α_S from Pion Decay Constant using Renormalization Group Optimized Perturbation

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(JLK, A. Neveu)

Workshop on High precision α_S measurements:

from LHC to FCC-ee, CERN, Oct. 12 2015

1. Introduction/Motivations
2. (Variationally) Optimized Perturbation (OPT)
3. Renormalization Group improvement of OPT (RGOPT)
4. “State-of-the art” $F_\pi/\Lambda^{QCD}_{\overline{\text{MS}}}$ determination and α_S
5. Summary and (preliminary) Workshop points

1. Introduction/Motivations

Goal: Peculiar resummations of perturbative expansions can give approximations to some nonperturbative parameters

Input: $F_\pi = 92.2 \pm .03 \pm .14$ ($\pi \to l\bar{\nu}(\gamma)$ decays) (PDG)

In a nutshell: estimate $F_\pi(m_q = 0)/\Lambda^{\rm QCD}_{\overline{\rm MS}}$ 'nonperturbatively', then $\Lambda^{n_f=3}_{\overline{\rm MS}} \to \alpha_S^{\overline{\rm MS}}$ (standard perturbative RG evolution).

How?: start from *perturbative* $F_\pi^2 \simeq m_q^2 \sum_{n,p} (\alpha_S)^n f_{np}(\ln m_q)$ (known at present to 4-loop order for any n_f)

Now $m_{quark} \to m$ *variational* mass (in a well-defined way), optimized consistently with RG properties$\equiv \rm RG(OPT)$.

$\Rightarrow m = \mathcal{O}(\Lambda^{QCD}) \Rightarrow F_\pi^{m_q=0}/\Lambda^{n_f=3}_{\overline{\rm MS}} \simeq 0.25 \pm .01 \to \alpha_S(m_Z) \simeq 0.1174 \pm .001 \pm .001$

●NB recently RGOPT applied to $\langle\bar{q}q\rangle$ at 3,4 -loops (using spectral density of Dirac operator) gives $\langle\bar{q}q\rangle^{1/3}_{m_q=0}(2\,\rm GeV) \simeq -(0.84 \pm 0.01)\Lambda_{\overline{\rm MS}}$ (JLK, A.Neveu 1506.07506)

Chiral Symmetry Breaking (χSB) Order parameters

Conventional wisdom: hopeless from standard perturbation:

1. $\langle \bar{q}q \rangle^{1/3}$, F_π,.. $\sim \mathcal{O}(\Lambda_{QCD}) \simeq$ 300 MeV
$\rightarrow \alpha_S$ (a priori) large $\rightarrow$ invalidates pert. expansion

2. $\langle \bar{q}q \rangle$, F_π,.. perturbative series $\sim (m_q)^d \sum_{n,p} \alpha_s^n \ln^p(m_q)$
vanish for $m_q \rightarrow 0$ at any pert. order (trivial chiral limit)

3. More sophisticated arguments e.g. (infrared) renormalons (factorially divergent pert. coeff. at large orders)

All seems to tell that χSB parameters are intrinsically NP

•Optimized pert. (OPT): circumvents at least 1., 2.,
and may give more clues to pert./NP bridge

2. (Variationally) Optimized Perturbation (OPT)

Trick: add and subtract a mass, consider $m\,\delta$ as interaction:

$$\mathcal{L}_{QCD}(g, m_q) \to \mathcal{L}_{QCD}(\delta\, g, m(1-\delta)) \quad (\alpha_S \equiv g/(4\pi))$$

$0 < \delta < 1$ interpolates between $\mathcal{L}_{free}$ and *massless* $\mathcal{L}_{int}$;
(quark) mass $m_q \to m$: arbitrary trial parameter

- Take any standard (renormalized) QCD pert. series, expand in δ *after:*

$$m_q \to m\,(1-\delta); \quad \alpha_S \to \delta\,\alpha_S$$

then take $\delta \to 1$ (to recover original massless theory):

BUT a m-dependence remains at any finite δ^k-order:
fixed typically by optimization (OPT):

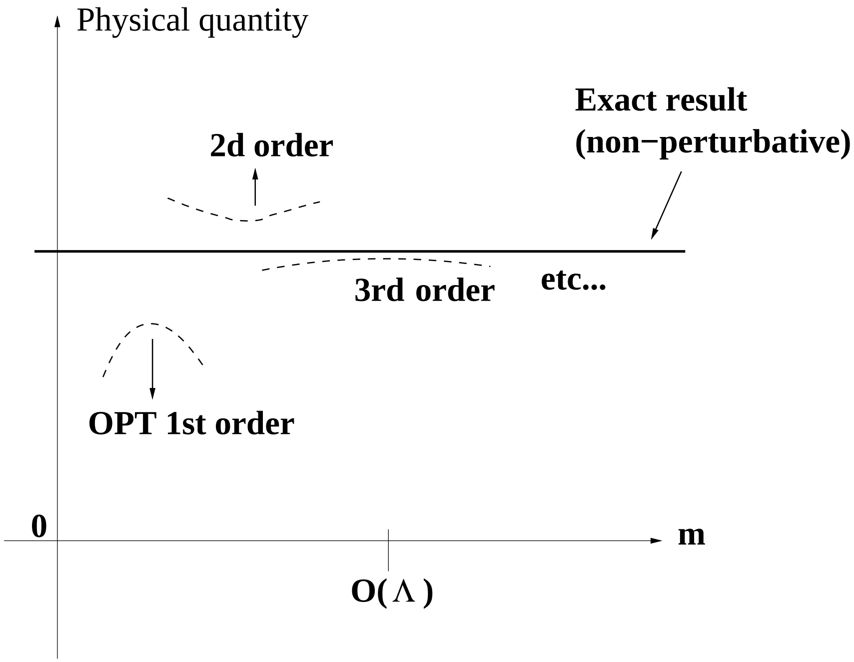
$$\frac{\partial}{\partial m}(\text{physical quantity}) = 0 \text{ for } m = \tilde{m}_{opt}(\alpha_S) \neq 0$$

Exhibit *dimensional transmutation*: $\tilde{m}_{opt} \sim \mu e^{-1/(\beta_0 \alpha_S)}$

But does this 'cheap trick' always work? and why?

Expected behaviour (Ideally...)

Expect *flatter* m-dependence at increasing δ orders...



But not quite what happens.. except for $\phi^4(D=1)$ (oscillator)

Higher orders: → what about convergence?

Main pb at higher order: OPT: $\partial_m(...)=0$ has multi-solutions (some complex!), how to choose right one??

Simpler model's support + properties

•Convergence proof of this procedure for $D = 1\ \lambda\phi^4$ oscillator (cancels large pert. order factorial divergences!) Guida et al '95

particular case of 'order-dependent mapping' Seznec+Zinn-Justin '79 (exponentially fast convergence for ground state energy $E_0 = const.\lambda^{1/3}$; good to % level at second δ-order)

•In renormalizable QFT, also produces factorial damping at large pert. orders (JLK, Reynaud '2002)

•Flexible, Renormalization-compatible, gauge-invariant: applications also at finite temperature (many variants: 'screened pert.', 'hard thermal loop resummation', ...)

(NB recently our RG(OPT) variant improves well-known problem of unstable thermal perturbation (JLK + M.B.Pinto 1507.03508; 1508.02610))

3. RG improved OPT (RGOPT)

Our main new ingredient (JLK, A. Neveu 2010):

Consider a physical quantity (perturbatively RG invariant), e.g. pole mass M (or here will be F_π):
in addition to OPT Eq: $\frac{\partial}{\partial m} M^{(k)}(m, g, \delta = 1)|_{m \equiv \tilde{m}} \equiv 0$

Require (δ-modified!) series at order δ^k to satisfy a standard perturbative Renormalization Group (RG) equation:

$$\mathrm{RG}\left(M^{(k)}(m, g, \delta = 1)\right) = 0$$

with standard RG operator: ($g = 4\pi\alpha_S$)

$$\mathrm{RG} \equiv \mu \frac{d}{d\,\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_m(g)\, m \frac{\partial}{\partial m}$$

$$\beta(g) \equiv -2b_0 g^2 - 2b_1 g^3 + \cdots\,, \quad \gamma_m(g) \equiv \gamma_0 g + \gamma_1 g^2 + \cdots$$

$\to$ Combined with OPT, RG Eq. takes a reduced form:

$$[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}]M^{(k)}(m,g,\delta=1)=0$$

Note: OPT+RG completely fix $m \equiv \tilde{m}$ and $g \equiv \tilde{g}$

- But $\Lambda_{\overline{\mathrm{MS}}}(g)$ satisfies by def. $[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}]\,\Lambda_{\overline{\mathrm{MS}}}\equiv 0$ consistently at a given pert. order for $\beta(g)$.

Thus equivalent to:

$$\frac{\partial}{\partial\, m}\left(\frac{M^k(m,g,\delta=1)}{\Lambda_{\overline{\mathrm{MS}}}(g)}\right)=0\ ;\quad \frac{\partial}{\partial\, g}\left(\frac{M^k(m,g,\delta=1)}{\Lambda_{\overline{\mathrm{MS}}}(g)}\right)=0 \text{ for } \tilde{m},\tilde{g}$$

- Sort of “virtual” (variational) fixed point (but $\beta(g)\neq 0$!)
- Optimal $\tilde{m},\tilde{g}=4\pi\tilde{\alpha}_S$ unphysical: true α_S from $\frac{F_\pi}{\Lambda_{\overline{\mathrm{MS}}}}(\tilde{m},\tilde{g})$
- Reproduces at first order exact nonpert results in simpler (e.g. Gross-Neveu) models

OPT + RG = RGOPT main new features

- Embarrassing freedom in interpolating Lagrangian, e.g.:

$$m \to m\,(1-\delta)^{a}$$

In most previous works: linear case $a = 1$ for 'simplicity'... but spoils RG invariance...

[exceptions: Bose-Einstein Condensate T_c shift, calculated from $O(2)\lambda\phi^4$, *requires* $a \neq 1$: gives real solutions +related to critical exponents (Kleinert,Kastening; JLK,Neveu,Pinto '04)

- OPT,RG Eqs: many solutions at increasing δ^k-orders

$\to$ Our approach restores RG +requires OPT, RG sol. to match standard perturbation (i.e. Asymptotic Freedom in QCD): $\alpha_S \to 0, \mu \to \infty$: $\tilde{g} = 4\pi\tilde{\alpha}_S \sim \frac{1}{2b_0 \ln \frac{\mu}{\tilde{m}}} + \cdots$

$\to$ At arbitrary order, AF-compatible RG + OPT branch, often unique, *only* appear for a critical universal a:

$$m \to m\,(1-\delta)^{\frac{\gamma_0}{2b_0}}; \qquad \text{(e.g. } \frac{\gamma_0}{2b_0}(\mathrm{QCD}, n_f = 3) = \frac{4}{9}\text{)}$$

$\to$ It removes spurious solutions incompatible with AF

4. Application: Pion decay constant F_π/Λ

Chiral Symmetry Breaking (CSB) $SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_{L+R}$
for n_f massless quarks. ($n_f = 2$, $n_f = 3$)

F_π given from (nonperturbative) definition at $p^2 \to 0$:

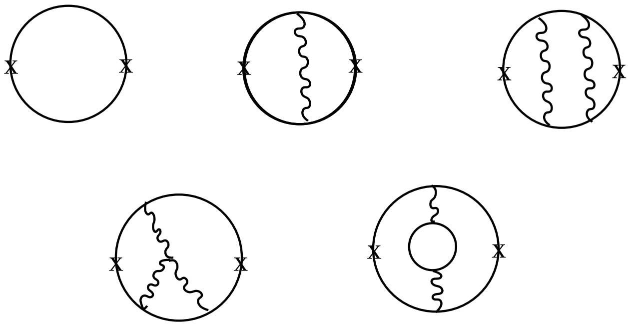
$$i\langle 0|T A^i_\mu(p) A^j_\nu(0)|0\rangle \equiv \delta^{ij} g_{\mu\nu} F^2_\pi + \mathcal{O}(p_\mu p_\nu)$$

where quark axial current: $A^i_\mu \equiv \bar{q}\gamma_\mu\gamma_5 \frac{\tau_i}{2}\, q$

$F_\pi \neq 0$: main (lowest order) CSB order parameter

$m_q \neq 0$: perturbative expansion known to 3,4 loops

(3-loop Chetyrkin et al ’95; 4-loop Maier et al ’08 ’09, +Maier, Marquard private comm.)



(Standard) perturbative available information

$$F_\pi^2(pert)_{\overline{\text{MS}}} = N_c \frac{m^2}{2\pi^2}\left[-L + \frac{\alpha_S}{4\pi}(8L^2 + \frac{4}{3}L + \frac{1}{6}) \right.$$
$$\left. +(\frac{\alpha_S}{4\pi})^2[f_{30}(n_f)L^3 + f_{31}(n_f)L + f_{32}(n_f)L + f_{33}(n_f)] + \mathcal{O}(\alpha_S^3)\right]$$

$L \equiv \ln \frac{m}{\mu}$, $n_f = 2(3)$

Note: finite part (after mass + coupling renormalization) not separately RG-inv: (i.e. $F_\pi^2 \sim \langle 0|T A^\mu A^\nu|0\rangle$ mixes with $m^2\, 1$ operator)

$\rightarrow$ (extra) renormalization by subtraction of the form: $S(m,\alpha_S) = m^2(s_0/\alpha_S + s_1 + s_2\alpha_S + ...)$ where s_i fixed requiring RG-inv order by order: $s_0 = \frac{3}{16\pi^3(b_0-\gamma_0)}$, $s_1 = ...$

Same well-known feature for $m\,\langle \bar{q}q\rangle$, related to vacuum energy, needs an extra (additive) renormalization in $\overline{\text{MS}}$-scheme to be RG invariant.

Warm-up calculation: pure RG approximation

2-loop + neglecting non-RG (non-logarithmic) terms:

$$F_\pi^2(\text{RG-1}, \mathcal{O}(g)) = 3\,\frac{m^2}{2\pi^2}\left[-L + \frac{\alpha_S}{4\pi}(8L^2 + \tfrac{4}{3}L) - \left(\frac{1}{8\pi(b_0-\gamma_0)\,\alpha_S} - \frac{5}{12}\right)\right]$$

$$\to F_\pi^2(m \to m(1-\delta)^{\gamma_0/(2b_0)}, \alpha_S \to \delta\alpha_S, \mathcal{O}(\delta))|_{\delta\to 1} =$$
$$3\,\frac{m^2}{2\pi^2}\left[-\frac{102\pi}{841\,\alpha_S} + \frac{169}{348} - \frac{5}{29}L + \frac{\alpha_S}{4\pi}(8L^2 + \tfrac{4}{3}L)\right]$$

OPT+RG: $\partial_m(F_\pi^2/\Lambda^2_{\overline{\rm MS}}), \partial_{\alpha_S}(F_\pi^2/\Lambda^2_{\overline{\rm MS}}) \equiv 0$: have a unique AF-compatible real solution: $\tilde{L} \equiv \ln\frac{\tilde{m}}{\mu} = -\frac{\gamma_0}{2b_0}\ ; \quad \tilde{\alpha}_S = \frac{\pi}{2}$

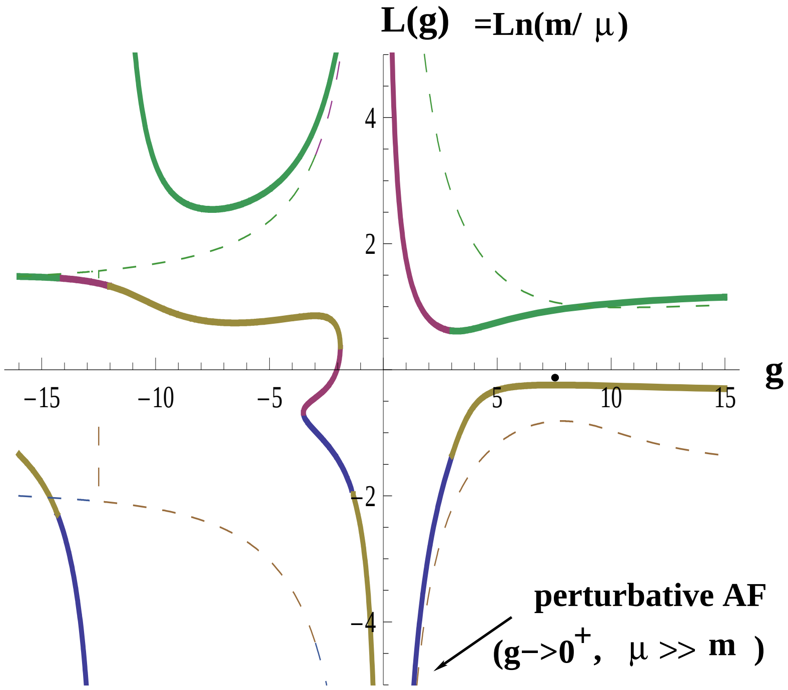
$\to F_\pi(\tilde{m}, \tilde{\alpha}_S) = (\frac{5}{8\pi^2})^{1/2}\tilde{m} \simeq 0.25\Lambda_{\overline{\rm MS}}$ (for $\Lambda^{1-loop}_{\overline{\rm MS}} = \mu\, e^{-1/(\beta_0\alpha_S)}$)

•Includes higher orders +non-RG terms: $\tilde{m}_{opt}$ remains $\mathcal{O}(\Lambda_{\overline{\rm MS}})$ (rather than $m \sim 0$): RG-consistent 'mass gap',

And OPT *stabilizes* $\alpha_S^{opt} \simeq .5$: more perturbative values

NB $\tilde{m}, \tilde{\alpha}_S$ variational parameters (not directly physical)

Exact F_π RG+OPT solutions at 4-loops ($\overline{\rm MS}$)



All branches of RG (thick) and OPT(dashed) solutions $Re[L \equiv \ln \frac{m}{\mu}(g)]$ to the δ-modified 3rd order (4-loop) perturbation ($g = 4\pi\alpha_S$). Unique AF compatible sol.: black dot

- However beyond lowest order, AF-compatibility and reality of solutions often incompatible...

But, complex solutions are artefacts of solving *exactly* the RG and OPT (polynomial in L) Eqs, in $\overline{\rm MS}$-scheme...

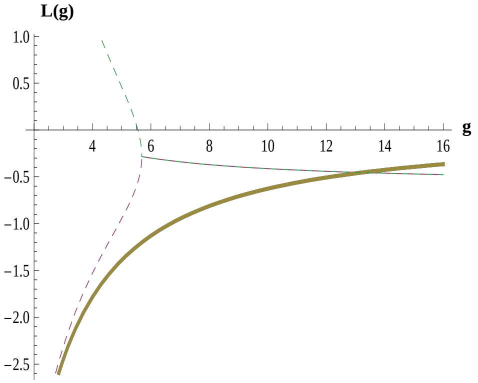
Recovering real AF-compatible solutions

Are there *perturbative* 'deformations' consistent with RG?:

Evidently: Renormalization scheme changes (RSC)

$$m \to m'(1 + B_1 g' + B_2 g'^2 + \cdots),\ g \to g'(1 + A_1 g' + A_2 g'^2 + \cdots)$$

$\mathcal{O}(\delta)$, $\overline{\text{MS}}$:



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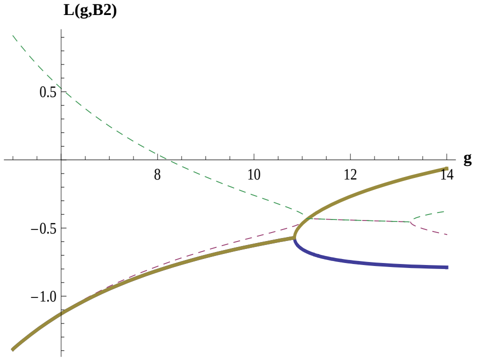

→ We require *contact* solution (thus closest to $\overline{\text{MS}}$):

$$\frac{\partial}{\partial g}\text{RG}(g, L, B_i)\frac{\partial}{\partial L}\text{OPT}(g, L, B_i) - \frac{\partial}{\partial L}\text{RG}\,\frac{\partial}{\partial g}\text{OPT} \equiv 0$$

RSC affects pert. coefficients, but with property:

$$F_\pi^{\overline{MS}}(\overline{m}, g; \overline{f}_{ij}) = F'_\pi(m', g'; f'_{ij}(B_i)) + g^{k+1}\text{remnant}(B_i)$$

→ differences *should* decrease with perturbative order

Results with theoretical uncertainties

Beside recovering real solution, RSC offer reasonably convincing uncertainty estimates: non-unique RSC
$\to$ we take differences between those as th. uncertainties

Table 1: Main optimized results at successive orders ($n_f = 3$)

δ^k order	nearest-to-$\overline{\text{MS}}$ RSC $\tilde{B}_i$	$\tilde{L}'$	$\tilde{\alpha}_S$	$\frac{F_0}{\overline{\Lambda}_{4l}}$ (RSC uncertainties)
δ, RG-2l	$\tilde{B}_2 = 2.38\,10^{-4}$	-0.523	0.757	$0.27 - 0.34$
δ^2, RG-3l	$\tilde{B}_3 = 3.39\,10^{-5}$	-1.368	0.507	$0.236 - 0.255$
δ^3, RG-4l	$\tilde{B}_4 = 1.51\,10^{-5}$	-1.760	0.374	$0.2409 - 0.2546$

$n_f = 2$: $\frac{F}{\overline{\Lambda}}(\delta^2) = 0.213 - 0.269$ ($\tilde{\alpha}_S = 0.46 - 0.64$)
$\frac{F}{\overline{\Lambda}}(\delta^3) = 0.2224 - 0.2495$ ($\tilde{\alpha}_S = 0.35 - 0.42$)

•Empirical stability/convergence exhibited, with
$2b_0\tilde{g}\ln(\tilde{m}/\mu) \simeq 1$ i.e. $\tilde{m}_{opt} \simeq \mu\, e^{-1/(2b_0\tilde{g})}$ (like first RG order)

More realistic: explicit symmetry breaking

- Need to "subtract" effect from explicit chiral symmetry breaking from genuine quark masses $m_u, m_d, m_s \neq 0$:
This relies at this stage on other (mainly lattice) results:

$\frac{F_\pi}{F} \sim 1.073 \pm 0.015$ [robust, $n_f = 2$ ChPT + lattice]

$\frac{F_\pi}{F_0} \sim 1.172(3)(43)$ (lattice MILC collaboration '10 using NNLO ChPT fits)

But quite different values by other collaborations

+ hint of slower convergence of $n_f = 3$ ChPT, e.g. Bernard, Descotes-Genon, Toucan '10

Alternative: implement explicit sym. break. within OPT

(to be less dependent of lattice/ChPT results):

$m \to m^{true}_{u,d,s} + m(1-\delta)^{\gamma_0/(2b_0)}$: promising but involved RG+OPT

Eqs. solving work in progress...,

+ missing full m_s dependence at 3- and 4-loop order)

Combined results with theoretical uncertainties:

Average different RSC +average δ^2 and δ^3 results:

$\overline{\Lambda}^{n_f=2}_{4-loop} \simeq 359^{+38}_{-26}|_{(\text{rgopt th})} \pm 5|_{(F_\pi/F)}$ MeV

$\overline{\Lambda}^{n_f=3}_{4-loop} \simeq 317^{+14}_{-7}|_{(\text{rgopt th})} \pm 13|_{(F_\pi/F_0)}$ MeV

To be compared with some recent lattice results, e.g.:

•'Schrödinger functional scheme' (ALPHA coll. Della Morte et al '12):

$\Lambda_{\overline{\text{MS}}}(n_f = 2) = 310 \pm 30$ MeV

•Twisted fermions (+NP power corrections) (Blossier et al '10):

$\Lambda_{\overline{\text{MS}}}(n_f = 2) = 330 \pm 23 \pm 22_{-33}$ MeV

•static potential (Karbstein et al '14): $\Lambda_{\overline{\text{MS}}}(n_f = 2) = 331 \pm 21$ MeV

Extrapolation to α_S at high (perturbative) q^2

Use only $\Lambda_{\overline{\text{MS}}}^{n_f=3}$ result, perform standard (perturbative 4-loop) evolution

$\Lambda_{\overline{\text{MS}}} \ll m_{charm} \ll m_{bottom} \cdots$

•In $\overline{\text{MS}}$-scheme non-trivial decoupling/matching: standard perturbative extrapolation

(3,4-loop with m_c, m_b thresholds, Chetyrkin et al '06):

$$\alpha_S^{n_f+1}(\mu) = \alpha_S^{n_f}(\mu)\left(1 - \frac{11}{72}\left(\frac{\alpha_S}{\pi}\right)^2 + (-0.972057 + .0846515 n_f)\left(\frac{\alpha_S}{\pi}\right)^3\right)$$

$$\rightarrow \overline{\alpha}_S(m_Z) = 0.1174^{+.0010}_{-.0005}(\text{rgopt th}) \pm .0010|_{(F_\pi/F_0)} \pm .0005_{evol}$$

$$\overline{\alpha}_S^{n_f=3}(m_\tau) = 0.308^{+.007}_{-.004} \pm .007 \pm .002_{evol}$$

Compare to 2014 world average:

$$\alpha_S(m_Z) = 0.1185 \pm 0.0006$$

5. Summary and Outlook

•OPT gives a simple procedure to resum perturbative expansions, using only perturbative information.

•Our RGOPT version includes 2 major differences w.r.t. most previous OPT approaches:

1) OPT+ RG optimization fix $\tilde{m}$ *and* $\tilde{g} = 4\pi\tilde{\alpha}_S$

2) Requiring RG invariance or AF-compatible solutions after interpolation uniquely fixes the latter $m \to m(1-\delta)^{\gamma_0/(2b_0)}$: discards spurious solutions and accelerates convergence.

($\mathcal{O}(10\%)$ accuracy at 1-2-loops, empirical stability shown at 3-loop)

(Preliminary) Workshop tasks

Our latest estimate:
$\alpha_S(m_Z) = 0.1174^{+.0010}_{-.0005}(\mathrm{rgopt\ th}) \pm .0010|_{(F_\pi/F_0)} \pm .0005_{evol}$

NB not trivial to combine properly our RGOPT *th errors* with Lattice (stat.+ syst.) uncertainties; and implement all this in world average

-Size of current exp/th uncertainties: missing higher orders, expected uncertainty in 10 years: new th developments, etc

•Direct F_π uncertainties ($\pi \to l\bar{\nu}(\gamma)$ decays): $F_\pi = 92.2 \pm .03(V_{ud}) \pm .14(th)$ (PDG)
almost negligible (at present) relative to other errors

•RGOPT error: here rather conservative: average of 3- and 4-loop results.
If only 4-loop: $\alpha_S(m_Z) = 0.1174 \pm .0008(\mathrm{rgopt\ th}) + \cdots$

•Possible improvements: going to 5-loops? recent works (e.g. Karlsruhe group) on this [two-point correlators have many other interests (Lattice, sum rules,...].
No doubts will be done before 10 years.

Likewise perturbative evolution error might reduce a little.

●Before such full 5-loop results, we could include 5-loop LL, NLL,NNLL,..approximations (available from RG properties).

●Most limiting at present: Lattice F_π/F_0 uncertainties: (apparently) no more precise lattice results recently (true chiral limit difficult) but progress surely soon.

●Alternative: implement explicit quark masses in our framework: IF works, $F_\pi/F_0 \sim 5\%$ uncertainty replaced by $\delta m_s \sim 2.7\%$ (Lattice FLAG 2014, also QCD sum rules)

●Expected improvements by the FCC-ee:

no idea in which way could influence our results

Backslides: Pre-QCD guidance: Gross-Neveu model

•$D = 2\ O(2N)$ GN model shares many properties with QCD (asymptotic freedom, (discrete) chiral sym., mass gap,..)

$$\mathcal{L}_{GN} = \bar{\Psi} i \not{\partial} \Psi + \frac{g_0}{2N} (\sum_1^N \bar{\Psi}\Psi)^2 \text{ (\textit{massless})}$$

Standard mass-gap (massless, large N approx.):

consider $V_{eff}(\sigma)$, $\sigma \sim \bar{\Psi}\Psi$;

$$\sigma \equiv M = \mu e^{-\frac{2\pi}{g}} \equiv \Lambda_{\overline{\text{MS}}}$$

•Mass gap known exactly for any N:

$$\frac{M_{exact}(N)}{\Lambda_{\overline{\text{MS}}}} = \frac{(4e)^{\frac{1}{2N-2}}}{\Gamma[1 - \frac{1}{2N-2}]}$$

(From $D = 2$ integrability: Bethe Ansatz) Forgacs et al ’91

massive GN model

Now consider *massive* case (still large N):

$M(m,g) \equiv m(1+g\ln\frac{M}{\mu})^{-1}$: Resummed mass ($g/(2\pi) \to g$)

$= m(1-g\ln\frac{m}{\mu}+g^2(\ln\frac{m}{\mu}+\ln^2\frac{m}{\mu})+\cdots)$ (pert. re-expanded)

• Only fully summed $M(m,g)$ gives right result, upon:

-identify $\Lambda \equiv \mu e^{-1/g}$; $\to M(m,g) = \frac{m}{g\ln\frac{M}{\Lambda}} \equiv \frac{\hat{m}}{\ln\frac{M}{\Lambda}}$;

-take *reciprocal*: $\hat{m}(F \equiv \ln\frac{M}{\Lambda}) = F\, e^F\, \Lambda \sim F$ for $\hat{m} \to 0$;

$$\to M(\hat{m}\to 0) \sim \frac{\hat{m}}{\hat{m}/\Lambda+\mathcal{O}(\hat{m}^2)} = \Lambda_{\overline{\text{MS}}}$$

never seen in standard perturbation: $M_{pert}(m \to 0) \to 0$

•But (RG)OPT gives $M = \Lambda_{\overline{\text{MS}}}$ at *first* (and any) δ-order

(at any order, OPT sol.: $\ln\frac{m}{\mu} = -\frac{1}{g}$, RG sol.: $g = 1$)

•At δ^2-order (2-loop), RGOPT $\sim 1-2\%$ from M_{exact}(any$N$)