α_s Determination from Hadronic τ Decays

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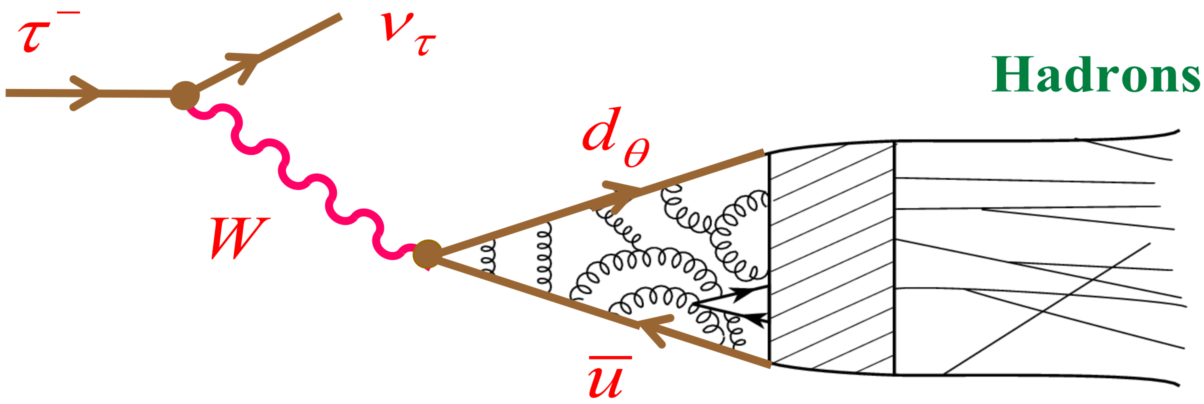
IFIC, Valencia

Workshop on high-precision alpha_s measurements: from LHC to FCC-ee
CERN, 12-13 October 2015

HADRONIC TAU DECAY



$$d_\theta = V_{ud}\, d + V_{us}\, s$$

Only lepton massive enough to decay into hadrons

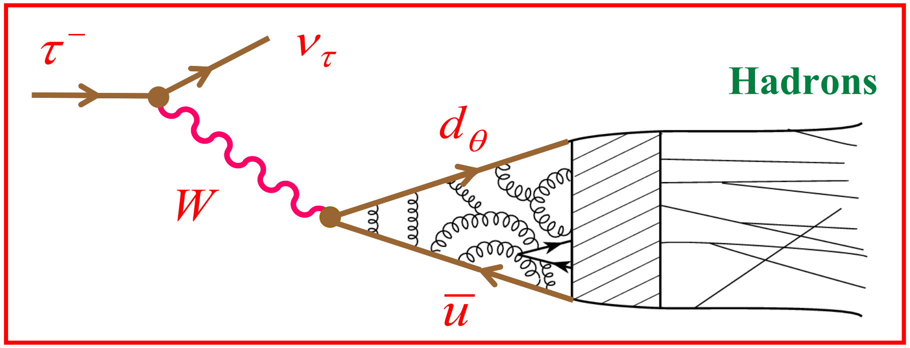
$$R_\tau \equiv \frac{\Gamma\left(\tau^- \to \nu_\tau + \text{Hadrons}\right)}{\Gamma\left(\tau^- \to \nu_\tau\, e^- \,\overline{\nu}_e\right)} \approx N_C \qquad ; \qquad R_\tau = \frac{1 - B_e - B_\mu}{B_e} = 3.637 \pm 0.011$$

$$R_\tau = \frac{1}{B_e^{\text{univ}}} - 1.97256 = 3.6410 \pm 0.0073 \qquad ; \qquad R_\tau = \frac{\text{Br}\left(\tau^- \to \nu_\tau + \text{Hadrons}\right)}{B_e^{\text{univ}}} = 3.6315 \pm 0.0081$$

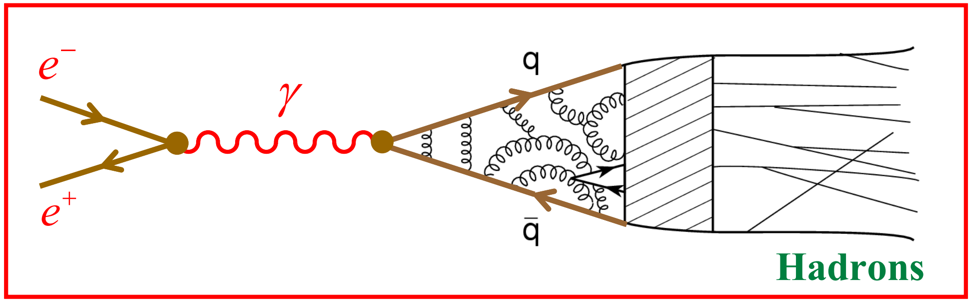
Only Lepton Massive Enough to Decay into Hadrons

$\tau^- \to \nu_\tau H^-$ **probes the hadronic V−A current**

$$\langle H^- | \bar{d}_\theta \gamma^\mu (1-\gamma_5) u | 0 \rangle$$



$e^+e^- \to H^0$ **probes the hadronic electromagnetic current**

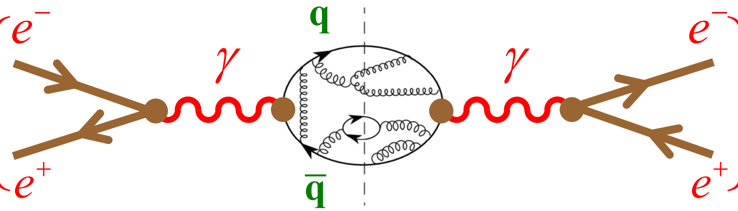


$$\langle H^0 | \sum_q Q_q \, \bar{q} \gamma^\mu q | 0 \rangle$$

Isospin:

$$\frac{\Gamma(\tau^- \to \nu_\tau V^-)}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)} = \frac{3\cos^2\theta_C}{2\pi\alpha^2} S_{\rm EW} \int_0^1 dx\, (1-x)^2 (1+2x)\, x\, \sigma^{I=1}_{e^+e^-\to V^0}(x m_\tau^2)$$

$$\sigma \sim \mathrm{Im}\left\{\right.$$

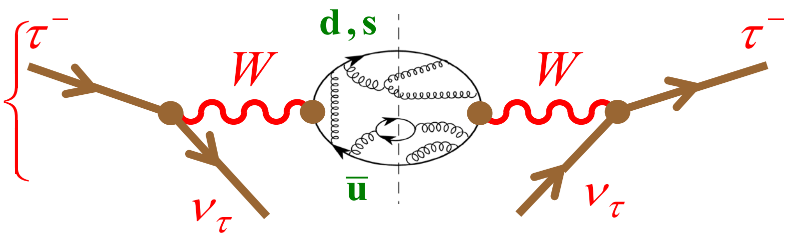


$$\left.\right\}$$

$$\frac{\sigma(e^+e^- \to \text{had})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi \, \mathrm{Im}\,\Pi_{\mathrm{em}}(s)$$

$$\Pi^{\mu\nu}_{\mathrm{em}}(q) \equiv i \int d^4x \; e^{iqx} \left\langle 0 \right| T[J^{\mu}_{\mathrm{em}}(x) J^{\nu}_{\mathrm{em}}(0)] \left| 0 \right\rangle = \left(-g^{\mu\nu} q^2 + q^{\mu} q^{\nu}\right) \Pi_{\mathrm{em}}(q^2)$$

$$\Gamma_{\tau \to \nu_\tau + \text{had}} \sim \mathrm{Im}\left\{\right.$$



$$\left.\right\}$$

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \nu_\tau + \text{had})}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)} = 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\,\Pi^{(1)}(s) + \mathrm{Im}\,\Pi^{(0)}(s)\right]$$

$$\Pi^{(J)}(s) \equiv \left|V_{ud}\right|^2 \left[\Pi^{(J)}_{ud,V}(s) + \Pi^{(J)}_{ud,A}(s)\right] + \left|V_{us}\right|^2 \left[\Pi^{(J)}_{us,V}(s) + \Pi^{(J)}_{us,A}(s)\right]$$

$$\Pi^{\mu\nu}_{ij,J}(q) \equiv i \int d^4x \; e^{iqx} \left\langle 0 \right| T[J^{\mu}_{ij}(x) J^{\nu}_{ij}(0)^\dagger] \left| 0 \right\rangle = \left(-g^{\mu\nu} q^2 + q^{\mu} q^{\nu}\right) \Pi^{(1)}_{ij,J}(q^2) + q^{\mu} q^{\nu} \, \Pi^{(0)}_{ij,J}(q^2)$$

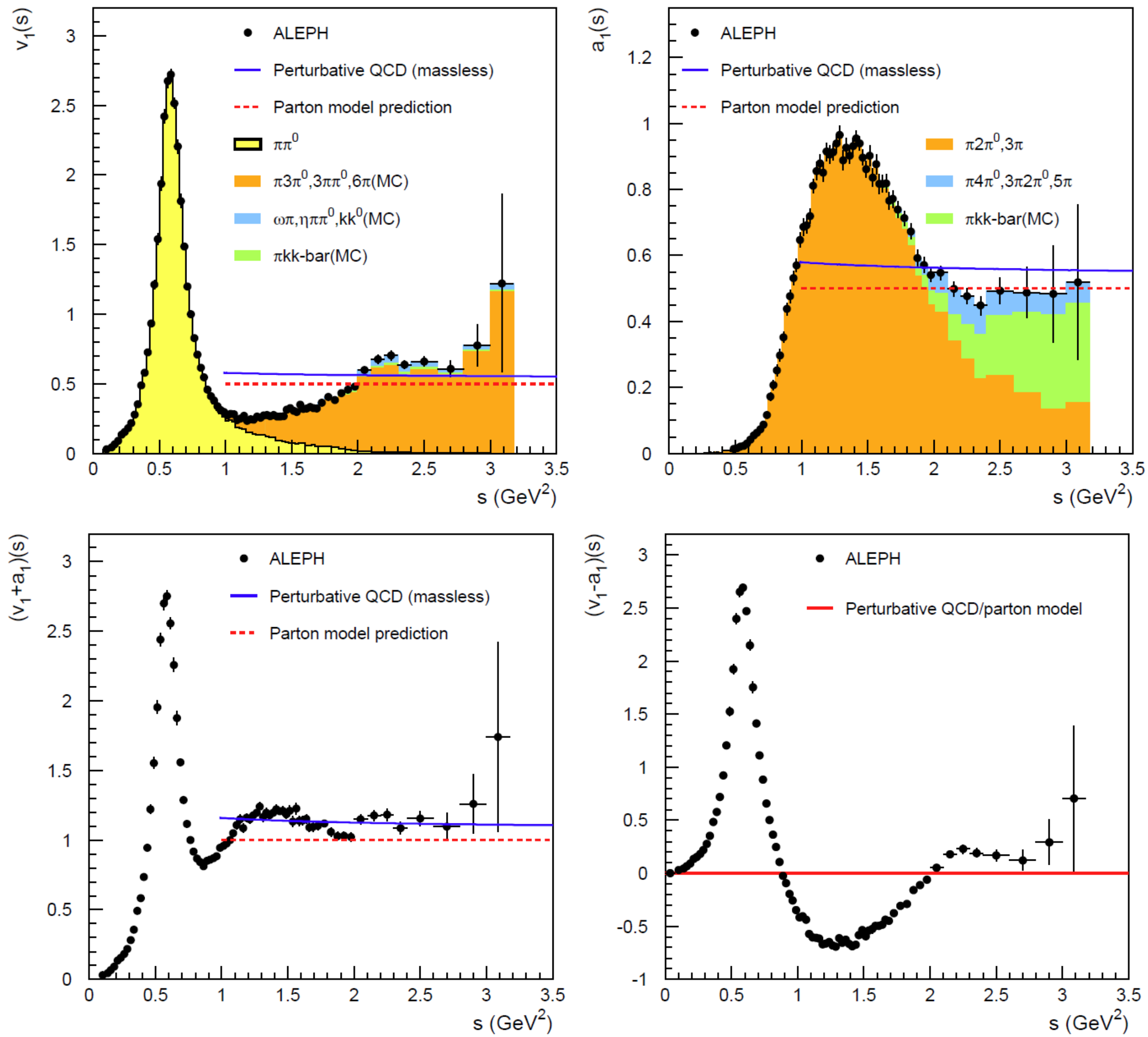
SPECTRAL FUNCTIONS

Davier et al, 1312.1501

$$v_1(s) = 2\pi \, \mathrm{Im}\, \Pi^{(0+1)}_{ud,V}(s)$$

$$a_1(s) = 2\pi \, \mathrm{Im}\, \Pi^{(0+1)}_{ud,A}(s)$$

Better data needed

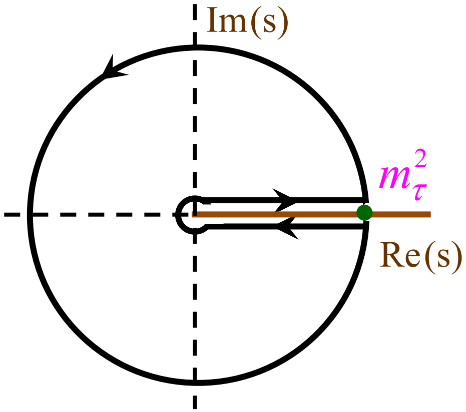


QCD Prediction of R_τ

Braaten-Narison-Pich'92

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \nu_\tau + \text{had})}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)} = 12\pi \int_0^1 dx\,(1-x)^2 \left[(1+2x)\,\mathrm{Im}\,\Pi^{(1)}(x m_\tau^2) + \mathrm{Im}\,\Pi^{(0)}(x m_\tau^2)\right]$$

$x \equiv s/m_\tau^2$



$$R_\tau = 6\pi i \oint_{|x|=1} dx\,(1-x)^2 \left[(1+2x)\,\Pi^{(0+1)}(x m_\tau^2) - 2x\,\Pi^{(0)}(x m_\tau^2)\right]$$

$$\Pi^{(J)}(s) = \sum_{D=2n} \frac{C_D^{(J)}(s,\mu)\,\langle O_D(\mu)\rangle}{(-s)^{D/2}}$$

OPE

QCD Prediction of R_τ

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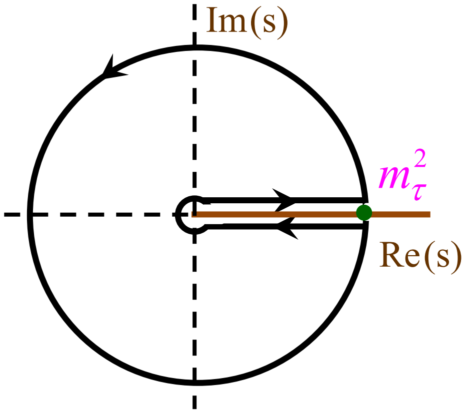

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OPE

$$R_\tau = N_C\, S_{\text{EW}} \left(1 + \delta_{\text{P}} + \delta_{\text{NP}}\right) = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

$S_{\text{EW}} = 1.0201\ (3)$; $\delta_{\text{NP}} = -0.0064 \pm 0.0013$

Marciano-Sirlin, Braaten-Li, Erler — Fitted from data (Davier et al)

$\delta_{\text{P}} = a_\tau + 5.20\, a_\tau^2 + 26\, a_\tau^3 + 127\, a_\tau^4 + \ldots \approx 20\%$; $a_\tau \equiv \alpha_s(m_\tau)/\pi$

Baikov-Chetyrkin-Kühn

Perturbative ($m_q=0$)

$$-s\,\frac{d}{ds}\,\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n \left(\frac{\alpha_s(-s)}{\pi}\right)^n$$

$K_0 = K_1 = 1$, $K_2 = 1.63982$, $K_3 = 6.37101$, $K_4 = 49.07570$ **Baikov-Chetyrkin-Kühn '08**

$$\delta_P = \sum_{n=1} K_n\, A^{(n)}(\alpha_s) = a_\tau + 5.20\, a_\tau^2 + 26\, a_\tau^3 + 127\, a_\tau^4 + \cdots$$

Le Diberder- Pich '92

$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-2x+2x^3-x^4)\left(\frac{\alpha_s(-s)}{\pi}\right)^n = a_\tau^n + \cdots \quad ; \quad a_\tau \equiv \alpha_s(m_\tau)/\pi$$

Perturbative $(m_q=0)$

$$-s\,\frac{d}{ds}\,\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n \left(\frac{\alpha_s(-s)}{\pi}\right)^n$$

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Baikov-Chetyrkin-Kühn '08

$$\Rightarrow \quad \delta_{\mathrm{P}} = \sum_{n=1} K_n\; A^{(n)}(\alpha_s) = a_\tau + 5.20\; a_\tau^2 + 26\; a_\tau^3 + 127\; a_\tau^4 + \cdots$$

Le Diberder- Pich '92

$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,(1-2x+2x^3-x^4)\left(\frac{\alpha_s(-s)}{\pi}\right)^n = a_\tau^n + \cdots \qquad ; \qquad a_\tau \equiv \alpha_s(m_\tau)/\pi$$

Power Corrections

$$\Pi^{(0+1)}_{\mathrm{OPE}}(s) \approx \frac{1}{4\pi^2}\sum_{n\geq 2}\frac{C_{2n}\langle O_{2n}\rangle}{(-s)^n}$$

Braaten-Narison-Pich '92

$$C_4\langle O_4\rangle \approx \frac{2\pi}{3}\langle 0|\alpha_s G^{\mu\nu}G_{\mu\nu}|0\rangle$$

$$\delta_{\mathrm{NP}} \approx \frac{-1}{2\pi i}\oint_{|x|=1} dx\;(1-3x^2+2x^3)\sum_{n\geq 2}\frac{C_{2n}\langle O_{2n}\rangle}{(-x m_\tau^2)^n} = -3\,\frac{C_6\langle O_6\rangle}{m_\tau^6} - 2\,\frac{C_8\langle O_8\rangle}{m_\tau^8}$$

Suppressed by m_τ^6 **[additional chiral suppression in** $C_6\langle O_6\rangle^{V+A}$ **]**

Perturbative Uncertainty on $\alpha_s(m_\tau)$

$$-s\,\frac{d}{ds}\,\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n\; a(-s)^n$$

$$\delta_P = \underbrace{\sum_{n=1} K_n\; A^{(n)}(\alpha_s)}_{\text{CIPT}} = \underbrace{\sum_{n=0} r_n\; a_\tau^n}_{\text{FOPT}}$$

$$r_n = K_n + g_n$$

$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,(1-2x+2x^3-x^4)\,a_\tau(-x m_\tau^2)^n = a_\tau^n + \cdots \qquad ; \qquad a_\tau \equiv \alpha_s(m_\tau)/\pi$$

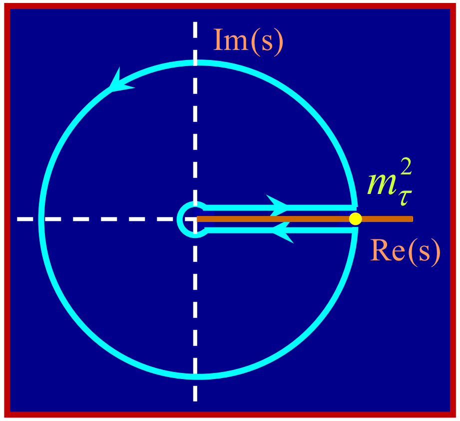
n	1	2	3	4	5
K_n	1	1.6398	6.3710	49.0757	
g_n	0	3.5625	19.9949	78.0029	307.78
r_n	1	5.2023	26.3659	127.079	

The dominant corrections come from the contour integration

Le Diberder- Pich 1992

Large running of α_s along the circle $s = m_\tau^2\, e^{i\varphi}$, $\varphi \in [0, 2\pi]$

$$A^{(n)}(a_\tau) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-2x+2x^3-x^4)\, a_\tau(-x m_\tau^2)^n = a_\tau^n + \cdots \quad ; \quad a_\tau \equiv \alpha_s(m_\tau)/\pi$$



$$A^{(1)}(a_\tau) = a_\tau - \frac{19}{24}\beta_1 a_\tau^2 + \left[\beta_1^2\left(\frac{265}{288} - \frac{\pi^2}{12}\right) - \frac{19}{24}\beta_2\right] a_\tau^3 + \cdots$$

$$a(-s) \simeq \frac{a_\tau}{1 - \frac{\beta_1}{2} a_\tau \log\left(-s/m_\tau^2\right)} = \frac{a_\tau}{1 - i\frac{\beta_1}{2} a_\tau \phi} = a_\tau \sum_n \left(i\frac{\beta_1}{2} a_\tau \phi\right)^n \quad ; \quad \phi \in [0, 2\pi]$$

FOPT expansion only convergent if $\alpha_\tau < 0.14$ (0.11) [at 1 (3) loops]

Experimentally $\alpha_\tau \approx 0.11$ ➡ **FOPT should not be used** (divergent series)

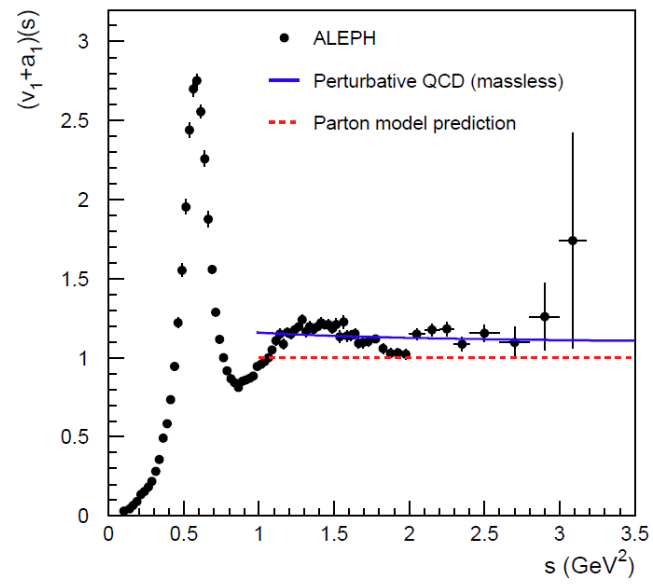
FOPT suffers a large renormalization-scale dependence **(Le Diberder- Pich , Menke)**

The difference between FOPT and CIPT grows at higher orders

Spectral Function Distribution

Moments:

$$R_\tau^{kl}(s_0) \equiv \int_0^{s_0} ds \left(1-\frac{s}{s_0}\right)^k \left(\frac{s}{m_\tau^2}\right)^l \frac{dR_\tau}{ds}$$

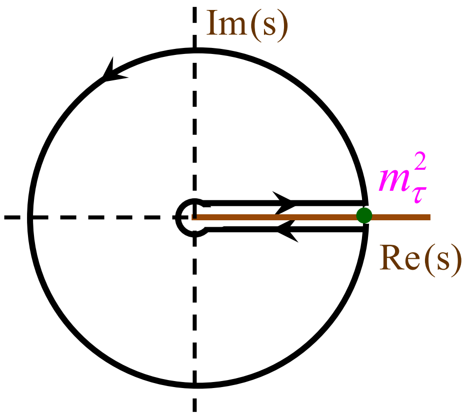


Sensitivity to power corrections (k,l)

The non-perturbative contribution to R_τ can be obtained from the invariant-mass distribution of the final hadrons

$$\delta_{NP} = -0.0064 \pm 0.0013$$ **Davier et al.** (ALEPH data)

R_τ suitable for a precise α_s determination



$$R_\tau = 6\pi i \oint_{|x|=1} dx\,(1-x)^2 \left[(1+2x)\,\Pi^{(0+1)}(x m_\tau^2) - 2x\,\Pi^{(0)}(x m_\tau^2) \right]$$

$$\Pi^{(J)}(s) = \sum_{D=2n} \frac{C_D^{(J)}(s,\mu)\,\langle O_D(\mu)\rangle}{(-s)^{D/2}}$$

OPE

- **m_τ large enough to safely use the OPE**
- **OPE only valid away from the real axis: $(1-x)^2$ pinched at $s = m_\tau^2$**
- **$m_{u,d} = 0$ ➡ $s\,\Pi^{(0)}(s) = 0$ ➡** $R_\tau = 6\pi i \oint_{|x|=1} dx\,(1-3x^2+2x^3)\,\Pi^{(0+1)}(x m_\tau^2)$

  ➡ $\delta_{NP} \sim 1/m_\tau^6$ Strong suppression of non-perturbative effects
- **D=6 OPE contributions have opposite sign for V & A. Cancellation**
- **δ_{NP} can be determined from data**

Recent $\alpha_s(m_\tau)$ Analyses

Reference	Method	δ_{NP}	δ_P	$\alpha_s(m_\tau)$	$\alpha_s(m_Z)$	
Baikov et al	CIPT, FOPT		0.1998 (43)	0.332 (16)	0.1202 (19)	
Davier et al'14	**CIPT, FOPT**	**– 0.0064 (13)**	**–**	**0.332 (12)**	**0.1199 (15)**	**ALEPH**
Beneke-Jamin	BSR + FOPT	– 0.007 (3)	0.2042 (50)	0.316 (06)	0.1180 (08)	
Maltman-Yavin	PWM + CIPT	+ 0.012 (18)	–	0.321 (13)	0.1187 (16)	
Menke	CIPT, FOPT		0.2042 (50)	0.342 (11)	0.1213 (12)	
Narison	CIPT, FOPT		–	0.324 (08)	0.1192 (10)	
Caprini-Fischer	BSR + CIPT		0.2037 (54)	0.322 (16)	–	
Abbas et al	IFOPT		0.2037 (54)	0.338 (10)		
Cvetič et al	β_{exp} + CIPT		0.2040 (40)	0.341 (08)	0.1211 (10)	
Boito et al	CIPT, DV	– 0.002 (12)	–	0.347 (25)	0.1216 (27)	OPAL
	FOPT, DV	– 0.004 (12)		0.325 (18)	0.1191 (22)	
	CIPT, DV	0.016 (10)	–	0.310 (14)	0.1174 (19)	ALEPH
	FOPT, DV	0.020 (9)		0.296 (10)	0.1155 (14)	
Pich'14	CIPT	– 0.0064 (13)	0.2009 (31)	0.341 (13)	0.1211 (14)	
	FOPT			0.319 (14)	0.1185 (17)	
Pich'14	**CIPT, FOPT**	**– 0.0064 (13)**	**0.2009 (31)**	**0.331 (13)**	**0.1200 (15)**	**R_τ**

CIPT: Contour-improved perturbation theory
FOPT: Fixed-order perturbation theory
BSR: Borel summation of renormalon series
IFOPT Improved FOPT
β_{exp}: Expansion in derivatives of α_s (β function)
PWM: Pinched-weight moments
CIPTm: Modified CIPT (conformal mapping)
DV: Duality violation (OPAL only)

Duality Violations

$$\int_0^{s_0} ds\;\; w(s)\,\mathrm{Im}\,\Pi(s) \;=\; \frac{i}{2}\oint_{|s|=s_0} ds\;\, w(s)\,\Pi(s) \;=\; \frac{i}{2}\oint_{|s|=s_0} ds\;\, w(s)\,\Pi(s)_{\mathrm{OPE}} \;+\; \delta_{\mathrm{DV}}$$

$$\delta_{\mathrm{DV}} = -\int_{s_0}^{\infty} ds\;\; w(s)\left(\mathrm{Im}\,\Pi(s) - \mathrm{Im}\,\Pi(s)_{\mathrm{OPE}}\right)$$

Catà et al., González-Alonso

- **δ_{DV} is negligible for R_τ : $w(s) \sim (s-s_0)^2$** — **Braaten-Narison-Pich '92**
- **More relevant for "non-pinched" weights w(s)** — **González-Alonso et al.**

Duality Violations

$$\int_0^{s_0} ds\;\; w(s)\,\mathrm{Im}\,\Pi(s) \;=\; \frac{i}{2}\oint_{|s|=s_0} ds\;\; w(s)\,\Pi(s) \;=\; \frac{i}{2}\oint_{|s|=s_0} ds\;\; w(s)\,\Pi(s)_{\mathrm{OPE}} \;+\; \delta_{\mathrm{DV}}$$

$$\delta_{\mathrm{DV}} = -\int_{s_0}^{\infty} ds\;\; w(s)\left(\mathrm{Im}\,\Pi(s) - \mathrm{Im}\,\Pi(s)_{\mathrm{OPE}}\right)$$

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Fitting the Spectral Function itself:

Boito et al.

- **Ansatz:** $\mathrm{Im}\,\Pi_{V,A}(s) = \kappa\, e^{-\gamma s} \sin[\alpha+\beta\, s]$ ($\kappa,\alpha,\beta,\gamma$ different for V & A)
- **9 parameters** (V, A, α_s) — Too many for a highly-correlated data set
- **Maximize DV taking** $w(s) = 1$
- **Fit s_0 dependence** $[s_{min} = (1.21\ \mathrm{GeV})^2 < s_0 < m_\tau^2]$ — But $M_{a1} = 1.23$ GeV !

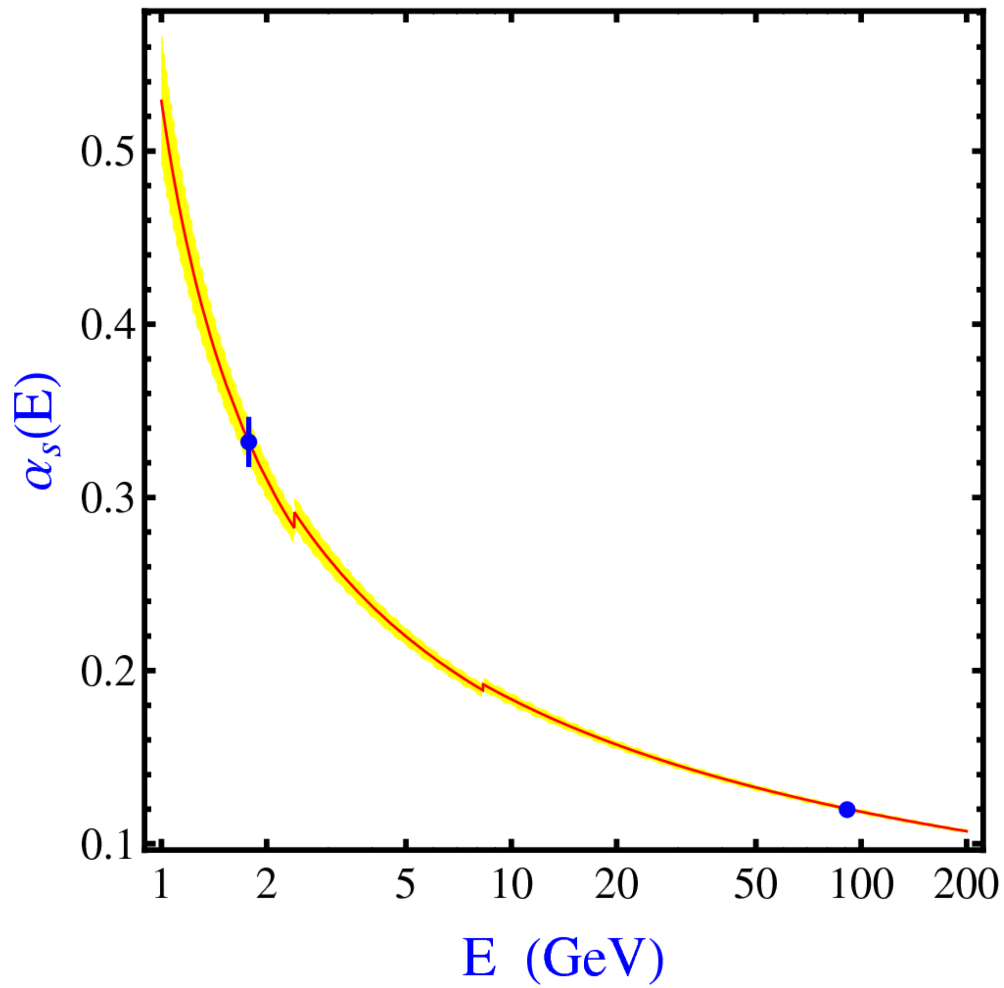
Big price: s_{min} too low, OPE not valid in real axis, δ_{NP} larger for separate V & A

In spite of all these caveats, quite reasonable values are obtained (V only):

$$\alpha_s(m_\tau^2)^{\mathrm{CIPT}} = \begin{cases} 0.347 \pm 0.025 & (\text{OPAL data}) \\ 0.310 \pm 0.014 & (\text{ALEPH data}) \end{cases}$$

[OPAL quotes 0.348 ± 0.021]
[ALEPH quotes 0.341 ± 0.008]

Present Status



$$\alpha_s(m_\tau^2) = 0.331 \pm 0.013$$

$$\alpha_s(M_Z^2) = 0.1200 \pm 0.0015$$

$$\alpha_s(M_Z^2)_{Z\,\mathrm{width}} = 0.1197 \pm 0.0028$$

The most precise test of Asymptotic Freedom

$$\alpha_s^\tau(M_Z^2) - \alpha_s^Z(M_Z^2) = 0.0003 \pm 0.0015_\tau \pm 0.0028_Z$$

SUMMARY

- ❑ **Very precise determination of α_s from τ decays**

$$\alpha_s(m_\tau^2) = 0.331 \pm 0.013 \quad \longrightarrow \quad \alpha_s(M_Z^2) = 0.1200 \pm 0.0015$$

- ❑ **Improvements require:**
  - **Better data (high statistics & precision)** **BF, FCC-ee**

    Accurate measurement of the spectral functions
  - **Better understanding of higher perturbative orders**

- ❑ **Many QCD tests with τ data: Chiral Dynamics**
- ❑ **τ data** ⟶ V_{us} , $(g-2)_\mu$ ...

Theoretical and experimental challenge

Backup Slides

Renormalons

$$D(s) \equiv -s\,\frac{d}{ds}\,\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n\; a(-s)^n$$

Asymptotic series

Borel Summation:

$$B(t) \equiv \sum_{n=0} K_{n+1}\,\frac{t^n}{n!} \quad \Longrightarrow \quad D(s) = \frac{1}{4\pi^2}\left\{1 + \int_0^\infty dt\; e^{-t/a(-s)}\; B(t)\right\}$$

However, $B(t)$ has pole singularities at

- $u \equiv -\beta_1 t/2 = +n \qquad (n \geq 2)$ **Infrared Renormalons**
- $u \equiv -\beta_1 t/2 = -n \qquad (n \geq 1)$ **Ultraviolet Renormalons**

IR - n Renormalon $\Longrightarrow$ **Ambiguity:** $\delta D(s) \sim \left(\frac{\Lambda^2}{-s}\right)^n$

Renormalon Hypothesis: Asymptotics already reached

Modelling a better behaved FOPT

(Beneke – Jamin)

- **Large higher-order K_n corrections could cancel the g_n ones**

  Happens in the "large-β_0" approximation (UV renormalon chain)

- D = 4 corrections very suppressed in R_τ

  ➡ **n = 2 IR renormalons can do the job ($K_n \approx - g_n$)**

- No sign of renormalon behaviour in known coefficients

  ➡ **n = -1,2,3 renormalons + linear polynomial**

  5 unknown constants fitted to K_n $(2 \leq n \leq 5)$. $K_5 = 283$ assumed

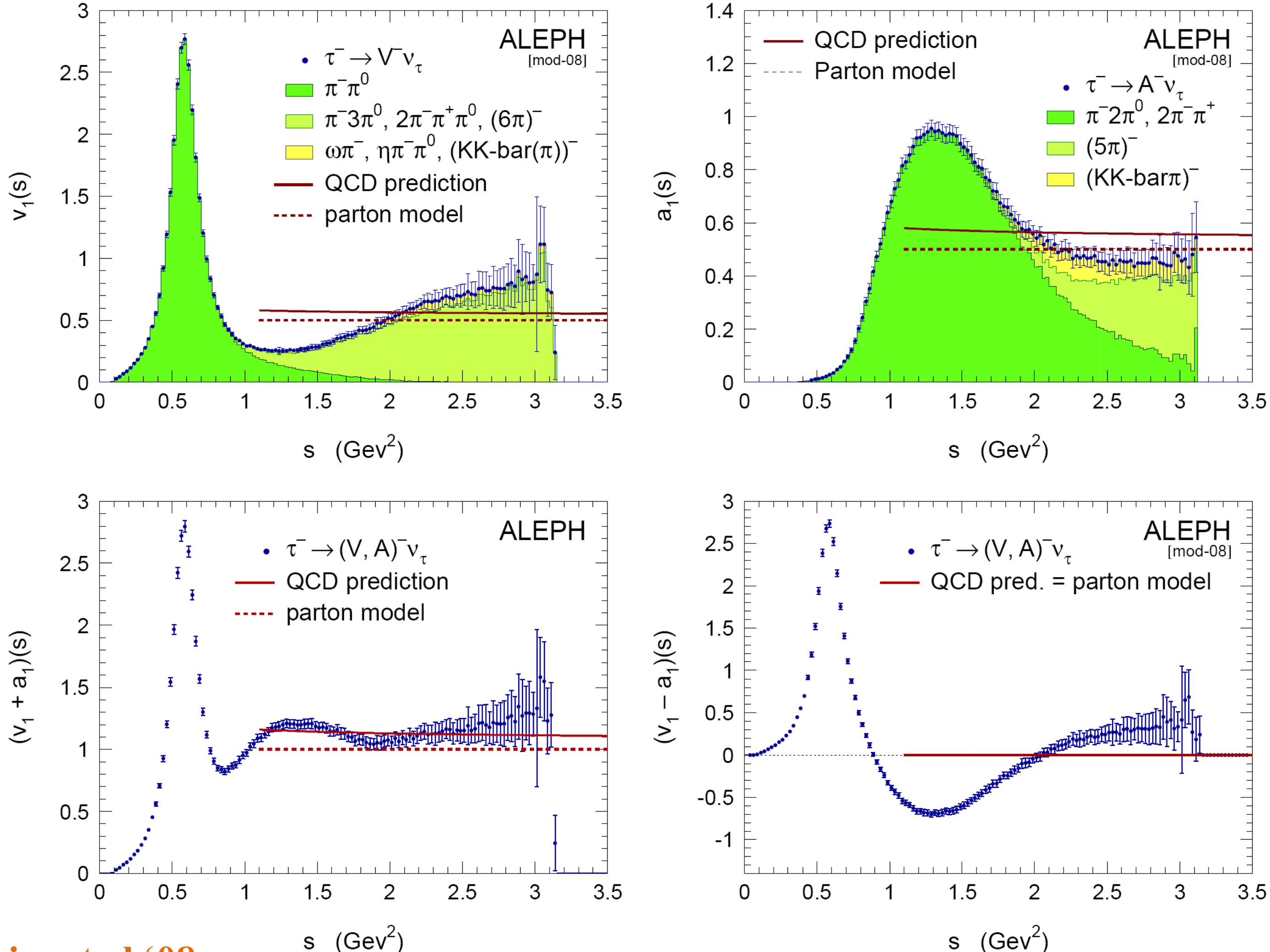
- **Borel summation:** large renormalon contributions. Smaller α

Nice model of higher orders. But too many different possibilities …

(Descotes-Genon – Malaescu)

SPECTRAL FUNCTIONS

$$v_1(s) = 2\pi\, \mathrm{Im}\, \Pi^{(0+1)}_{ud,V}(s) \qquad a_1(s) = 2\pi\, \mathrm{Im}\, \Pi^{(0+1)}_{ud,A}(s)$$



BF data needed

Davier et al ‘08