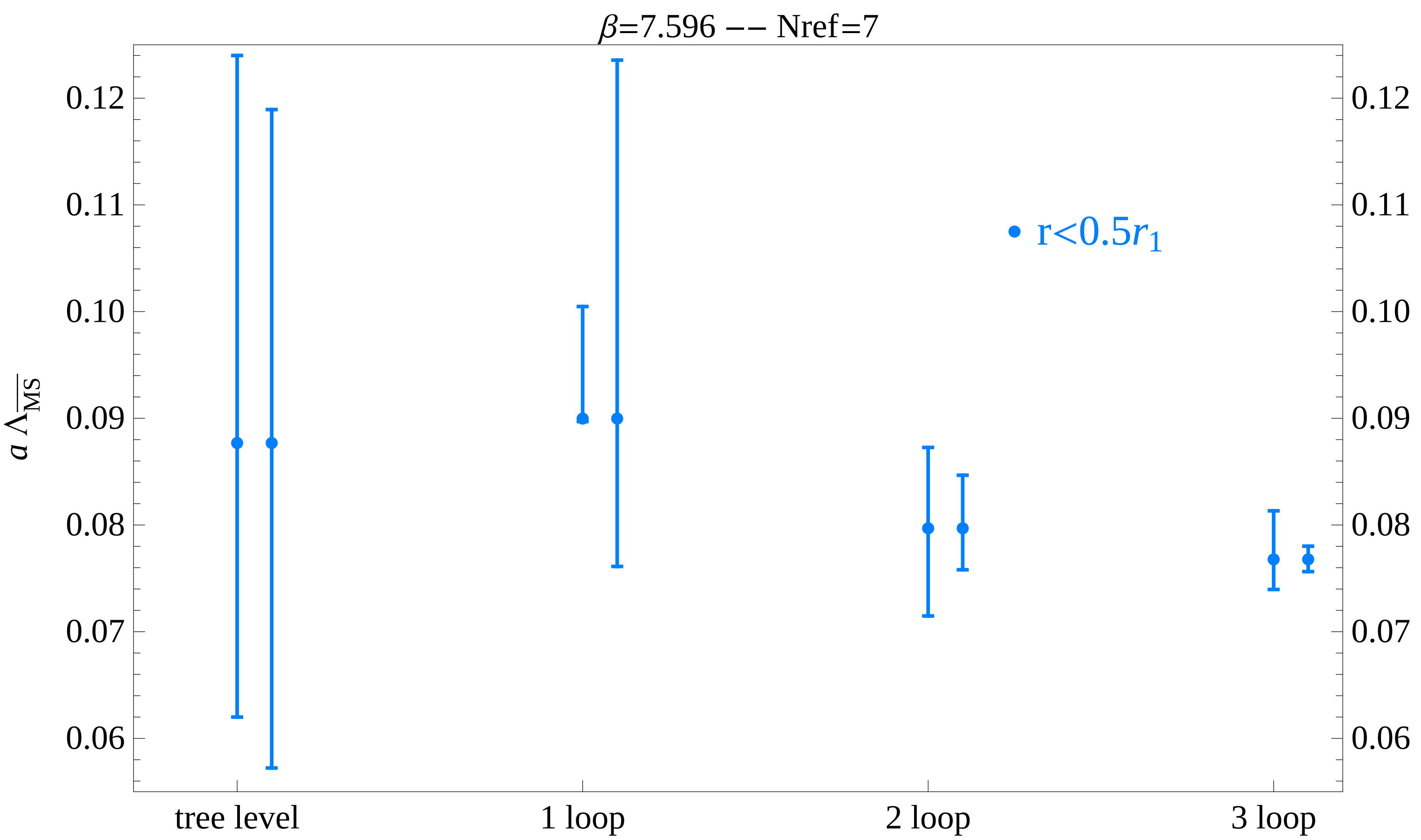
- Estimate pert. uncertainty: Repeat fits with scale variation, and adding $\pm(C_F/r^2)\alpha_{\rm s}^{n+2}$


β=7.596 –– N_{ref}=7
r<0.75r_1
r<0.7r_1
r<0.65r_1
r<0.6r_1
r<0.55r_1
r<0.5r_1
r<0.45r_1
χ^2/d.o.f.
0
5
10
15
20
25
tree level
1 loop
2 loop
3 loop


β=7.596 –– Nref=7
$a\,\Lambda_{\overline{\mathrm{MS}}}$
0.12
0.11
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0.09
0.08
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0.06
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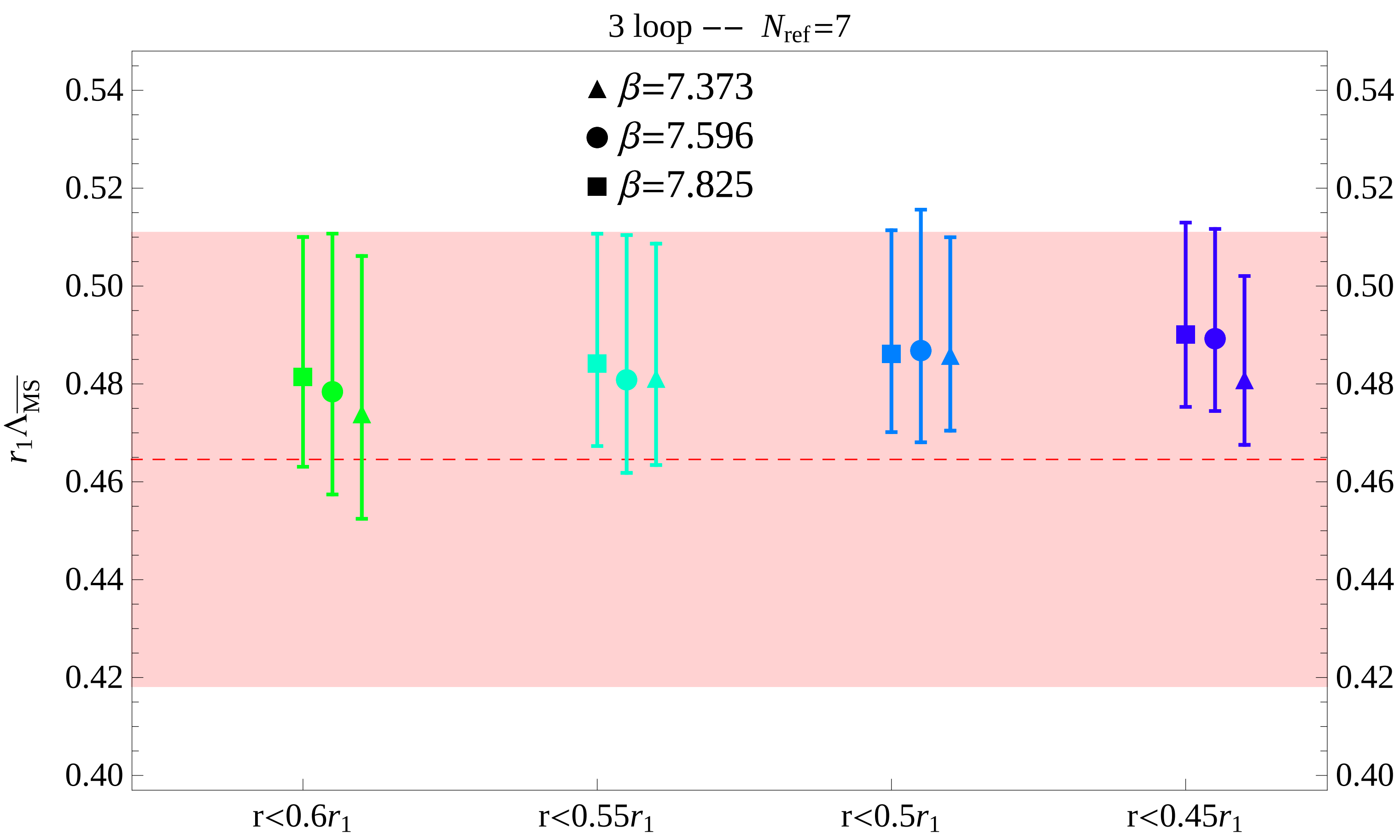
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$r_1\Lambda_{\overline{MS}}$
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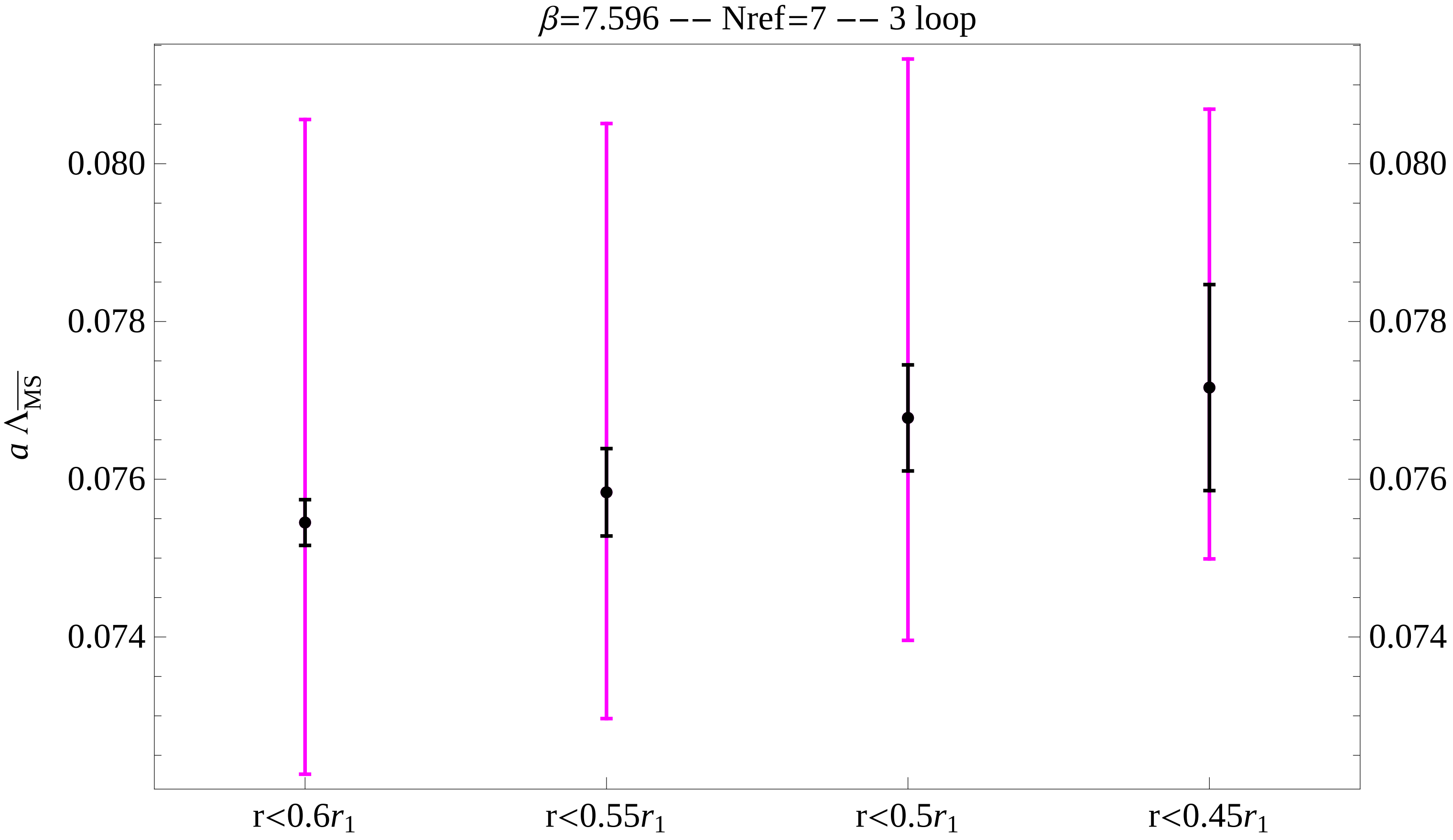
β=7.596 -- Nref=7 -- 3 loop
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Cross checks

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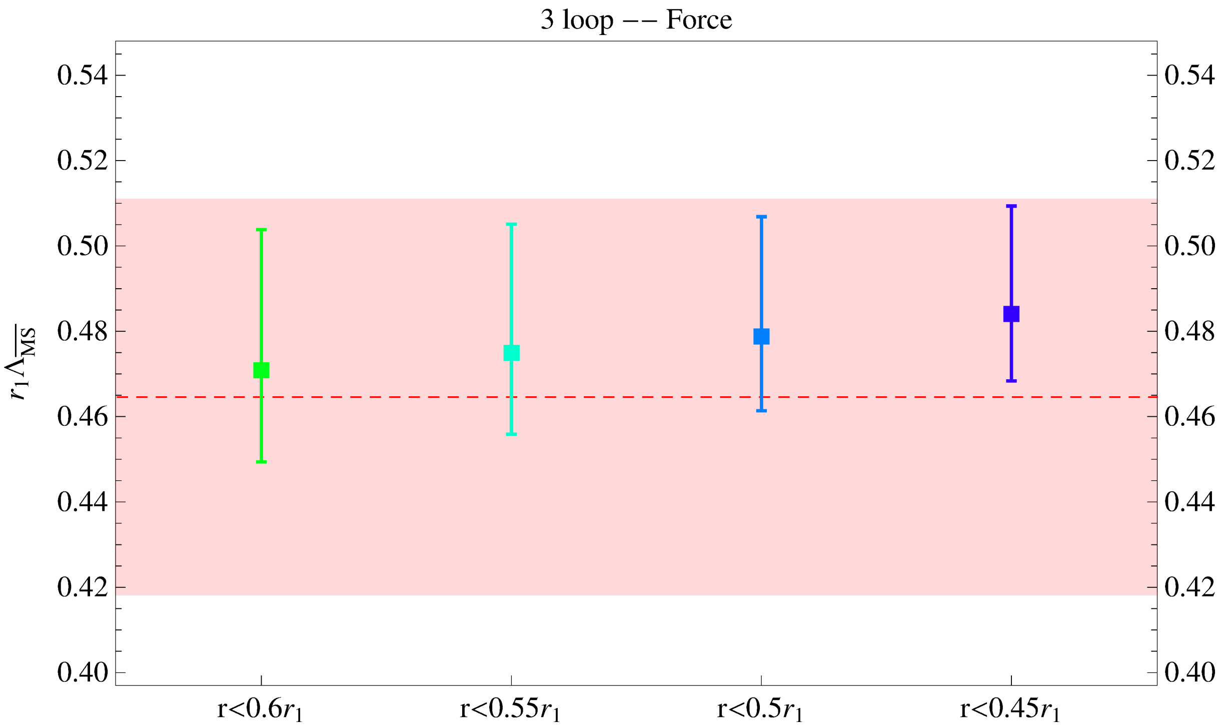
- Compare with lattice data for the force



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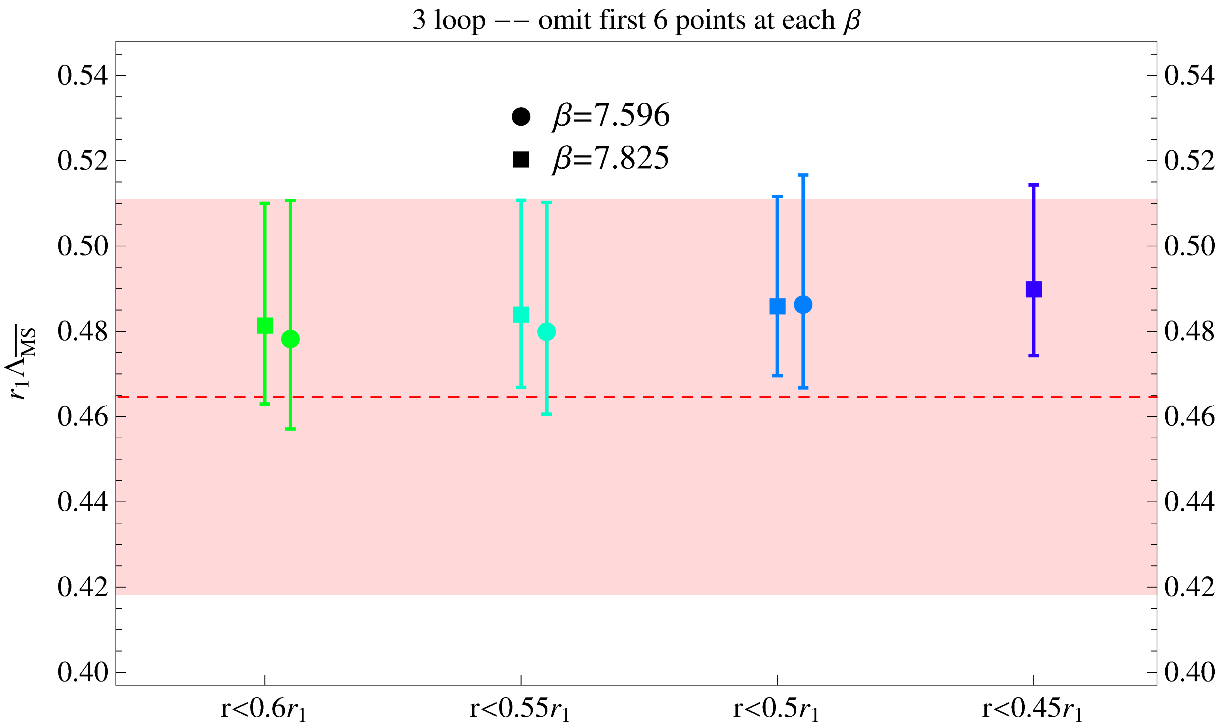
- Exclude lattice points with larger systematic (discretization) uncertainties, i.e. use only points were these are negligible



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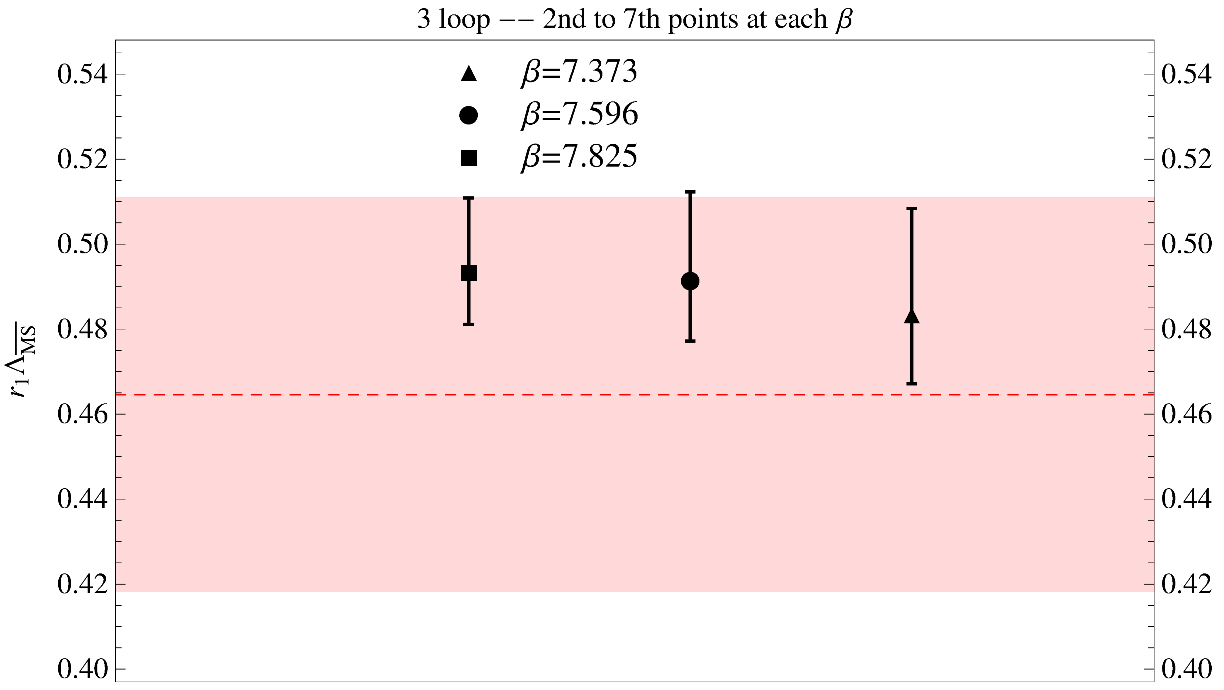

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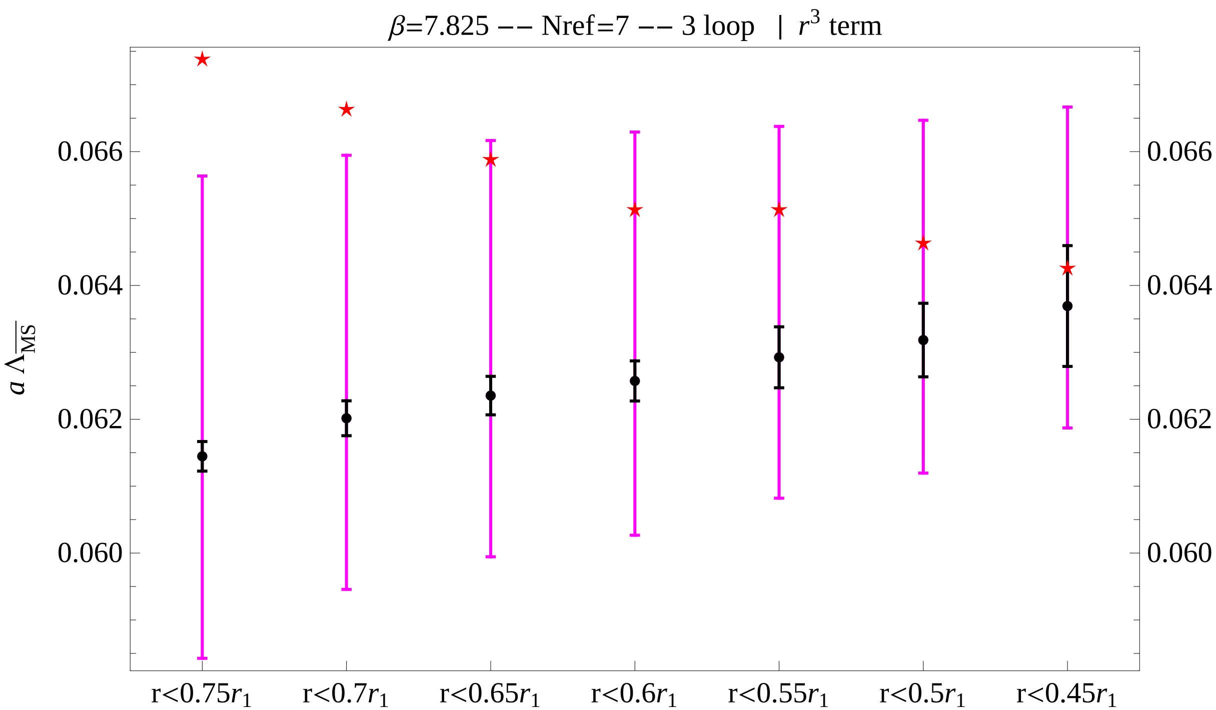

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All the results perfectly compatible with each other. It shows that the extraction is robust

Result for α_s

We take the 3-loop + leading ultrasoft log res. accuracy result

$$r_1\Lambda_{\overline{\rm MS}} = 0.495^{+0.028}_{-0.018} \to \Lambda_{\overline{\rm MS}} = 315^{+18}_{-12}\ \text{MeV}$$

$$\alpha_s(1.5\ \text{GeV}, n_f = 3) = 0.336^{+0.012}_{-0.008}$$

$$\to \alpha_s(M_Z, n_f = 5) = 0.1166^{+0.0012}_{-0.0008}$$

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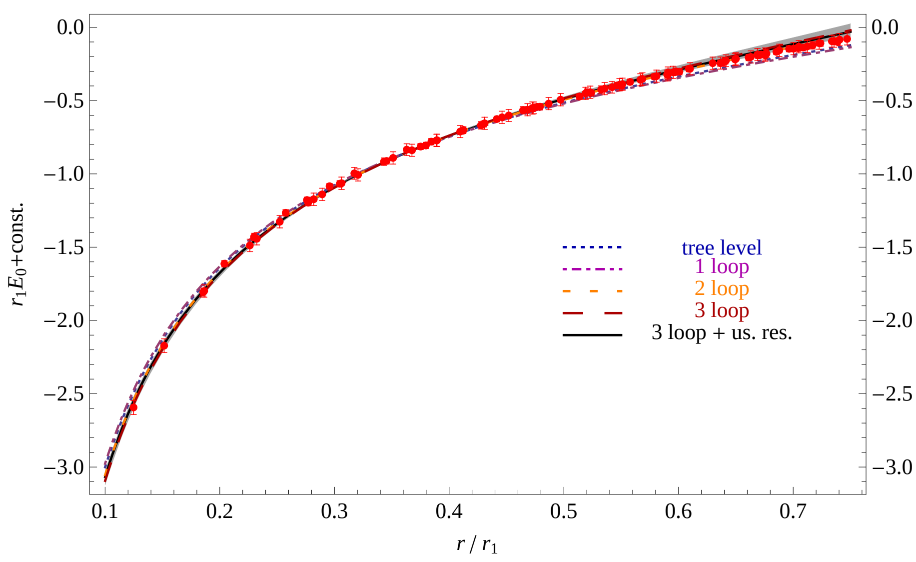
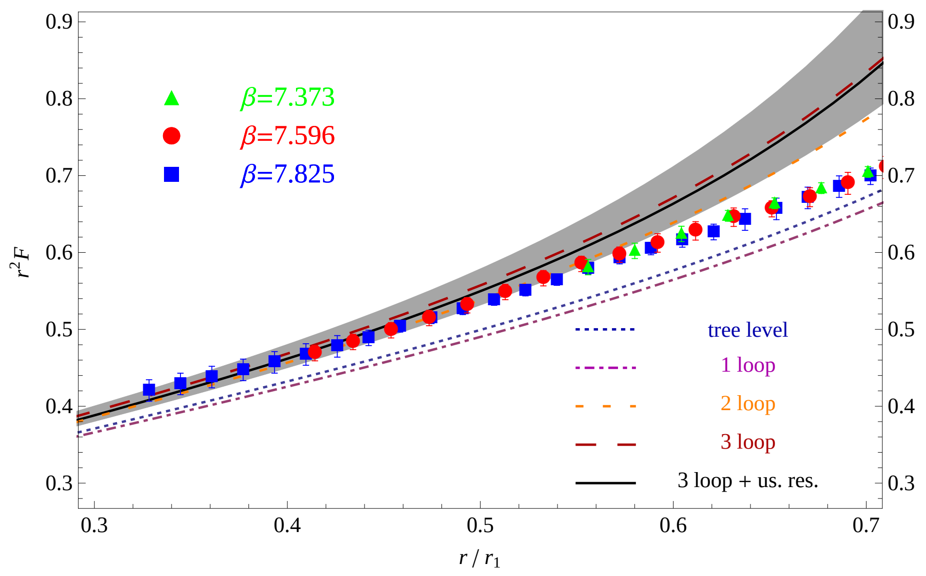
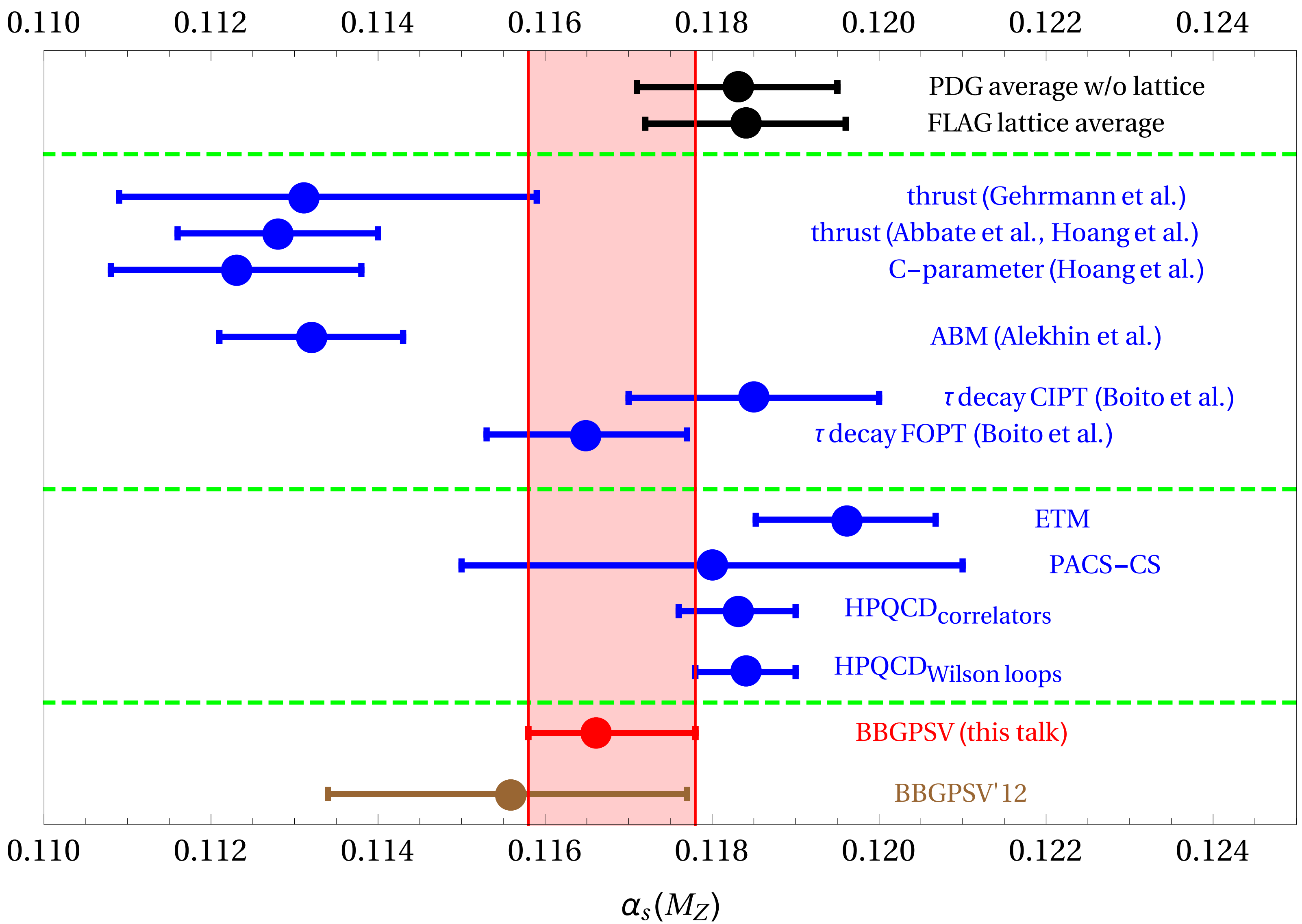




Comparison with other results



Conclusions

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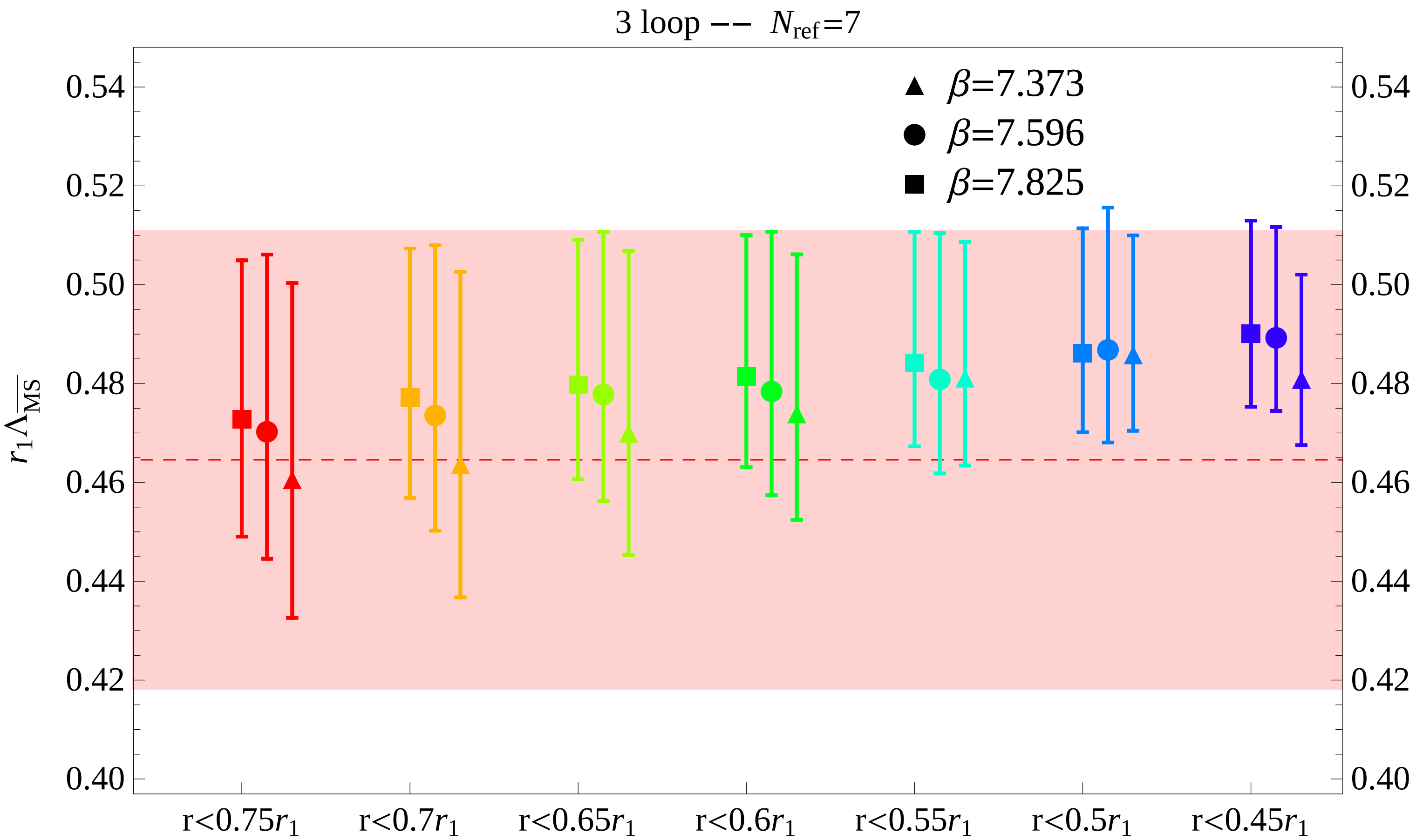
Backup slides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r<0.75r_1
r<0.7r_1
r<0.65r_1
r<0.6r_1
r<0.55r_1
r<0.5r_1
r<0.45r_1

Lattice

$E_0(r)$ calculated on the lattice in $n_f = 2+1$ flavor QCD

Bazavov *et al.* (HotQCD Coll.)'14

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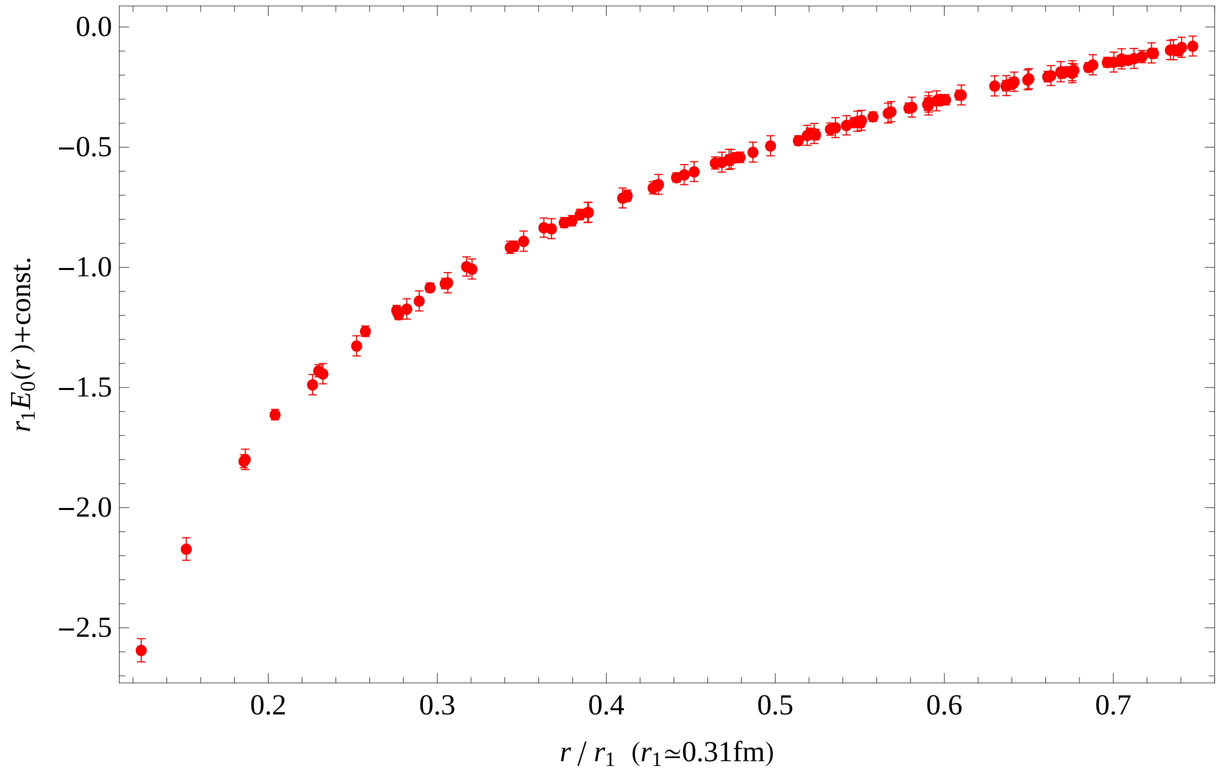

Lattice

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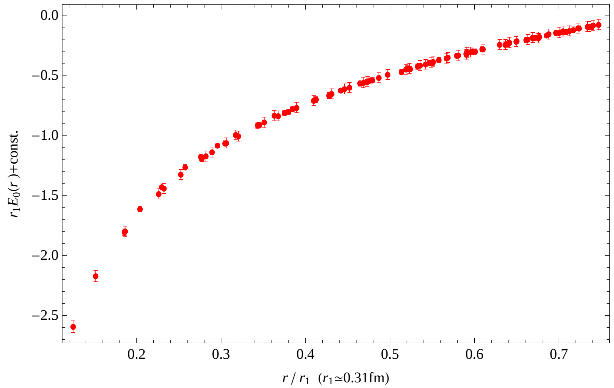
Bazavov *et al.* (HotQCD Coll.)'14



Combination of tree-level improved gauge action and HISQ action

Lattice

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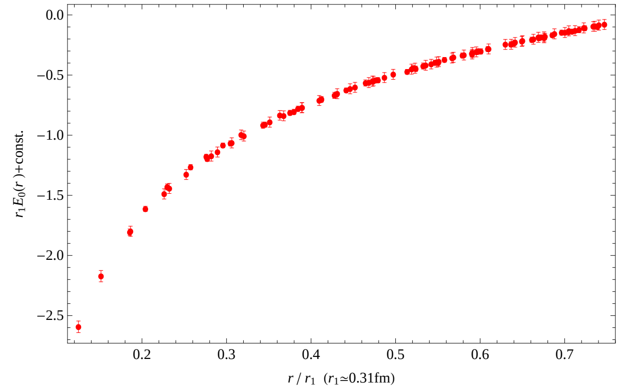
Bazavov *et al.* (HotQCD Coll.)'14



Combination of tree-level improved gauge action and HISQ action

m_s phys. value; $m_l = m_s/20$ corresponding to $m_\pi \sim 160$ MeV

Lattice

$E_0(r)$ calculated on the lattice in $n_f = 2 + 1$ flavor QCD

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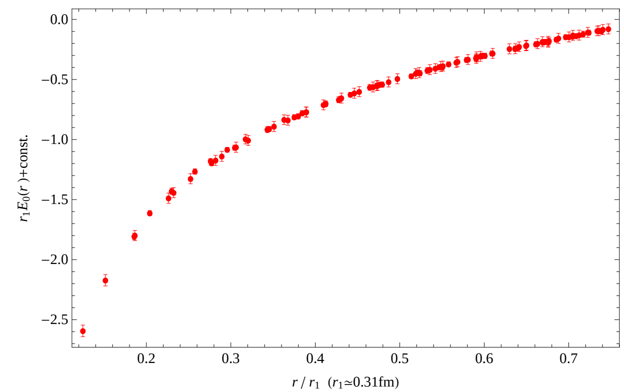

Combination of tree-level improved gauge action and HISQ action

m_s phys. value; $m_l = m_s/20$ corresponding to $m_\pi \sim 160$ MeV

Energy calculated in units of r_1

$$r^2 \frac{dE_0(r)}{dr}\Big|_{r=r_1} = 1$$

Lattice data for several gauge couplings
$\beta = 7.150, 7.280, 7.373, 7.596, 7.825,$
the smallest lattice spacing is $a = 0.041$ fm

To compare results at different β, need to normalize to common value at a certain distance (or take numerical derivative)

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To compare results at different β, need to normalize to common value at a certain distance (or take numerical derivative)

Need to correct for lattice artifacts, and estimate corresponding systematic errors

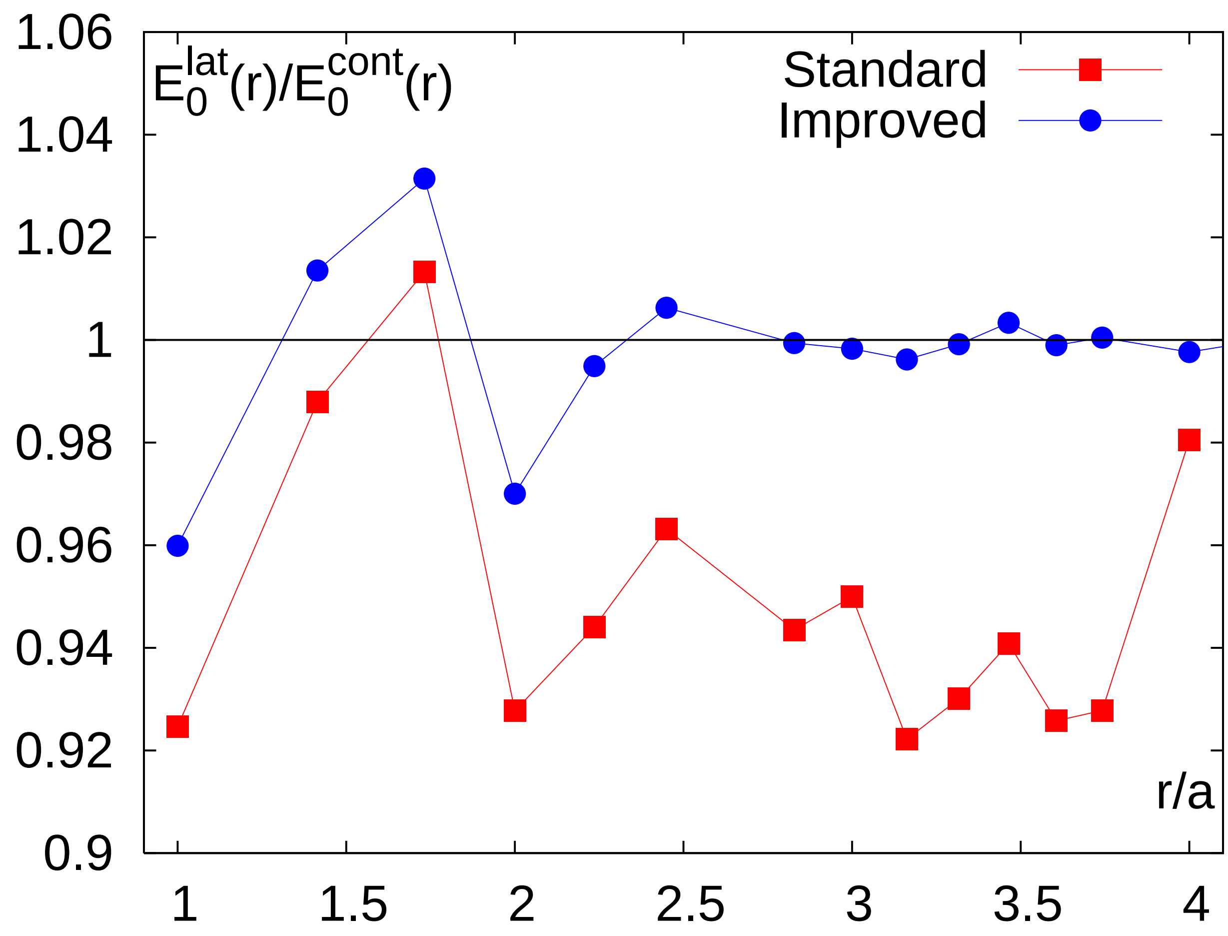
Replace r by improved distance $r_I = (4\pi C_L(r))^{-1}$

Necco Sommer'01

$$C_L(r) = \int \frac{d^3k}{(2\pi)^3} D_{00}(k_0 = 0, \vec{k}) e^{i\vec{k}\vec{r}}$$

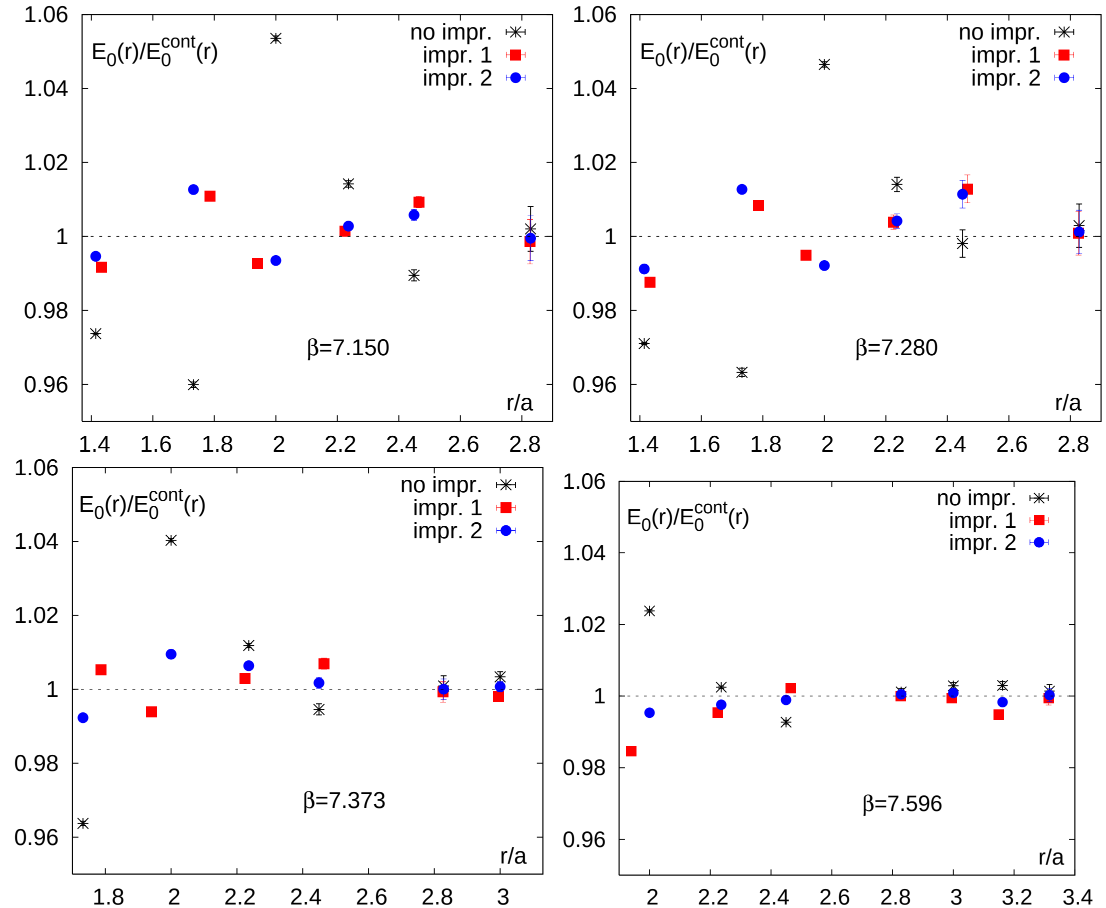
(D_{00} is the tree-level gluon propagator on the lattice)

Tree level



Discretization effects $\lesssim 1\%$ for $r/a > 2$

To estimate cutoff effects in actual calculation, need continuum estimate of E_0. Assume cutoff effects negligible for $r/a > 2$, fit $\beta = 7.825$ results to Coulomb plus linear plus constant form, to get continuum estimate.



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Correct for residual cutoff effects, divide value of lattice static energy for six first points at each lattice spacing by correction factors. Estimated via an iterative procedure, taking ratios of lattice values to continuum estimate.

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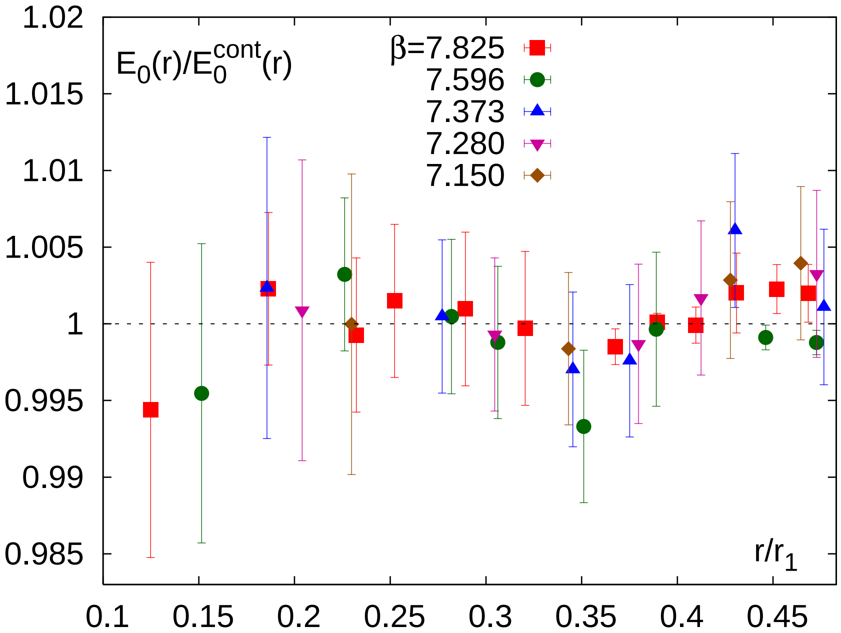
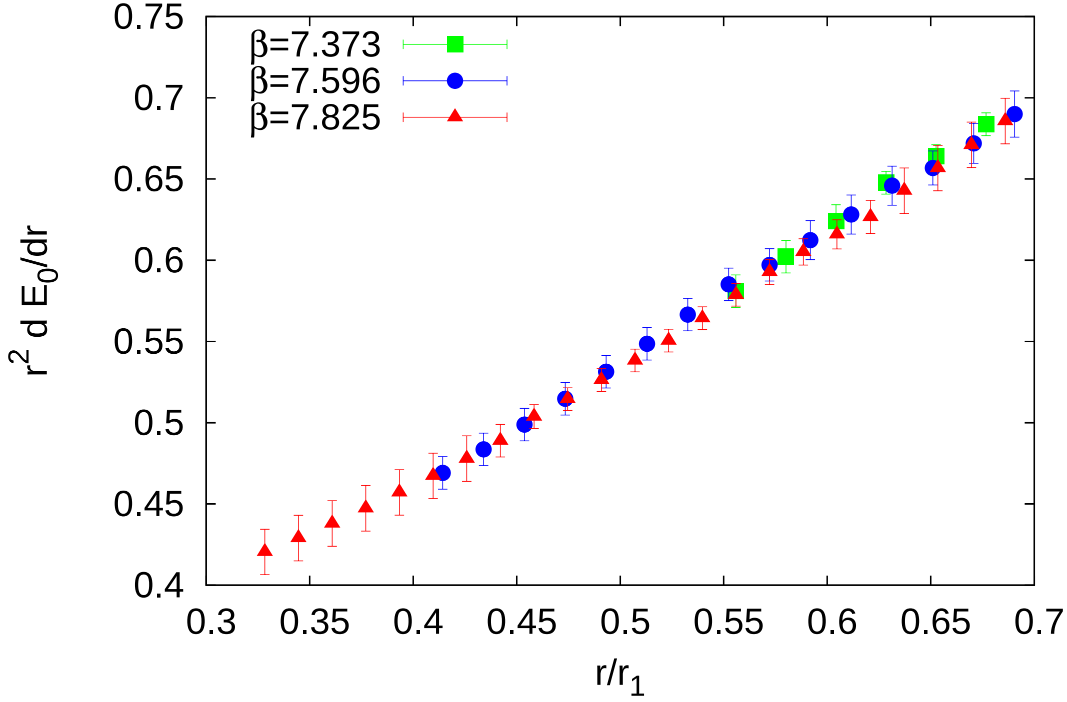


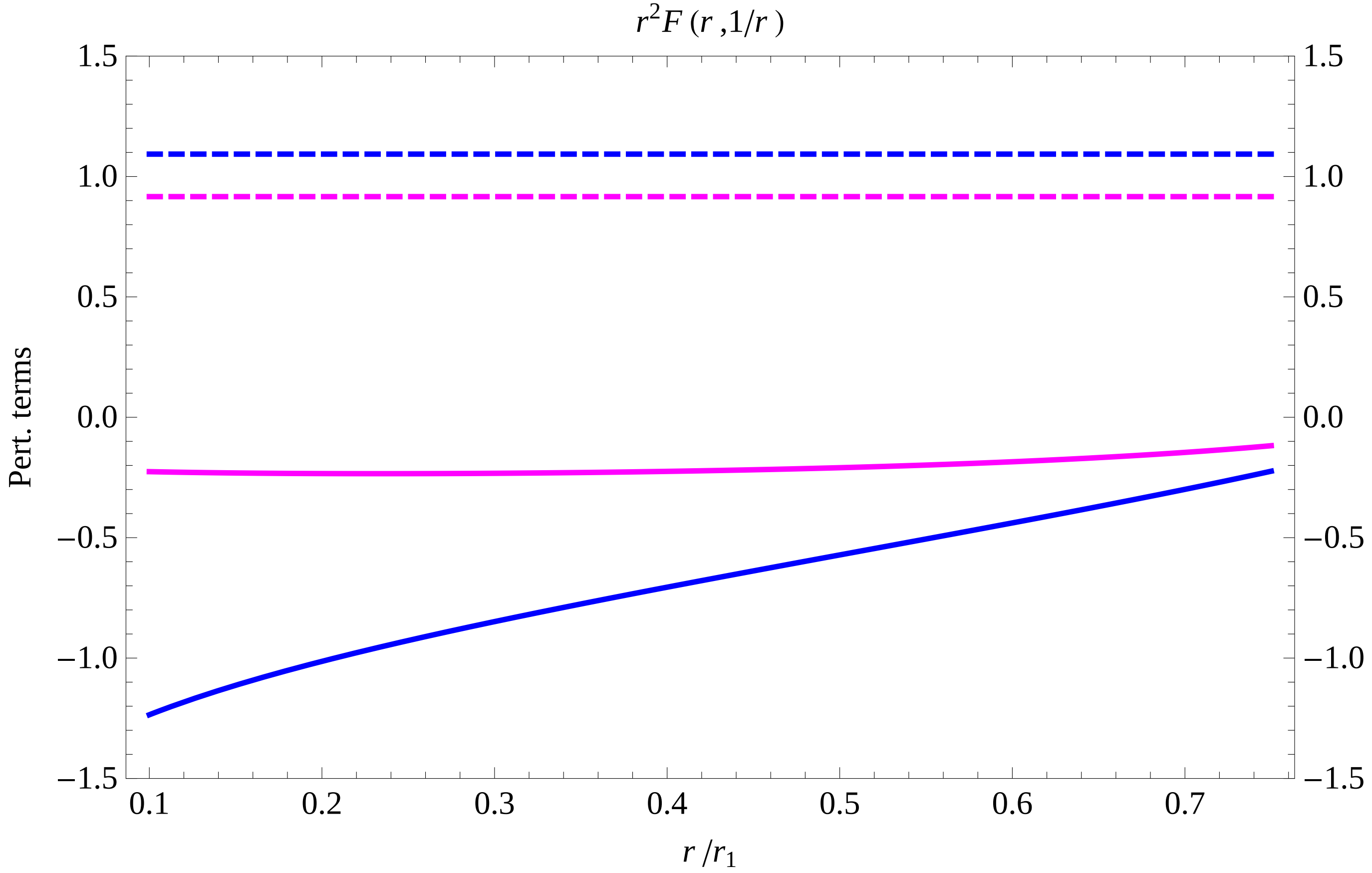
Can also calculate force from the lattice data. Use only $r/a > 2$ to avoid problems with lattice artifacts. Obtained with smoothing splines

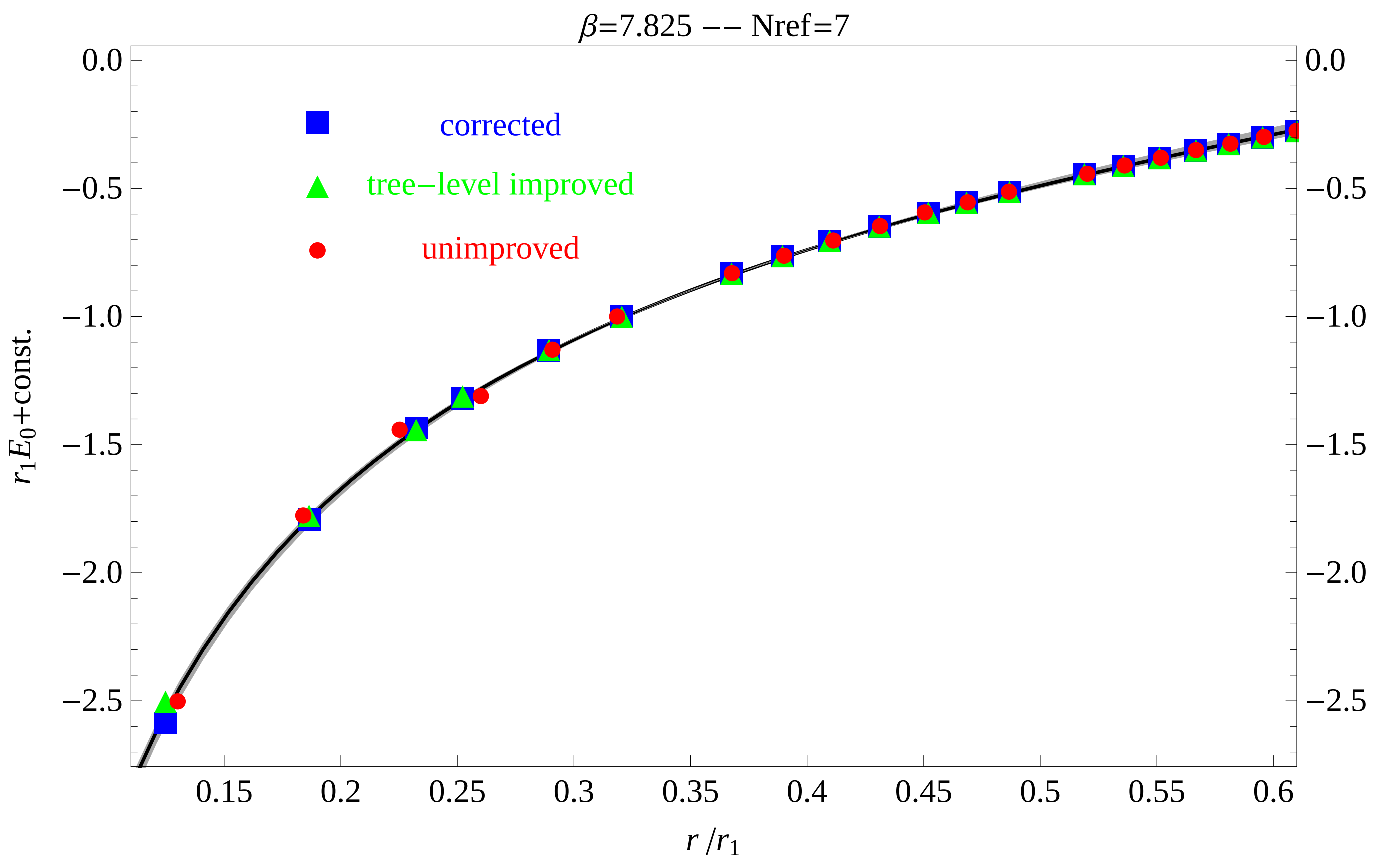


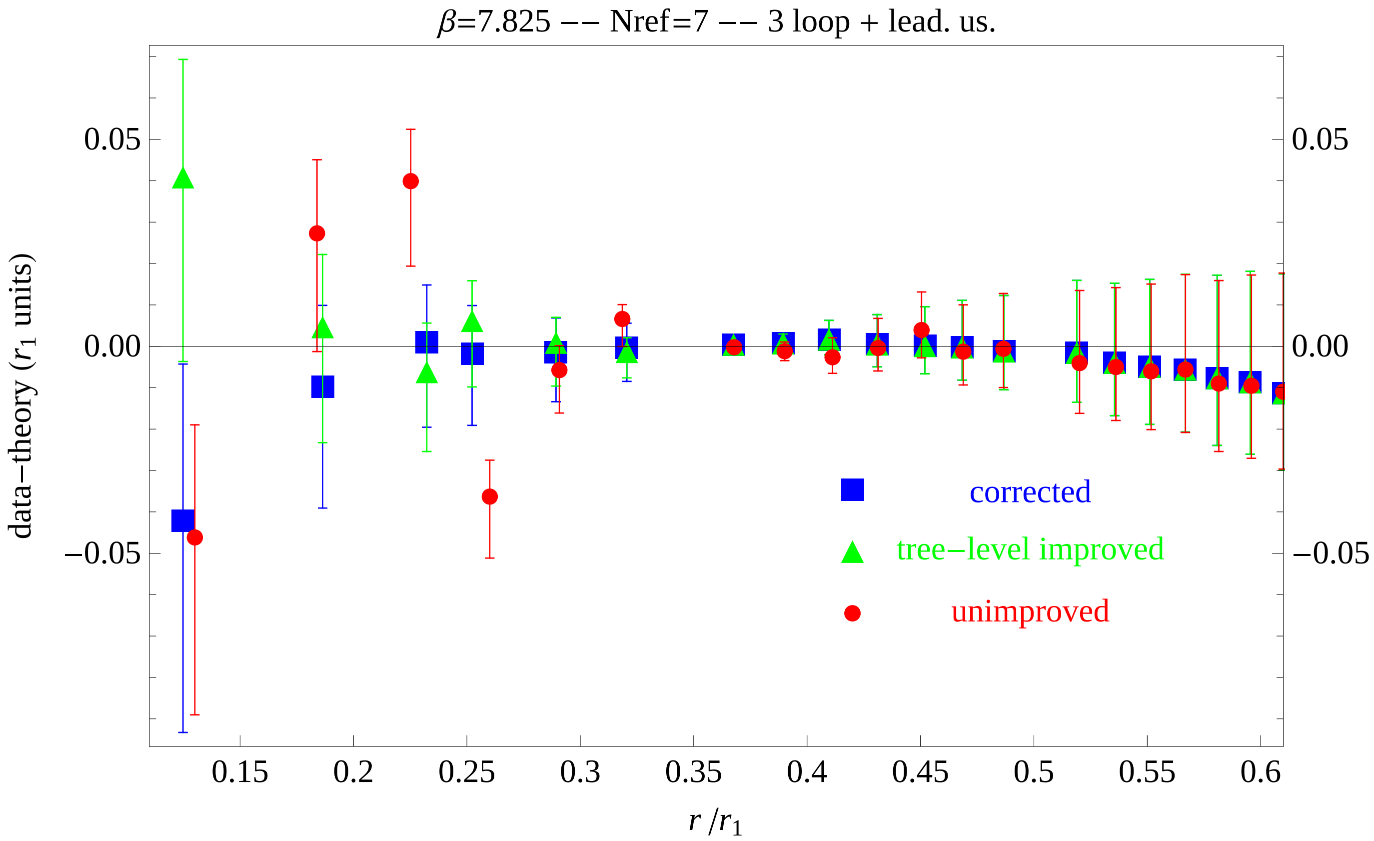
Correction factors for the static energy

β	$r/a=1$	$r/a=\sqrt{2}$	$r/a=\sqrt{3}$	$r/a=2$	$r/a=\sqrt{5}$	$r/a=\sqrt{6}$
7.150	0.980	0.995	1.007	0.988	1.000	1.010
7.280	0.980	0.997	1.008	0.992	1.000	1.013
7.373	0.980	0.998	1.009	0.994	0.995	1.005
7.596	0.980	0.995	1.005	0.994	1.000	1.001
7.825	0.968	0.992	1.005	0.994	0.998	1.001

Size of ultrasoft effects




β=7.825 -- Nref=7
corrected
tree-level improved
unimproved
r_1E_0+const.
r/r_1
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6


β=7.825 –– Nref=7 –– 3 loop + lead. us.
data−theory (r_1 units)
r/r_1
0.05
0.00
−0.05
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
corrected
tree−level improved
unimproved