Non-perturbative effects in Calabi-Yau compactifications

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arXiv:1304.0766

The problem

Superstring theory

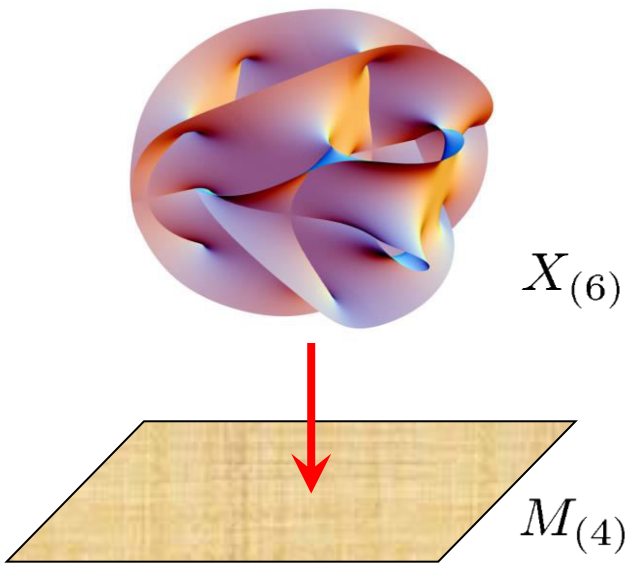
extended objects in
10-dimensional spacetime

low energy limit

& compactification

Effective field theory in $M_{(4)}$

$$X_{(10)} = M_{(4)} \times X_{(6)}$$



$\mathcal{N}$=2 supersymmetry

Type II string theory

compactification
on a *Calabi-Yau*

ungauged $\mathcal{N}$=2 supergravity
in 4d coupled to matter

The goal: to find the complete non-perturbative effective action in 4d for type II string theory compactified on arbitrary CY

(in the 2-derivative approximation)

Motivation:

- Non-perturbative structure of string theory

  instantons, S-duality, mirror symmetry,…

- Preliminary step towards phenomenological models

  non-perturbative effects and moduli stabilization, inflationary models

- BPS black holes
- Relations to $\mathcal{N}$=2 gauge theories, wall-crossing, topological strings…
- Hidden integrability
- Extremely rich mathematical structure

Plan of the talk:

1. Compactified string theory and its moduli spaces
2. Twistorial description of Quaternion-Kähler manifolds
3. Quantum corrections to the HM moduli space
4. Twistorial description of the quantum corrections
5. NS5-branes & Integrability

hypermultiplet moduli space

Ungauged $\mathcal{N}$=2 supergravity

vector multiplets (Abelian gauge fields & scalars)	supergravity multiplet (metric & graviphoton)	hypermultiplets (only scalars)

In our approximation: bosonic action = $\frac{1}{2}R$ + kinetic terms + ~~scalar potential~~

only in *gauged* supergravity

$$\mathcal{L}_{\rm kin} = \underbrace{\frac{1}{2}\Im\mathcal{N}_{IJ}(z)F^I_{\mu\nu}F^{\mu\nu,J} + \frac{1}{2}\Re\mathcal{N}_{IJ}(z)F^I_{\mu\nu}(\star F)^{\mu\nu,J} - g^{\rm VM}_{i\bar{\jmath}}(z)\partial_\mu z^i\partial^\mu\bar{z}^{\bar{\jmath}}}_{\text{VM sector}} - \underbrace{g^{\rm HM}_{\alpha\beta}(q)\partial_\mu q^\alpha\partial^\mu\bar{q}^\beta}_{\text{HM sector}}$$

$\mathcal{N}_{IJ}$ is determined by the metric $g^{\rm VM}_{i\bar{\jmath}}$ ➡ The low-energy action is determined by the geometry of the *moduli space* parameterized by scalars of *vector* and *hypermultiplets:*

$$\boxed{\mathcal{M}_{VM} \quad z^i,\ i=1,\ldots,n_V \quad g^{\rm VM}_{i\bar{\jmath}}(z)} \times \boxed{\mathcal{M}_{HM} \quad q^\alpha,\ \alpha=1,\ldots,4n_H \quad g^{\rm HM}_{\alpha\beta}(q)}$$

VM moduli space

$$\mathcal{M}_{VM} \text{ — } $$ ***(projective) special Kähler manifold***

Special Kähler manifold: a Kähler manifold whose Kähler potential is determined by a holomorphic function $F(z)$ – *holmorphic prepotential*

Two versions:

- $K(z,\bar z) = \mathrm{i}\left(\bar z^i F_i - z^i \bar F_i\right)$ – *rigid special Kähler* (relevant in gauge theories)
- $K(z,\bar z) = -\log\left[\mathrm{i}\left(\bar z^I F_I - z^I \bar F_I\right)\right]$ – *local (or projective) special Kähler* (relevant in supergravity)

$z^I = (1, z^i)$

$F(z)$ — homogeneous of degree 2

In our case: $F(z)$ is classically exact (no corrections in string coupling g_s) and determined by topological data on Calabi-Yau

Example: prepotential in type IIA

$$F(z) = -\kappa_{ijk}\frac{z^i z^j z^k}{6z^0} + \chi_{CY}\frac{\zeta(3)(z^0)^2}{2(2\pi\mathrm{i})^3} - \frac{(z^0)^2}{(2\pi\mathrm{i})^3}\sum_{q_i\gamma^i\in H_2^+(CY)} n_{q_i}^{(0)}\,\mathrm{Li}_3\left(e^{2\pi\mathrm{i}q_i z^i/z^0}\right)$$

$\mathcal{M}_{VM}$ — **known**

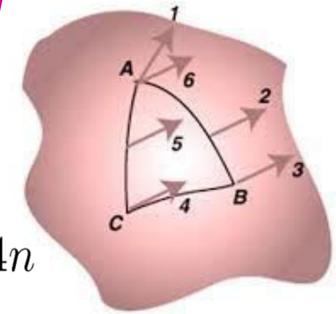
HM moduli space

$\mathcal{M}_{HM}$ — ***quaternion-Kähler manifold***

$\mathcal{M}$ — **QK** ⟷ *Holonomy group*

$$\mathrm{Hol}(\mathcal{M}) = \mathrm{Sp}(n) \times \mathrm{Sp}(1) \subset \mathrm{O}(4n)$$

$$\dim\mathcal{M} = 4n$$



Special holonomy manifolds (Berger's list)

$\mathrm{Hol}(\mathcal{M})$	$\dim\mathcal{M}$	type of manifold	appearance in physics
$\mathrm{SO}(n)$	n	orientable	
$\mathrm{U}(n)$	$2n$	Kähler	$\mathcal{N}$=1 theories
$\mathrm{SU}(n)$	$2n$	Calabi-Yau	compactification manifold
$\mathrm{Sp}(n)\times\mathrm{Sp}(1)$	$4n$	quaternion-Kähler	$\mathcal{N}$=2 supergravity
$\mathrm{Sp}(n)$	$4n$	hyperkähler	$\mathcal{N}$=2 gauge theories
G_2	7	G_2 manifold	M-theory
$\mathrm{Spin}(7)$	8	Spin(7) manifold	F-theory

$\mathcal{M}_{HM}$
- complicated type of geometry
- receives all types of g_s-corrections

$\mathcal{M}_{HM}$ — **unknown**

The (concrete) goal: **to find the non-perturbative geometry of $\mathcal{M}_{HM}$**

Twistor approach

How to describe instanton corrections to the HM metric? ⟷ How to parametrize QK manifolds?

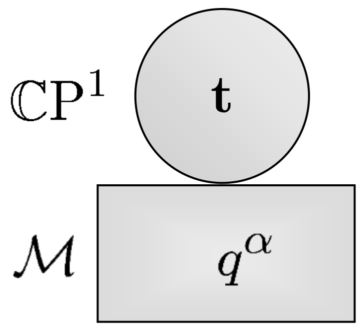
The idea: one should work at the level of the *twistor space*

<u>Quaternionic structure</u>

quaternion algebra of *almost* complex structures

$$J^i J^j = \varepsilon^{ijk} J^k - \delta^{ij}$$

Twistor space



Connection on the $\mathbb{C}\mathrm{P}^1$ bundle

$$Dt = \mathrm{d}t + p^+ - \mathrm{i}t p^3 + t^2 p^-$$

su(2)-part of the Levi-Civita connection

Twistor spaces carries:

- *integrable* complex structure
- holomorphic *contact structure*

$\mathcal{X} \sim \frac{Dt}{t}$ – nowehere vanishing holomorphic 1-form

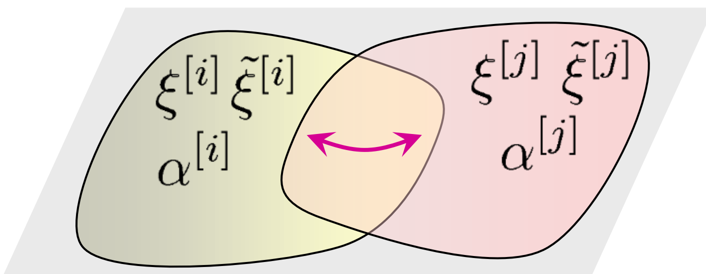
Holomorphicity

Darboux coordinates and contact Hamiltonians

Darboux coordinates for contact structure:

$$\mathcal{X} \equiv \mathrm{d}\alpha + \xi^\Lambda \,\mathrm{d}\tilde\xi_\Lambda$$

The geometry is determined by
contact transformations
between sets of Darboux coordinates



$$\mathcal{X}^{[i]} \mapsto \mathcal{X}^{[j]} = \lambda \mathcal{X}^{[i]}$$

generated by
holomorphic functions
(contact Hamiltonians)

$$h^{[ij]}(\xi_{[i]}, \tilde\xi^{[i]}, \alpha^{[i]})$$

Gluing conditions

$$\begin{pmatrix} \xi^\Lambda_{[j]} \\ \tilde\xi^{[j]}_\Lambda \\ \alpha^{[j]} \end{pmatrix} = e^{\{h^{[ij]},\,\cdot\,\}} \begin{pmatrix} \xi^\Lambda_{[i]} \\ \tilde\xi^{[i]}_\Lambda \\ \alpha^{[i]} \end{pmatrix}$$

contact bracket

$$\{h, \xi^\Lambda\} = -\,\partial_{\tilde\xi_\Lambda} h + \xi^\Lambda \partial_\alpha h$$

$$\{h, \tilde\xi_\Lambda\} = \partial_{\xi^\Lambda} h$$

$$\{h, \alpha\} = h - \xi^\Lambda \partial_{\xi^\Lambda} h$$

The metric is uniquely defined by a set of *holomorphic* functions on the twistor space

From Hamiltonians towards the metric

Gluing conditions

Solve integral equations for Darboux coordinates

$$\xi^{\Lambda}_{[i]}(q^\alpha,t) = A^\Lambda + t^{-1}Y^\Lambda - t\bar{Y}^\Lambda + \frac{1}{2}\sum_j \oint_{C_j}\frac{dt'}{2\pi i t'}\,\frac{t'+t}{t'-t}\left[\left(e^{\{h^{[ij]},\cdot\}}-1\right)\xi^{\Lambda}_{[i]}(t')\right]$$

$$\tilde{\xi}^{[i]}_{\Lambda}(q^\alpha,t) = \cdots$$

Extract the SU(2) part of the Levi-Civita connection

$$d\alpha + \xi^\Lambda d\tilde{\xi}_\Lambda = \mathcal{X} \;\sim\; \frac{Dt}{t} = \frac{dt}{t} + t^{-1}p^+ - ip^3 + tp^-$$

expanding around t=0

Find the su(2) part of the curvature

$$\omega^3 = dp^3 - 2ip^+ \wedge p^-$$

Find the almost complex structure J^3

(basis of (1,0)-forms π^X)

metric on $\mathcal{M}$

$$g(v,v') = \omega^3(v, J^3 v')$$

$$\omega^3 = i g_{X\bar{Y}}\pi^X \wedge \bar{\pi}^Y$$

$$ds^2 = 2g_{X\bar{Y}}\pi^X \otimes \bar{\pi}^Y$$

Symmetries

All isometries of $\mathcal{M}$ can be lifted to *holomorphic* isometries on twistor space.
They are realized as *contact* transformations. $\varrho\cdot\mathcal{X}=\lambda_\varrho\mathcal{X}$

Transformation property of the contact bracket

$$\varrho\cdot\{h,f\}=\{\lambda_\varrho^{-1}\varrho\cdot h,\varrho\cdot f\}$$

Linear constraint ensuring that ϱ is a symmetry transformation

$$\varrho\cdot h^{[i]}=\lambda_\varrho h^{[j]}+\text{reg.}$$

It is sufficient to find contact Hamiltonians consistent with all symmetries

Example: S-duality in type IIB

lift to the twistor space

$$\xi^0\mapsto\frac{a\xi^0+b}{c\xi^0+d}\qquad \xi^a\mapsto\frac{\xi^a}{c\xi^0+d}$$

$$\tilde\xi_a\mapsto\tilde\xi_a+\frac{c}{2(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c$$

$$\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix}\mapsto\begin{pmatrix}d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix}+\text{non-linear terms}$$

contact transformation

$$\mathcal{X}\mapsto\frac{\mathcal{X}}{c\xi^0+d}$$

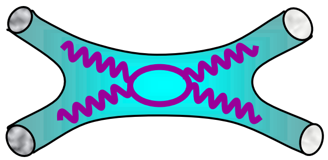
Condition the isometric action of SL(2,$\mathbb{Z}$)

$$h^{[i]}_{m,n}\mapsto\frac{h^{[i]}_{m',n'}}{c\xi^0+d}+\text{reg.}$$

$$\begin{pmatrix}m'\\ n'\end{pmatrix}=\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}m\\ n\end{pmatrix}$$

Quantum corrections

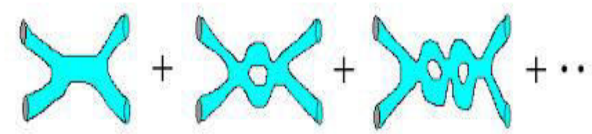
Quantum effects on the worldsheet



Classical HM metric

(KK reduction from 10d)
c-map

Quantum effects in the target space



α'- corrections **+** g_s - corrections

perturbative + worldsheet instantons

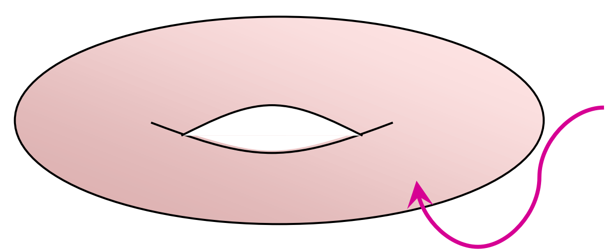
Captured by the prepotential $F(z)$
(still c-map)

perturbative corrections

one-loop correction controlled by χ_X

non-perturbative corrections

Instantons –
(Euclidean) world volumes
wrapping non-trivial cycles of CY



brane

Antoniadis, Minasian, Theisen, Vanhove ’03
Robles-Llana, Saueressig, Vandoren ’06,
S.A. ‘07

Instanton corrections

Type IIA

d_{cycle}	
1	×
3	D2
5	×

D-brane instantons

$$e^{-2\pi|Z_\gamma|/g_s-2\pi \mathrm{i}(q_\Lambda\zeta^\Lambda-p^\Lambda\tilde\zeta_\Lambda)}$$

Type IIB

d_{cycle}	
0	D(-1)
2	D1
4	D3
6	D5

$\gamma=(p^\Lambda,q_\Lambda)$ — D-brane charge

$Z_\gamma(z)=q_\Lambda z^\Lambda-p^\Lambda F_\Lambda(z)$

similar to instantons in Yang-Mills

$$g_s\sim g_{\mathrm{YM}}^2$$

+

NS5-brane instantons

$$e^{-2\pi|k|\mathcal{V}/g_s^2-\mathrm{i}\pi k\sigma}$$

k — NS5-brane charge

$\mathcal{V}$ — CY volume

pure stringy effects

Instanton contributions to the HM moduli space: what we have found

Type IIA		Type IIB	
		D1-D(-1)	to all orders
D2	to all orders		
		D3	one-instanton approximation
NS5	?	D5-NS5	one-instanton approximation

Chain of dualities:

{α', 1-loop } $\xrightarrow{\text{S-duality}}$ {α', D(-1), D1}

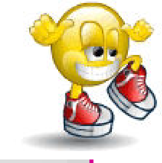
{A-D2} $\xleftarrow{\text{mirror}}$ {α', D(-1), D1}

{A-D2} $\xrightarrow{\text{e/m duality}}$ {all D2}

{all D2} $\xrightarrow{\text{mirror}}$ {α', D(-1), D1, D3, D5}

{α', D(-1), D1, D3, D5} $\xrightarrow{\text{S-duality}}$ {α', D(-1), D1, D3, D5, NS5}

{D2, NS5} $\xleftarrow{\text{mirror}}$ {α', D(-1), D1, D3, D5, NS5}



Perturbative HM moduli space

Twistorial description of $\mathcal{M}_{HM}$ at tree level

$$\boxed{h^{[+]} = F(\xi) \qquad h^{[-]} = \bar{F}(\xi)}$$

c-map

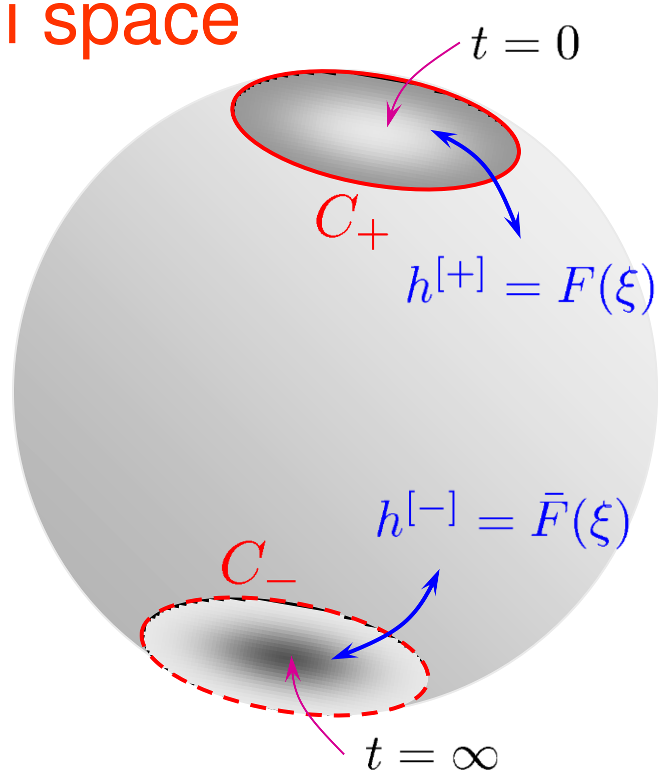
Solution of "integral" equations:

$$\begin{aligned}
\xi^\Lambda &= \zeta^\Lambda + \frac{\tau_2}{2}\left(t^{-1} z^\Lambda - t\,\bar{z}^\Lambda\right) \\
\tilde{\xi}_\Lambda &= \tilde{\zeta}_\Lambda + \frac{\tau_2}{2}\left(t^{-1} F_\Lambda(z) - t\,\bar{F}_\Lambda(\bar{z})\right) \\
\alpha &= \sigma + \frac{\tau_2}{2}\left(t^{-1} W(z) - t\,\bar{W}(\bar{z})\right) + \frac{\mathrm{i}\chi_X}{24\pi}\log t
\end{aligned}$$

where $W(z) \equiv F_\Lambda(z)\zeta^\Lambda - z^\Lambda \tilde{\zeta}_\Lambda$

$\tau_2 \sim g_s^{-1}$

reproduce the known HM metric



One-loop correction

appears as a singular boundary condition for Darboux coordinate

S.A.,Pioline,Saueressig, Vandoren '08, S.A. '09

D-instantons in Type IIA

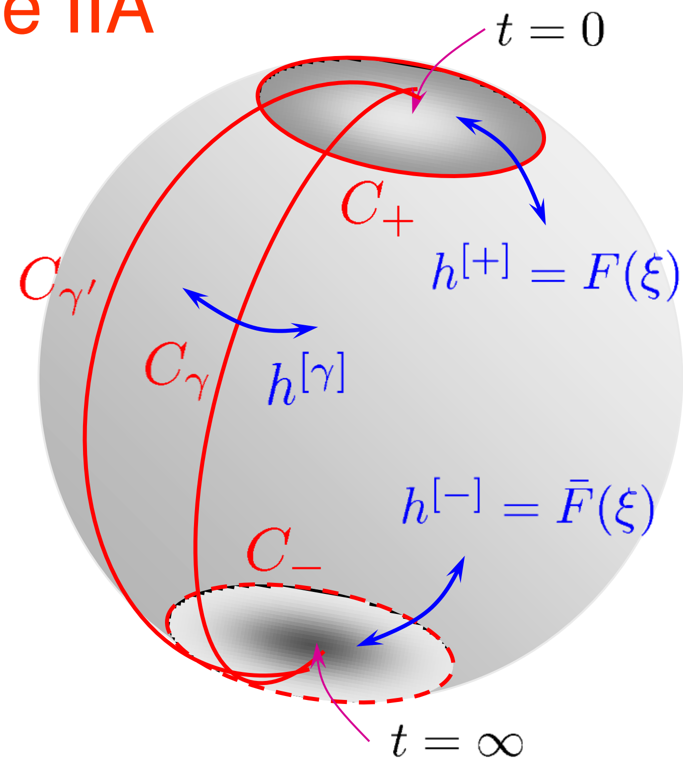
D2-instanton of charge $\gamma = (q_\Lambda, p^\Lambda)$

Contour on $\mathbb{C}\mathrm{P}^1$: ray C_γ along $\theta_\gamma = \arg Z_\gamma$

Contact Hamiltonian:

$$\boxed{h^{[\gamma]}(\xi, \tilde\xi) = \frac{\Omega(\gamma)}{(2\pi)^2}\, \mathrm{Li}_2\left(e^{2\pi \mathrm{i}(q_\Lambda \xi^\Lambda - p^\Lambda \tilde\xi_\Lambda)}\right)}$$

$\Omega(\gamma)$ — generalized DT invariants $\quad \mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$



Integral equations for Darboux coordinates can be solved explicitly for *mutually local* charges $\langle\gamma, \gamma'\rangle = 0$, e.g. only electric charges $\gamma = (q_\Lambda, 0)$

Explicit QK metric — non-trivial deformation of the c-map

S.A.,Banerjee '14

Integrability of D-instantons

- Darboux coordinates are determined by *Thermodynamic Bethe Ansatz* eqs.
- Geometric potentials on the moduli space — *free energy and YY-functional*
- S-matrix satisfies all axioms of integrability

S.A. '12

Instanton corrections in Type IIB

Type IIB formulation must be manifestly invariant under S-duality group

Quantum corrections in type IIB:

- α'-corrections: pert.
- pert. g_s-corrections: 1ℓ
- instanton corrections: D(-1) D1 D3 D5 NS5

w.s.inst.

SL(2,$\mathbb{Z}$) duality

using S-duality

Robles-Llana,Roček,Saueressig, Theis,Vandoren '06

S.A.,Banerjee '14

Twistorial description:

$$h^{\rm D1}_{m,n}(\xi)=\frac{\mathrm{i}}{(2\pi)^3}\sum_{q_a\ge 0} n^{(0)}_{q_a}\,\frac{e^{-2\pi\mathrm{i} m q_a\xi^a}}{m^2(m\xi^0+n)}$$

$$h^{\rm D1}_{0,n}(\xi)=\frac{\mathrm{i}(\xi^0)^2}{(2\pi)^3}\sum_{q_a\ge 0} n^{(0)}_{q_a}\,\frac{e^{2\pi\mathrm{i} n q_a\xi^a/\xi^0}}{n^3}$$

$n^{(0)}_{q_a}$ — Gopakumar-Vafa invariants of genus zero

S.A.,Saueressig, '09

from mirror symmetry

In the one-instanton approximation the mirror of the type IIA construction is consistent with S-duality

requires technique of *mock modular forms*

manifestly S-duality invariant formulation is unknown

S.A.,Manschot,Pioline '12

Open problem: NS5-instantons

How NS5-brane instantons look like in type IIA?

- These are true stringy effects
  (no relation to SUSY gauge theories in contrast to D-instantons)
- Resolution of a curvature singularity
  (expected to be regularized by NS5-instantons)
- Resummation of divergent series over brane charges
  (expected to be cured by NS5-instantons)
- Relation to the topological string wave function

$$\boxed{\sum_{\gamma} h_{\rm NS5}^{[\gamma]} \sim \sum_{n^\Lambda} e^{-\alpha+n^\Lambda\tilde\xi_\Lambda}\,\Psi_{\mathbb{R}}^{\rm top}(\xi^\Lambda+n^\Lambda)}$$

A-model topological string partition function **= wave function**

- Can the integrable structure of D-instantons in Type IIA be extended to include NS5-brane corrections?

Inclusion of NS5-instantons $\overset{?}{=}$ quantum deformation

NS5-branes & Integrability

- NS-axion corresponds to the central element in the Hiesenberg algebra

$$[P^\Lambda, Q_\Sigma] = -2\delta^\Lambda_\Sigma K \qquad P^\Lambda = \partial_{\tilde\zeta_\Lambda} - \zeta^\Lambda \partial_\sigma \qquad Q_\Lambda = -\partial_{\zeta^\Lambda} - \tilde\zeta_\Lambda \partial_\sigma \qquad K = \partial_\sigma$$

$$\Downarrow$$

$$k \neq 0: \ \text{Inst. cor.} \ \sim e^{2\pi i k \sigma} \Theta^{(k)}_\gamma(\zeta, \tilde\zeta)$$

– non-abelian Fourier expansion ("quantum torus")

- Universal hypermultiplet (S.A. '12)

Contact Hamilt. generating NS5-brane instantons $\longleftrightarrow$ Baker-Akhiezer function of Toda hierarchy

$$8\pi k = \hbar^{-1}$$

NS5-brane charge

quantization parameter

- Wall-crossing:

there is a natural extension involving *quantum dilog*

Conjecture

In Type IIA formulation the NS5-brane instantons are encoded by $\mathrm{q\text{-}Li}_2$

NS5-branes & Integrability

- NS-axion corresponds to the central element in the Hiesenberg algebra

$$[P^\Lambda, Q_\Sigma] = -2\delta^\Lambda_\Sigma K \qquad P^\Lambda = \partial_{\tilde\zeta_\Lambda} - \zeta^\Lambda\partial_\sigma \qquad Q_\Lambda = -\partial_{\zeta^\Lambda} - \tilde\zeta_\Lambda\partial_\sigma \qquad K = \partial_\sigma$$

$$k \neq 0 : \ \text{Inst. cor.} \ \sim e^{2\pi i k\sigma}\Theta^{(k)}_\gamma(\zeta, \tilde\zeta)$$

– non-abelian Fourier expansion
("quantum torus")

- Universal hypermultiplet (S.A. '12)

Contact Hamilt. generating NS5-brane instantons $\longleftrightarrow$ One-fermion wave function

$$8\pi k = \hbar^{-1}$$

NS5-brane charge

quantization parameter

- Wall-crossing:

there is a natural extension involving *quantum dilog*

<u>Conjecture</u>

In Type IIA formulation the NS5-brane instantons are encoded by q-Li_2

THANK YOU!