SuSy Supertalk

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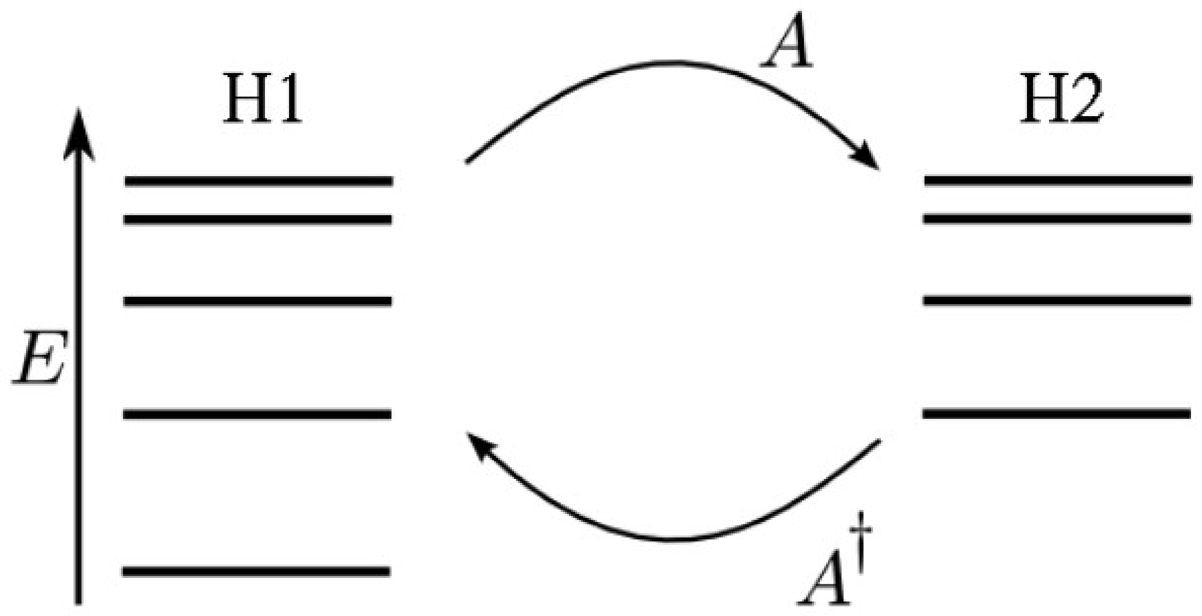
SuSy *Revisited*

$$\mathbb{A} = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \qquad \mathbb{A}^\dagger = \frac{-\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$



$$V(x) = W^2 - \frac{\hbar}{\sqrt{2m}}W'(x) \qquad V_2(x) = W^2 + \frac{\hbar}{\sqrt{2m}}W'(x)$$

(*Un*)*Broken* SuSy

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)}$$

$$\Rightarrow \psi_0 = N\exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x W(y)dy\right)$$

- Caso ψ_0 seja de quadrado-integrável, existe o estado fundamental de energia nula

$\Longrightarrow$ Unbroken SuSy

Hierarquia de Hamiltonianos

$$\hbar = 2m = 1$$

$$H_1 = A_1^\dagger A_1 + E_0^{(1)} = -\frac{d^2}{dx^2} + V_1(x) \qquad W_1(x) = -\frac{d\,\ln\psi_0^{(1)}}{dx}$$

$$H_2 = A_1 A_1^\dagger + E_0^{(1)} = -\frac{d^2}{dx^2} + V_2(x) \qquad W_2(x) = -\frac{d\,\ln\psi_0^{(2)}}{dx}$$

$$V_2(x) = W_1^2 + W_1' + E_0^{(1)} = V_1(x) + 2W_1' = V_1(x) - 2\frac{d^2}{dx^2}\ln\psi_0^{(1)}$$

$$\psi_n^{(2)} = \left(E_{n+1}^{(1)} - E_0^{(1)}\right)^{-1/2} A_1 \psi_{n+1}^{(1)}$$

Hierarquia de Hamiltonianos

$$H_2 = A_1 A_1^\dagger + E_0^{(1)} = A_2^\dagger A_2 + E_1^{(1)}$$

Repetindo o processo...

$$H_3 = A_2 A_2^\dagger + E_1^{(1)} = -\frac{d^2}{dx^2} + V_3(x)$$

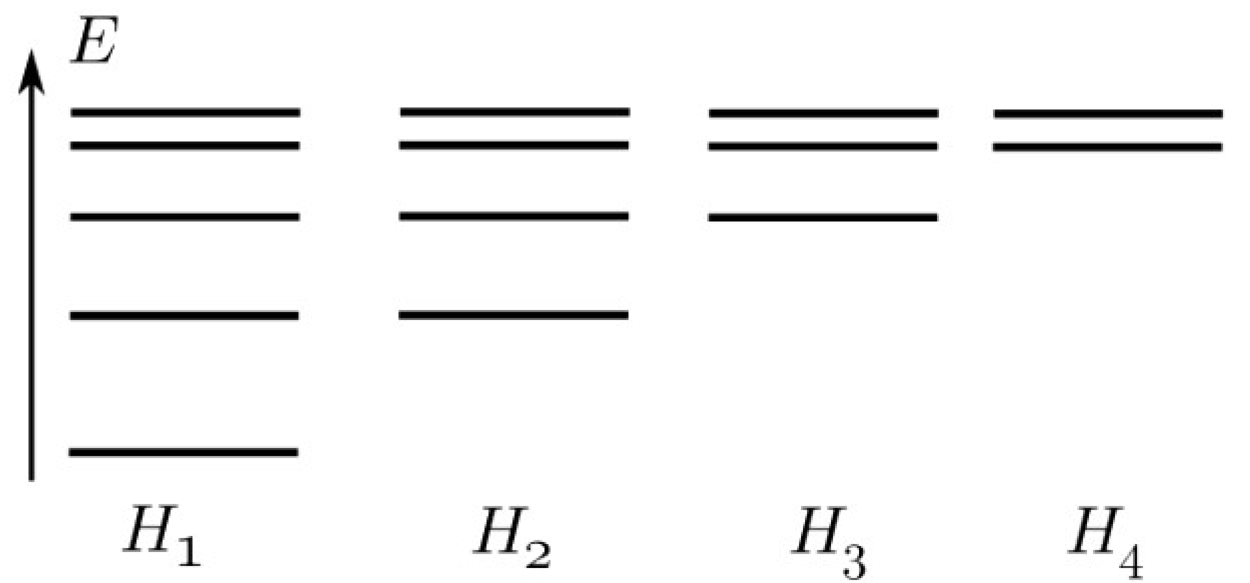
$$\begin{aligned} V_3(x) &= W_2^2 + W_2' + E_1^{(1)} = V_2(x) - 2\frac{d^2}{dx^2}\ln\psi_0^{(2)} \\ &= V_1(x) - 2\frac{d^2}{dx^2}\ln(\psi_0^{(1)}\psi_0^{(2)}). \end{aligned}$$

Hierarquia de Hamiltonianos

… *m-1* vezes

$$H_m = A_m^\dagger A_m + E_{m-1}^{(1)} = -\frac{d^2}{dx^2} + V_m(x)$$

$$\begin{aligned} E_n^{(m)} &= E_{n+1}^{(m-1)} = \dots = E_{n+m-1}^{(1)} \, , \\ \psi_n^{(m)} &= (E_{n+m-1}^{(1)} - E_{m-2}^{(1)})^{-1/2} \dots (E_{n+m-1}^{(1)} - E_0^{(1)})^{-1/2} A_{m-1} \dots A_1 \psi_{n+m-1}^{(1)} \\ V_m(x) &= V_1(x) - 2\frac{d^2}{dx^2} \ln(\psi_0^{(1)} \dots \psi_0^{(m-1)}) \end{aligned}$$



Shape Invariant Potentials (SIP)

Shape invariance condition

$$V_2(x;a_1) = V_1(x;a_2) + R(a_1) \qquad a_2 = f(a_1)$$

Shape invariance + Unbroken SuSy

$$\mathbb{H}_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x;a_1)$$

$$\mathbb{H}_2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x;a_1) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x;a_2) + R(a_1)$$

...

$$H_s = -\frac{d^2}{dx^2} + V_1(x;a_s) + \sum_{k=1}^{s-1} R(a_k)$$

$$H_{s+1} = -\frac{d^2}{dx^2} + V_1(x;a_{s+1}) + \sum_{k=1}^{s} R(a_k) = -\frac{d^2}{dx^2} + V_2(x;a_s) + \sum_{k=1}^{s-1} R(a_k)$$

Shape Invariant Potentials (*SIP*)

$$E_0^{(s)} = \sum_{k=1}^{s-1} R(a_k)$$

Como são parceiros SuSy:

$$E_n^{(1)}(a_1) = \sum_{i=1}^{n} R(a_i); \qquad E_0^{(1)} = 0$$

$$\psi_k^{(1)}(x;a_1) \propto \mathbb{A}^\dagger(x;a_1)\mathbb{A}^\dagger(x;a_2)...\mathbb{A}^\dagger(x;a_k)\psi_0^{(1)}(x;a_{k+1})$$

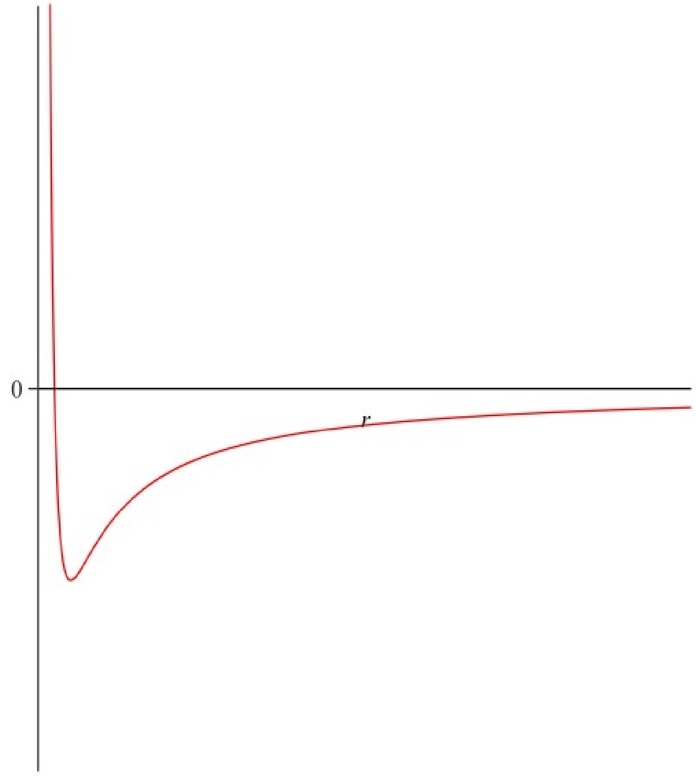
Exemplos

- Potencial de Coulomb (Átomo de Hidrogénio)

- Potencial de Morse (molécula diatómica)

Potencial de Coulomb

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$



Equação radial de Schrödinger:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u(r) = E_0 u(r)$$

Potencial efectivo com *shift* na energia:

$$\tilde{V}(r) = \left[-\frac{1}{4}\frac{e^2}{\pi\epsilon_0}\right]\frac{1}{r} + \left[\frac{\hbar^2 l(l+1)}{2m}\right]\frac{1}{r^2} - E_0$$

Potencial de Coulomb

$$\tilde{V}(r) = W(r)^2 - \frac{\hbar}{\sqrt{2m}} W'(r)$$

Ansatz para o superpotencial: $$W(r) = C - \frac{D}{r}$$

$$\begin{aligned}\tilde{V}(r) =& C^2 - \frac{2CD}{r} + \frac{D^2}{r^2} - \frac{D}{r^2}\frac{\hbar}{\sqrt{2m}} \\ =& C^2 - \frac{1}{r}2CD + \frac{1}{r^2}(D^2 - \frac{\hbar}{\sqrt{2m}}D)\end{aligned}$$

Por comparação:

$$E_0 = -C^2 \qquad -2CD = -\frac{e^2}{4\pi\epsilon_0} \qquad D^2 - \frac{\hbar}{\sqrt{2m}}D = \frac{\hbar^2}{2m}l(l+1)$$

Potencial de Coulomb

Resolvendo o sistema:

$$C = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{2 \cdot 4\pi\epsilon_0(l+1)} \qquad D = \frac{\hbar}{\sqrt{2m}}(l+1)$$

$$E_0 = -C^2 = -\frac{e^4}{4 \cdot 16\pi^2\epsilon_0^2(l+1)^2}\frac{2m}{\hbar^2}$$

Substituindo:

$$W(r) = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{2 \cdot 4\pi\epsilon_0(l+1)} - \frac{\left(\frac{\hbar}{\sqrt{2m}}(l+1)\right)}{r}$$

É possível, então, obter $V_2(r)$:

$$V_2(r) = \left[-\frac{1}{4}\frac{e^2}{\pi\epsilon_0}\right]\frac{1}{r} + \left[\frac{\hbar^2(l+1)(l+2)}{2m}\right]\frac{1}{r^2} + \left[\frac{e^4 m}{32\pi^2\hbar^2\epsilon_0^2(l+1)^2}\right]$$

Potencial de Coulomb

Comparando os potenciais parceiros:

$$\tilde{V}(r) = \left[-\frac{1}{4}\frac{e^2}{\pi\epsilon_0}\right]\frac{1}{r} + \left[\frac{\hbar^2 l(l+1)}{2m}\right]\frac{1}{r^2} + \frac{e^4}{4\cdot 16\pi^2\epsilon_0^2(l+1)^2}\frac{2m}{\hbar^2}$$

$$V_2(r) = \left[-\frac{1}{4}\frac{e^2}{\pi\epsilon_0}\right]\frac{1}{r} + \left[\frac{\hbar^2(l+1)(l+2)}{2m}\right]\frac{1}{r^2} + \left[\frac{e^4 m}{32\pi^2\hbar^2\epsilon_0^2(l+1)^2}\right]$$

$$a_2 = f(a_1) \Rightarrow f(l) = l+1$$

Obtém-se:

$$R(l) = \frac{e^4 m(2l+3)}{32\pi^2\hbar^2\epsilon_0^2(l+1)^2(l+2)^2}$$

Potencial de Coulomb

"Des-*shiftando*":

$$E_n = E_0 + \sum_{i=1}^{n} \frac{e^4 m(2(l+i-1)+3)}{32\pi^2\hbar^2\epsilon_0^2(l+i)^2(l+i+1)^2}$$

Definindo $n'=n+l$:

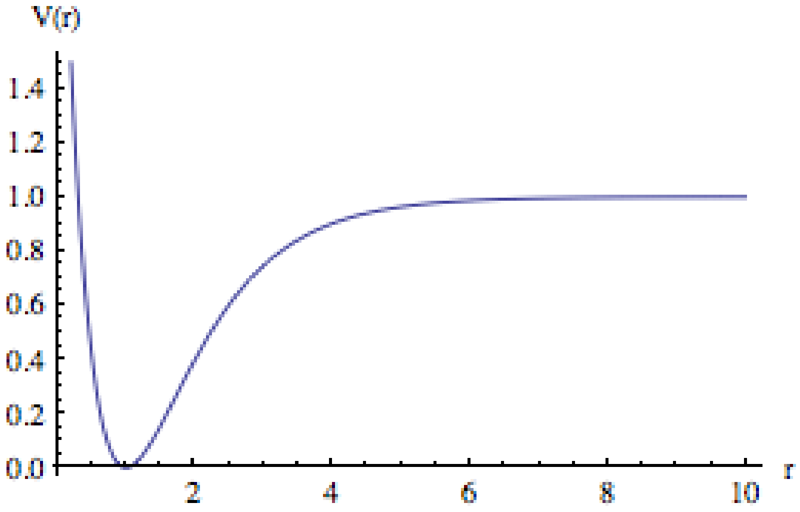
$$E_{n'} = \frac{e^4 m}{32\pi^2\hbar^2\epsilon_0^2(n'+1)^2}$$

Potencial de Morse

$$V_1(x) = A^2 + e^{-2x} - \left(2A + \frac{\hbar}{\sqrt{2m}}\right)e^{-x}$$

$$V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}\,W'(x)$$

$$\text{Ansatz:}\quad W(x) = B + Ce^{-x}$$



$$V(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}\,W'(x) = \left(B + Ce^{-x}\right)^2 - \frac{\hbar}{\sqrt{2m}}\cdot(-1)\,e^{-x}$$

$$= B^2 + C^2e^{-2x} + 2BCe^{-x} + \frac{\hbar}{\sqrt{2m}}\,e^{-x}$$

$$\Leftrightarrow B^2 + \left(2BC + \frac{\hbar}{\sqrt{2m}}\right)e^{-x} + C^2e^{-2x} = A^2 - e^{-x}\left(2A + \frac{\hbar}{\sqrt{2m}}\right)e^{-x} + e^{-2x}$$

Potencial de Morse

$$\Rightarrow \begin{cases} B^2 = A^2 \\ C^2 = 1 \\ 2BC + \frac{\hbar}{\sqrt{2m}} = 2A + \frac{\hbar}{\sqrt{2m}} \end{cases} \Rightarrow C = -1,\ B = A$$

$$W(x) = A - e^{-x}$$

Verificação:

$$V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x) = \left(A - e^{-x}\right)^2 - \frac{\hbar}{\sqrt{2m}} e^{-x}$$

$$= A^2 + e^{-2x} - \left(2A + \frac{\hbar}{\sqrt{2m}}\right) e^{-x}$$

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x) = A^2 + e^{-2x} - \left(2A - \frac{\hbar}{\sqrt{2m}}\right) e^{-x}$$

Potencial de Morse

Shape Invariance Condition:

$$V_2(x;A_1) = V_1(x;A_2) + R(A_1)$$

$$\Leftrightarrow A_1^2 + e^{-2x} - \left(2A_1 - \frac{\hbar}{\sqrt{2m}}\right)e^{-x} = A_2^2 + e^{-2x} - \left(2A_2 + \frac{\hbar}{\sqrt{2m}}\right)e^{-x} + R(A_1)$$

$$\Rightarrow \begin{cases} e^{-2x} = e^{-2x} \\ \left(2A_1 - \frac{\hbar}{\sqrt{2m}}\right)e^{-x} = \left(2A_2 + \frac{\hbar}{\sqrt{2m}}\right)e^{-x} \end{cases}$$

$$\Leftrightarrow 2A_1 - 2A_2 = 2\frac{\hbar}{\sqrt{2m}} \Leftrightarrow A_2 = A_1 - \frac{\hbar}{\sqrt{2m}}$$

Potencial de Morse

Substituindo na Condição:

$$(A_1^2 - A_2^2) \cdot -\left(2A_1 - \frac{\hbar}{\sqrt{2m}} - 2A_2 - \frac{\hbar}{\sqrt{2m}}\right) = R(A_1)$$

$$\Leftrightarrow \left(A_1^2 - \left(A_1 - \frac{\hbar}{\sqrt{2m}}\right)^2\right) - \left(2A_1 - 2\left(A_1 - \frac{\hbar}{\sqrt{2m}}\right) - \frac{2\hbar}{\sqrt{2m}}\right)e^{-x} = R(A_1)$$

$$\Leftrightarrow R(A_1) = A_1^2 - A_1^2 + 2\frac{A_1\hbar}{\sqrt{2m}} - \frac{\hbar^2}{2m} = -\frac{\hbar^2}{2m} + 2\frac{A_1\hbar}{\sqrt{2m}}$$

$$R(A_2) = -\frac{\hbar^2}{2m} + 2A_2\frac{\hbar}{\sqrt{2m}} = -\frac{\hbar^2}{2m} + \frac{2\hbar}{\sqrt{2m}}\left(A_1 - \frac{\hbar}{\sqrt{2m}}\right)$$

$$= -\frac{3\hbar^2}{2m} + 2A_1\frac{\hbar}{\sqrt{2m}}$$

Potencial de Morse

$$R(A_3) = -\frac{\hbar^2}{2m} + 2A_3\frac{\hbar}{\sqrt{2m}} = -\frac{\hbar^2}{2m} + 2\left(A_2 - \frac{\hbar}{\sqrt{2m}}\right)\frac{\hbar}{\sqrt{2m}}$$

$$= -\frac{3\hbar^2}{2m} + \frac{2\hbar}{\sqrt{2m}}\left(A_1 - \frac{\hbar}{\sqrt{2m}}\right) = -\frac{5\hbar^2}{2m} + 2A_1\frac{\hbar}{\sqrt{2m}}$$

$$\Rightarrow R(A_i) = (2i-1)\frac{\hbar^2}{2m} - 2A\frac{\hbar}{\sqrt{2m}}$$

Para obter os valores de Energia:

$$E_n = \sum_{i=1}^{n} R(A_i) = \sum_{i=1}^{n}\left(2A\frac{\hbar}{\sqrt{2m}} - \frac{\hbar^2}{2m}(i-1)\right) = 2\frac{A\hbar}{\sqrt{2m}}.n - \sum_{i=1}^{n}\frac{\hbar^2}{2m}(i-1)$$

$$\Leftrightarrow E_n = \frac{2\hbar}{\sqrt{2m}}A.n - \frac{\hbar^2}{2m}.n^2$$

Potencial de Morse

$$\psi_0(x) = \mathcal{N} \exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x W(x)dx\right)$$

$$= \mathcal{N} \exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x (A - e^{-x})dx\right)$$

$$= \mathcal{N} \exp\left(-\frac{\sqrt{2m}}{\hbar}\left(Ax + e^{-x} + K\right)\right)$$

$$= \tilde{\mathcal{N}} \exp\left(-\frac{\sqrt{2m}}{\hbar}Ax\right)\exp\left(-\frac{\sqrt{2m}}{\hbar}e^{-x}\right)$$

Através desta função de onda, podemos extrair as restantes

$$\psi_1(x;A_1) \propto \hat{A}^+(x;A_1)\psi_0(x;A_2) = \left[\frac{-\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x;A_1)\right]\psi_0\left(x; A - \frac{\hbar}{\sqrt{2m}}\right)$$

. . .

$$\psi_n(x;A_1) \propto \hat{A}^+(x;A_1)\hat{A}^+(x;A_2)\ldots\hat{A}^+(x;A_n)\psi_0(x;A_{n+1})$$