Double beta decay matrix elements in IBM-2 with isospin restoration

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**0νββ and 2νββ nuclear matrix elements in the interacting boson model with isospin restoration**

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We introduce a method for isospin restoration in the calculation of nuclear matrix elements (NMEs) for 0νββ and 2νββ decay within the framework of the microscopic interacting boson model (IBM-2). With this method, we calculate the NMEs for all processes of interest in 0νββ and 2νββ decay and in 0νββ decay and in 0νECβ+, R0νECEC, 2νββ, 2νECβ+, and 2νECEC. With this method, the Fermi matrix elements for 2νββ vanish, and those for 0νββ are considerably reduced.

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**I. INTRODUCTION**

The question of whether neutrinos are Majorana or Dirac particles, and of what are their masses and phases in the mixing matrix, remains one of the most important in physics today. A direct measurement of the average mass can be obtained from the observation of the neutrinoless double-β decay (0νββ)

\[ ^A Z X^N ightarrow ^A Z \pm 2 Y N \mp 2 + 2 e^\pm, \]

(1)

two scenarios of which are shown in Fig. 1.

Several experiments are under way to detect this decay, and others are in the planning stage (for a review, see, e.g., Ref. [1]). The half-life for this decay can be written as

\[ \left[ x_{1/2}^{0\nu} \right]^{-1} = G_{0\nu} |M_{0\nu}|^2 |f(m_i, U_{ei})|^2, \]

(2)

where \( G_{0\nu} \) is a phase-space factor, \( M_{0\nu} \) is the nuclear matrix element, and \( f(m_i, U_{ei}) \) contains physics beyond the standard model through the masses \( m_i \) and the mixing matrix elements \( U_{ei} \) of neutrino species.

Concomitant with the neutrinoless modes, there is also the process allowed by the standard model, 2νββ, depicted in Fig. 2. For this process, the half-life can be, to a good approximation, factorized in the form

\[ \frac{\tau}{\tau_{2\nu}} \approx \frac{G_{2\nu}}{G_{0\nu}} M_{2\nu}^2, \]

or approximately into the product of a phase-space factor and a nuclear matrix element, which then are the crucial ingredients of any double-β decay calculation.

To extract physics beyond the standard model, contained in the function \( f \) in Eq. (2), we need an accurate calculation of both \( G_{0\nu} \) and \( M_{0\nu} \). These calculations will serve the purpose of extracting the neutrino mass \( \langle m_{\nu} \rangle \) if 0νββ is observed and of guiding searches if 0νββ is not observed.

Recently we have started a systematic evaluation of phase-space factors (PSFs) and nuclear matrix elements (NMEs) for all processes of interest. The results for NMEs are presented in Refs. [3-7], and those for PSFs are presented in Refs. [2,7,8]. The calculations for the NMEs have been carried out within the framework of the microscopic interacting boson model (IBM-2).

Having completed the calculations in all nuclei of interest, we have now readdressed them with the purpose of providing as accurate as possible results. As shown in Table XV of Ref. [5], the Fermi matrix elements \( M_F^{(2\nu)} \) for 2νββ decay in IBM-2 do not vanish in cases where protons and neutrons occupy the same major shell. Similarly, the Fermi matrix elements \( M_F^{(0\nu)} \) for 0νββ decay are large when protons and neutrons are in the same major shell, as one can see from Table VII.
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$N_\nu$ and $N_\pi$ bosons are introduced for $2N_\nu$ valence protons and $2N_\pi$ valence neutrons, respectively.

Bosons can be only in two states with positive parity

$$L = 0 \implies s_\nu^\dagger, s_\pi^\dagger,$$

$$L = 2 \implies d_\nu^\dagger, d_\pi^\dagger.$$

Bosons are allowed to interact with one and two body interactions.
The IBM-2 Hamiltonian

\[ H = \epsilon d_\pi d_\pi^\dagger \cdot \tilde{d}_\pi + \epsilon d_\nu d_\nu^\dagger \cdot \tilde{d}_\nu + \kappa Q_\pi \cdot Q_\nu + M + \omega_{\pi\pi} L_\pi \cdot L_\pi + \frac{1}{2} \sum_{L=0,2,4} c^{(L)}_\pi \left( d_\pi^\dagger \times d_\pi^\dagger \right)^{(L)} \cdot \left( \tilde{d}_\pi \times \tilde{d}_\pi \right)^{(L)} + \omega_{\nu\nu} L_\nu \cdot L_\nu + \frac{1}{2} \sum_{L=0,2,4} c^{(L)}_\nu \left( d_\nu^\dagger \times d_\nu^\dagger \right)^{(L)} \cdot \left( \tilde{d}_\nu \times \tilde{d}_\nu \right)^{(L)} + \omega_{\nu\pi} L_\nu \cdot L_\pi \]

\[ Q_\rho = s_\rho^\dagger \tilde{d}_\rho + d_\rho^\dagger \tilde{s}_\rho + \chi_\rho \left( d_\rho^\dagger \times \tilde{d}_\rho \right)^{(2)} \quad L_\rho = \sqrt{10} \left( d_\rho^\dagger \times \tilde{d}_\rho \right)^{(1)} \quad \rho = \pi, \nu \]

\[ M = \frac{1}{2} \xi_2 \left( d_\pi^\dagger s_\nu^\dagger - d_\nu^\dagger s_\pi^\dagger \right) \cdot \left( \tilde{d}_\pi s_\nu - \tilde{d}_\nu s_\pi \right) - \sum_{K=1,3} \xi_K \left( d_\nu^\dagger \times d_\nu^\dagger \right)^{(K)} \cdot \left( \tilde{d}_\nu \times \tilde{d}_\nu \right)^{(K)} \]
IBM-2 and isospin

• For heavy nuclei the valence protons occupy orbits full of neutrons

\[ T_- |\psi\rangle = 0 \Rightarrow T = |M_T|_{\text{max}} = \frac{1}{2} |N - Z| \]

• For nuclei where protons and neutrons are in the same major shell, but with different character as particles or holes, isospin symmetry is violated only of order

\[ \frac{1}{\Omega} \]

• For the rest of the cases (lighter nuclei) IBM-2 does not produce states with definite isospin

\[ \Rightarrow \text{IBM-3 and IBM-4 are required} \]

Source: J. P. Elliot, Prog. Part. Nucl. Phys. 25, 325 (1990)
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Nuclear Matrix Elements

\[
\begin{align*}
\left[ \tau_{1/2}^{(0\nu)} \right]^{-1} &= G_{0\nu} |M_{0\nu}|^2 |f(m_i, U_{ei})|^2, \\
\left[ \tau_{1/2}^{(2\nu)} \right]^{-1} &= G_{2\nu} |m_e c^2 M_{2\nu}|^2,
\end{align*}
\]

\[
M_{0\nu} = \left\langle F; J_F \, | -h_{0\nu}^F + h_{0\nu}^{GT} + h_{0\nu}^T | l; 0_1^+ \right\rangle
\]

\[
M_{2\nu} = \left\langle F; J_F \, | -h_{2\nu}^F + h_{2\nu}^{GT} | l; 0_1^+ \right\rangle
\]

\[
h_X^{F,GT,T} = -\frac{1}{4} \sum_{\alpha} \sum_{\alpha'} \sum_{\alpha''} (-1)^J \sum_{\alpha'''} (\alpha\pi\alpha'\pi\alpha\nu\alpha'\nu) \sum_{\alpha''''} (-1)^{\alpha'''} G_X^{F,GT,T} (\alpha\pi\alpha'\pi\alpha\nu\alpha'\nu; J)
\]

\[
\sqrt{1 + (-1)^J \delta_{\alpha\pi\alpha'\pi} \sqrt{1 + (-1)^J \delta_{\alpha\nu\alpha'\nu}}}
\]

\[
\left( \pi^\dagger_{\alpha\pi} \times \pi^\dagger_{\alpha'\pi} \right)^{(J)} \cdot \left( \tilde{\nu}_{\alpha\nu} \times \tilde{\nu}_{\alpha'\nu} \right)^{(J)}
\]

\[
\alpha_\rho = (n_\rho l_\rho j_\rho), \quad \rho = \nu, \pi, \quad X = 0\nu, 0\nu_h, 2\nu
\]

The closure approximation have been avoided in computing $2\nu$ NMEs in IBM:

N. Yoshida and F. Iachello,
Prog. Theor. Exp. Phys. 2013 043D01

IBM is not restricted to just one major shell: the number of single particle levels can be extended to include spin-orbit partners in the formulation.
Two body matrix elements I

\[ G_X^{F,GT,T} (\alpha_\pi \alpha'_\pi \alpha_\nu \alpha'_\nu ; J) = \left\langle \alpha_\pi \alpha'_\pi ; JM \left| h_X^{F,GT,T} \right| \alpha_\nu \alpha'_\nu ; JM \right\rangle \]

\[ h_X^{F,GT,T} = h_X^{(s_1,s_2,\lambda)} \]

\[ = \frac{1}{2} \sum_{n,n'} \tau_n^+ \tau_n' \left( \Sigma_n^{(s_1)} \times \Sigma_n^{(s_2)} \right)^{(\lambda)} \cdot H_X (r_{nn'}) C^{(\lambda)} (\Omega_{nn'}) , \]

\[ h^F \rightarrow h_X^{(0,0,0)}, \quad h^{GT} \rightarrow h_X^{(1,1,0)}, \quad h^T \rightarrow h_{0\nu}^{(1,1,2)} \]

\[ \Sigma_n^{(0)} = 1, \quad \Sigma_n^{(1)} = \tilde{\sigma}_n, \quad C^{(\lambda)} (\Omega) = \sqrt{4\pi/(2\lambda + 1)} Y^{(\lambda)} (\Omega) \]
Two body matrix elements II

\[ G_X^{F,G,T} (\alpha_\pi \alpha'_\pi \alpha_\nu \alpha'_\nu; J; J) \equiv G_X^{(s_1,s_2,\lambda)} (\alpha_\pi \alpha'_\pi \alpha_\nu \alpha'_\nu; J; J) = \]

\[ \sum_{k_1=|l_\pi-l_\nu|}^{l_\pi+l_\nu} \sum_{k_2=|l'_\pi-l'_\nu|}^{l'_\pi+l'_\nu} \sum_{k=k_{\min}}^{k_{\max}} \langle k_1 - k_2 + \lambda \rangle \hat{k}_1 \hat{k}_2 \langle k_1 0 k_2 0 | \lambda 0 \rangle \]

\[ \times (-1)^{s_2+k_1} \left\{ \begin{array}{ccc} k_1 & s_1 & k \\ s_2 & k_2 & \lambda \end{array} \right\} \langle -1 \rangle \left\{ \begin{array}{ccc} j'_\pi + j_\nu + J \\ j'_\nu & j_\nu & k \end{array} \right\} \]

\[ \times \hat{k}_j \pi \hat{j}_\nu \left\{ \begin{array}{ccc} 1/2 & l_\pi & j_\pi \\ 1/2 & l_\nu & j_\nu \\ s_1 & k_1 & k \end{array} \right\} \hat{k}_j \pi \hat{j}_\nu \left\{ \begin{array}{ccc} 1/2 & l'_\pi & j'_\pi \\ 1/2 & l'_\nu & j'_\nu \\ s_2 & k_2 & k \end{array} \right\} \]

\[ \times \langle \frac{1}{2} \mid \Sigma^{(s_1)} \mid \frac{1}{2} \rangle (-1)^{-k_1} \hat{l}_\pi \langle \hat{l}_\pi 0 k_1 0 | l_\nu 0 \rangle \]

\[ \times \langle \frac{1}{2} \mid \Sigma^{(s_2)} \mid \frac{1}{2} \rangle (-1)^{-k_2} \hat{l}_\pi \langle \hat{l}_\pi 0 k_2 0 | l'_\nu 0 \rangle \]

\[ \times R_X^{(s_1,s_2,\lambda)} (n_\pi, l_\pi, n'_\pi, l'_\pi, n_\nu, l_\nu, n'_\nu, l'_\nu) \]
Radial Integrals

\[
\int_0^\infty h^{(s_1, s_2, \lambda)}_X(p) \, p^2 \, dp \times \int_0^\infty R_{X, k_1, k_2}^{(s_1, s_2, \lambda)}(n_{\pi}, l_{\pi}, n'_{\pi}, l'_{\pi}, n_{\nu}, l_{\nu}, n'_{\nu}, l'_{\nu}) =
\int_0^\infty R_{n_{\pi} l_{\pi}}(r_1) R_{n_{\nu} l_{\nu}}(r_1) j_{k_1}(pr_1) r_1^2 \, dr_1
\times \int_0^\infty R_{n'_{\pi} l'_{\pi}}(r_2) R_{n'_{\nu} l'_{\nu}}(r_2) j_{k_2}(pr_2) r_2^2 \, dr_2
\]

\[
h^{(s_1, s_2, \lambda)}_X(p) = v_X(p) \underbrace{\tilde{h}^{(s_1, s_2, \lambda)}(p)}_{+FNS +SRC +HOC};
\]

\[
v_X(p) = \begin{cases} 
\frac{\delta(p)}{p^2} & \text{for } 2\nu \\
\frac{2}{\pi} \frac{1}{p(p+\tilde{A})} & \text{for light } 0\nu \\
\frac{2}{\pi} \frac{1}{m_em_p} & \text{for heavy } 0\nu 
\end{cases}
\]

The transition operator in IBM-2

\[ h^{F,GT,T}_X = h^{F,GT,T}_{X,ss} \tilde{s}^\dagger_{\pi} \cdot \tilde{s}_{\nu} + h^{F,GT,T}_{X,dd} \tilde{d}^\dagger_{\pi} \cdot \tilde{d}_{\nu} \]

Expansion coefficients

\[ h^{F,GT,T}_{X,ss} = - \sum_{j_\pi} \sum_{j_\nu} G^{F,GT,T}_X (\alpha_\pi \alpha'_\pi \alpha_\nu \alpha'_\nu; J = 0) \ A^{(01)}_{j_\pi} \tilde{A}^{(01)}_{j_\nu} \]

\[ h^{F,GT,T}_{X,dd} = - \frac{1}{2} \sum_{j_\pi, j'_\pi} \sum_{j_\nu, j'_\nu} \sqrt{1 + \delta_{j_\pi j'_\pi}} \sqrt{1 + \delta_{j_\nu j'_\nu}} \]

\[ \times G^{F,GT,T}_X (\alpha_\pi \alpha'_\pi \alpha_\nu \alpha'_\nu; J = 2) \ B^{(01)}_{j_\pi j'_\pi} \tilde{B}^{(01)}_{j_\nu j'_\nu} \]
Isospin correction of the transition operator

Monopole term removed

\[ R_{X,k_1,k_2}^{(s_1,s_2,\lambda)} \rightarrow R_{X,k_1,k_2}^{(s_1,s_2,\lambda)} - \delta_{k_10} \delta_{k_20} \delta_{k0} \delta_{\lambda0} \delta_{\alpha_\pi \alpha_\nu} \delta_{\alpha'_\pi \alpha'_\nu} R_{X,0,0}^{(s_1,s_2,0)} \]

Consequences:

\[ M_{2\nu}^F \sim 0 \text{ and } M_{0\nu}^F \text{ are strongly reduced} \]

\[ M_{2\nu}^{GT} \text{ and } M_{0\nu}^{GT,T} \text{ does not change} \]

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Fermi NMEs for $\beta^-\beta^- (0\nu)$ to the ground state

![Graph showing Fermi NMEs for $\beta^-\beta^- (0\nu)$ to the ground state. The x-axis represents atomic numbers from 48 to 238, and the y-axis represents the absolute value of the Fermi matrix element $|M_{0\nu}|$. The graph compares new values with old values.](image-url)
NMEs for $\beta^-\beta^- (0\nu)$ to the ground state
NMEs for $\beta^-\beta^- (0\nu)$ to the first excited state

| $|M_{0\nu}|$ |
|----------------|
| 48  |
| 76  |
| 82  |
| 96  |
| 100 |
| 110 |
| 116 |
| 124 |
| 128 |
| 130 |
| 136 |
| 148 |
| 150 |
| 154 |
| 160 |
| 198 |
| 232 |
| 238 |

- **Old Values**
- **New Values**
Comparison with other approaches
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Conclusions

- We have cancelled $2\nu$ Fermi NMEs in those cases where isospin was broken and the $0\nu$ Fermi NMEs was strongly reduced.
- The Gamow-Teller and Tensor NMEs remain with the same values.
- The same effects are present in the NMEs for $0\nu\beta^+\beta^+$ and the other channels: $\{0, 2\} \nu EC\beta$, $R0\nu ECEC$ and $2\nu ECEC$.
- These effects are less pronounced in the NME for heavy neutrino exchange.
THANK YOU FOR YOUR ATTENTION
FIG. 4. (Color online) Multipole decomposition of the matrix element $M_{0^+}^{0\nu}$. The results with the old and new parametrizations are compared. Note the dominant effect for the $0^+$ multipole and the relatively small effects for the other multipoles. This is the case of $^{76}$Ge.

Short Range Correlations

In momentum space

\[
h^{F,GT}(p) = \frac{2}{\pi} \int_0^\infty j_0(pr)H^{F,GT}(r)r^2dr
\]

\[
h^T(p) = \frac{2}{\pi} \int_0^\infty j_2(pr)H^T(r)r^2dr
\]

Jastrow function

\[
H^{F,GT,T}(r) \rightarrow H^{F,GT,T}(r)f^2(r)
\]

\[
f(r) = 1 - Ce^{-Ar^2}(1 - Br^2)
\]
### SRC Parametrizations

<table>
<thead>
<tr>
<th>Name</th>
<th>$A$ (fm$^{-2}$)</th>
<th>$B$ (fm$^{-2}$)</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miller-Spencer</td>
<td>1.10</td>
<td>0.68</td>
<td>1.00</td>
</tr>
<tr>
<td>Argonne</td>
<td>1.59</td>
<td>1.45</td>
<td>0.92</td>
</tr>
<tr>
<td>CD-Bonn</td>
<td>1.52</td>
<td>1.88</td>
<td>0.46</td>
</tr>
</tbody>
</table>

\[ h = \sum_{n,n'} \tau^\dagger_n \tau^\dagger_{n'} \left[ -h^F(p) + h^{GT}(p) \bar{\sigma}_n \cdot \bar{\sigma}_{n'} + h^T(p) S^p_{nn'} \right] \]

\[ S^p_{nn'} = 3 \left[ \left( \vec{\sigma}_n \cdot \hat{p} \right) \left( \vec{\sigma}_{n'} \cdot \hat{p} \right) \right] - \vec{\sigma}_n \cdot \vec{\sigma}_{n'} \]

\[ h^{F,GT,T}(p) = \nu(p) \tilde{h}^{F,GT,T}(p); \quad \nu(p) = \begin{cases} 
\frac{2}{\pi} \frac{1}{p(p+\tilde{A})} & \text{for light } \nu \\
\frac{2}{\pi} \frac{1}{m_e m_p} & \text{for heavy } \nu
\end{cases} \]
<table>
<thead>
<tr>
<th>HOC term</th>
<th>$\tilde{h}(p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{h}_V^F$</td>
<td>$F_{V\gamma}^2 g_A^2 \frac{g_V^2/g_A^2}{(1+p^2/M_V^2)^4}$</td>
</tr>
<tr>
<td>$\tilde{h}_{A\gamma}^{GT}$</td>
<td>$A_{\gamma\gamma}^2 g_A^2 \frac{1}{(1+p^2/M_A^2)^4}$</td>
</tr>
<tr>
<td>$\tilde{h}_A^{GT}$</td>
<td>$g_A^2 \left[ -\frac{2}{3} \left(\frac{1}{1+p^2/M_A^2}\right)^4 \frac{p^2}{p^2+m_{\pi}^2} \left(1 - \frac{m_{\pi}^2}{M_A^2}\right) \right]$</td>
</tr>
<tr>
<td>$\tilde{h}^{GT}_{PP}$</td>
<td>$g_A^2 \left[ \frac{1}{\sqrt{3}} \left(\frac{1}{1+p^2/M_A^2}\right)^2 \frac{p^2}{p^2+m_{\pi}^2} \left(1 - \frac{m_{\pi}^2}{M_A^2}\right) \right]^2$</td>
</tr>
<tr>
<td>$\tilde{h}^{GT}_{MM}$</td>
<td>$g_A^2 \left[ \frac{2}{3} g_V^2 \frac{1}{g_A^2 (1+p^2/M_V^2)^4} \frac{\kappa_{A}^{2} p^2}{4m_{p}^2} \right]$</td>
</tr>
<tr>
<td>$\tilde{h}_A^{T}$</td>
<td>$- h_{A\gamma}^{GT}$</td>
</tr>
<tr>
<td>$\tilde{h}_{PP}^{T}$</td>
<td>$- \tilde{h}_{A\gamma}^{GT}$</td>
</tr>
<tr>
<td>$\tilde{h}_{MM}^{T}$</td>
<td>$\frac{1}{2} h_{A\gamma}^{GT}$</td>
</tr>
</tbody>
</table>
Example: $M_{GT}^{(2\nu)}$

$$\sum_N \frac{\langle F | \tau^+ \vec{\sigma} | N \rangle \langle N | \tau^+ \vec{\sigma} | I \rangle}{\frac{1}{2} Q_{\beta\beta} + m_e c^2 + E_N - E_I} \rightarrow \frac{\langle F | \tau^+ \tau^+ \vec{\sigma} \cdot \vec{\sigma} | I \rangle}{\frac{1}{2} Q_{\beta\beta} + m_e c^2 + \langle E_N \rangle - E_I}$$
Boson expansion of the coupled pairs operators

\[
\begin{align*}
\left( \pi_{j\pi}^\dagger \times \pi_{j\pi}^\dagger \right)^{(0)} & \quad \longmapsto \quad A_{j\pi}^{(01)} s_{\pi}^\dagger + A_{j\pi}^{(11)} s_{\pi}^\dagger \left( d_{\pi}^\dagger \tilde{d}_{\pi} \right)^{(0)} + \ldots \\
\left( \pi_{j\pi}^\dagger \times \pi_{j'\pi}^\dagger \right)^{(2)} & \quad \longmapsto \quad B_{j\pi j'\pi}^{(01)} d_{\pi}^\dagger \\
& \quad \quad + \quad B_{j\pi j'\pi}^{(11)} s_{\pi}^\dagger \left( s_{\pi}^\dagger \tilde{d}_{\pi} \right)^{(2)} + B_{j\pi j'\pi}^{(12)} s_{\pi}^\dagger \left( d_{\pi}^\dagger \tilde{d}_{\pi} \right)^{(2)} \\
& \quad \quad + \quad \ldots
\end{align*}
\]

\[
\begin{align*}
\left( \tilde{\nu}_{j\nu} \times \tilde{\nu}_{j'\nu} \right)^{(0)} & \quad \longmapsto \quad \tilde{A}_{j\nu}^{(01)} \tilde{s}_{\nu} + \tilde{A}_{j\nu}^{(11)} \tilde{s}_{\nu} \left( d_{\nu}^\dagger \tilde{d}_{\nu} \right)^{(0)} + \ldots \\
\left( \tilde{\nu}_{j\nu} \times \tilde{\nu}_{j'\nu} \right)^{(2)} & \quad \longmapsto \quad \tilde{B}_{j\nu j'\nu}^{(01)} \tilde{d}_{\nu} \\
& \quad \quad + \quad \tilde{B}_{j\nu j'\nu}^{(11)} \left( d_{\nu}^\dagger \tilde{s}_{\nu} \right)^{(2)} \tilde{s}_{\nu} + \tilde{B}_{j\nu j'\nu}^{(12)} \left( d_{\nu}^\dagger \tilde{d}_{\nu} \right)^{(2)} \tilde{s}_{\nu} \\
& \quad \quad + \quad \ldots
\end{align*}
\]
\( h^{F,GT,T} \)

\[ \mapsto \begin{align*}
  & h_{s-s}^{F,GT,T} s_{\pi}^{\dagger} \cdot \tilde{s}_{\nu} + h_{d-d}^{F,GT,T} d_{\pi}^{\dagger} \cdot \tilde{d}_{\nu} \\
  + & h_{d-dss}^{F,GT,T} d_{\pi}^{\dagger} \cdot d_{\nu}^{\dagger} \tilde{s}_{\nu} \tilde{s}_{\nu} + h_{d-dds}^{F,GT,T} d_{\pi}^{\dagger} \cdot \left( d_{\nu}^{\dagger} \tilde{d}_{\nu} \right)^{(2)} \tilde{s}_{\nu} \\
  + & h_{ssd-d}^{F,GT,T} s_{\pi}^{\dagger} s_{\pi}^{\dagger} \tilde{d}_{\pi} \cdot \tilde{d}_{\nu} + h_{sdd-d}^{F,GT,T} s_{\pi}^{\dagger} \left( d_{\nu}^{\dagger} \tilde{d}_{\pi} \right)^{(2)} \cdot \tilde{d}_{\nu} \\
  + & h_{ssd-dss}^{F,GT,T} s_{\pi}^{\dagger} s_{\pi}^{\dagger} d_{\pi} \cdot d_{\nu}^{\dagger} \tilde{s}_{\nu} \tilde{s}_{\nu} \\
  + & h_{ssd-dds}^{F,GT,T} s_{\pi}^{\dagger} s_{\pi}^{\dagger} \tilde{d}_{\pi} \cdot \left( d_{\nu}^{\dagger} \tilde{d}_{\nu} \right)^{(2)} \tilde{s}_{\nu} \\
  + & h_{sdd-dss}^{F,GT,T} s_{\pi} \left( d_{\pi}^{\dagger} \tilde{d}_{\pi} \right)^{(2)} \cdot d_{\nu}^{\dagger} \tilde{s}_{\nu} \tilde{s}_{\nu} \\
  + & h_{sdd-dds}^{F,GT,T} s_{\pi} \left( d_{\pi}^{\dagger} \tilde{d}_{\pi} \right)^{(2)} \cdot \left( d_{\nu}^{\dagger} \tilde{d}_{\nu} \right)^{(2)} \tilde{s}_{\nu} \\
  + & \ldots
\end{align*} \]
Different orders in the boson expansion of the Fermi NME

![Graph showing different orders in the boson expansion of the Fermi NME.](image)

- $h_{s-s}^F \langle F | s_\pi^+ \tilde{s}_v | I \rangle$
- $h_{d-d}^F \langle F | d_\pi^+ \tilde{d}_v | I \rangle$
- NLO terms
Calculation of $\alpha_j$ and $\beta_{jj'}$

\[
S^\dagger = \sum_j \alpha_j \sqrt{\frac{\Omega_j}{2}} (c_j^\dagger \times c_j^\dagger)^{(0)}, \quad D^\dagger = \sum_{j \leq j'} \beta_{jj'} \frac{1}{\sqrt{1 + \delta_{jj'}}} (c_j^\dagger \times c_{j'}^\dagger)^{(2)}
\]

\[
H_{SDI} = \sum_j \varepsilon_j + A_T V_{SDI}
\]

- Single particle energies $\varepsilon_j$ taken from spectra of nuclei with one nucleon of valence
- $A_T$ fitted to reproduce the $2^+_1 - 0^+_g$s in nuclei with 2 nucleons of valence.
- $\alpha_j$ and $\beta_{jj'}$ are extracted from the lowest $0^+$ and $2^+$ states obtained diagonalizing $H_{SDI}$.