CHAPTER 5

REPRESENTATIONS OF THE SYMMETRIC GROUPS

The symmetric or permutation groups S_n and their representations are important for a number of reasons. First, recall that all finite groups of order n are subgroups of S_n (Cayley's Theorem, Sec. 2.3). Knowledge of the representations of S_n can, therefore, be useful in the study of representations of other finite groups. Secondly, as we shall see, the irreducible representations of S_n provide a valuable tool to analyze the irreducible representations of the important classical continuous groups—such as GL(m), U(m), and SU(m) (m = integer)—through tensor analysis. Furthermore, permutation symmetry is of direct relevance to physical systems consisting of identical particles. Therefore, the study of group theory in both mathematics and physics requires a reasonable familiarity with the representations of S_n.

We have discussed, in previous chapters, the representations of the simpler symmetric groups S_2 ($=C_2$) and S_3 for the purpose of illustrating general group representation theory methods. In this chapter, we shall construct all inequivalent irreducible representations of S_n for an arbitrary n. Section 1 describes the one dimensional representations of S_n from two alternative points of view. The basic tools for the general analysis are introduced in Secs. 2 and 3; they consist of Young diagrams, Young tableaux, and the associated symmetrizers, anti-symmetrizers, and irreducible symmetrizers. The case of S_3 is worked out in detail to illustrate the use of these tools in a concrete example, and to motivate the general method. The main theorems on the irreducible representations of S_n are discussed in Sec. 4. It is shown that the irreducible symmetrizers provide the necessary projection operators (called idempotents) to generate all such representations on the group algebra space. In fact, this method leads to the complete decomposition of the regular representation of S_n. The last section (Sec. 5) explores the power of the same symmetrizers applied to another important area of group representation theory—the analysis of finite dimensional irreducible representations of the general linear group GL(m). This significant application of the Young symmetrizers is based on the intricate complementary roles played by the two groups S_n and GL(m) on the space of nth-rank tensors in m-dimensional space. Tensors of specific symmetries and symmetry classes, which have a wide range of use, are introduced. This section will provide the basis for a systematic study of the representations of the classical groups to be given in Chap. 13.

For the sake of clarity, we shall emphasize the precise presentation of the methodology and the results rather than details of the derivations. To this end, most of the technical material and lengthy proofs which are not needed in any other parts of the book will be relegated to Appendix IV. For those who are mainly interested in the applications of Young tableaux, it will not be detrimental to skip these proofs and take the stated theorems on faith. Since use is made of properties of idempotents on the group algebra space in the discussion of the theorems, some knowledge of the content of Appendix III is desirable. If a full understanding of the details is desired, then Appendix III is a prerequisite and Appendix IV is a corequisite.

5.1 One-Dimensional Representations

Every symmetric group S_n has a non-trivial invariant subgroup A_n consisting of all even permutations. (An *even permutation* is one which is equivalent to an even number of simple transpositions.) This subgroup is called the *alternating group*. The factor group S_n/A_n is isomorphic to C_2. It follows that every S_n has two one-dimensional irreducible representations which are induced by the two representations of C_2. [cf. Table 3.1] The first is the identity representation. The second assigns to each permutation p the number $(-1)^p$ which is 1 if p is "even" and -1 if p is "odd". We shall refer to $(-1)^p$ as the *parity* of the permutation p.

An alternative way of deriving the one-dimensional representations of S_n is by means of the idempotents (i.e. projection operators on the group algebra; cf. Appendix III).

Theorem 5.1: The *symmetrizer* $s = \sum_p p$ and the *anti-symmetrizer* $a = \sum_p (-1)^p p$ of the group S_n are essentially idempotent and primitive.

Proof: Using the rearrangement lemma, it is straightforward to show that $qs = sq = s$ for all $q \in S_n$. It follows then that $ss = n!\, s$, and $sqs = ss = n!\, s$. According to the discussions of Appendix III, s is essentially idempotent and primitive. Similarly, for the anti-symmetrizer, we have $qa = aq = (-1)^q a$, which implies $aa = n!\, a$ and $aqa = (-1)^q n!\, a$ for all $q \in S_n$. The same result then follows. QED

According to Appendix III.3, s and a generate irreducible representations of S_n on the group algebra. Since $sqa = sa = 0$ for all $q \in S_n$, the two representations are inequivalent. The basis vectors of the irreducible representations are of the form $|q\, s\rangle$ and $|q\, a\rangle$ respectively. But since $qs = s$ and $qa = (-1)^q a$ for all $q \in S_n$, both representations are one-dimensional and the matrix elements are 1 and $(-1)^q$ respectively. Thus we reproduce the previous results.

5.2 Partitions and Young Diagrams

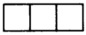
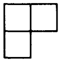
In order to generate primitive idempotents for all the irreducible representations of S_n, it is convenient to introduce Young Diagrams. We begin with the idea of "partitions".

Definition 5.1 (Partition of n, Young Diagram): (i) A *partition* $\lambda \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ of the integer n is a sequence of positive integers λ_i, arranged in descending order, whose sum is equal to n: $\lambda_i \geq \lambda_{i+1}$, $i = 1, \ldots, r-1$; and $\sum_{i=1}^{r} \lambda_i = n$.

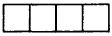
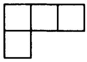
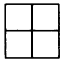
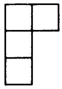
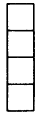
(ii) Two partitions λ, μ are *equal* if $\lambda_i = \mu_i$ for all i.
(iii) $\lambda > \mu$ ($\lambda < \mu$) if the first non-zero number in the sequence $(\lambda_i - \mu_i)$ is positive (negative).

(iv) A partition λ is represented graphically by a *Young Diagram* which consists of n squares arranged in r rows, the ith one of which contains λ_i squares.

Example 1: For the case $n = 3$, there are three distinct partitions: $\{3\}$, $\{2, 1\}$, and $\{1, 1, 1\}$. The corresponding Young diagrams are

  and  respectively .

Example 2: For the case of $n = 4$, there are five distinct partitions: $\{4\}$, $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$, and $\{1, 1, 1, 1\}$. The corresponding Young diagrams are

    and  .

There is a one-to-one correspondence between the partitions of n and the classes of group elements in S_n. Recall that every class S_n is characterized by a given "cycle structure" consisting of, say, ν_1 1-cycles, ν_2 2-cycles, ... etc. Since the numbers $1, 2, \ldots, n$ fill all the cycles, we must have: $n = \nu_1 + 2\nu_2 + 3\nu_3 + \cdots = (\nu_1 + \nu_2 + \cdots) + (\nu_2 + \nu_3 + \cdots) + (\nu_3 + \cdots) + \cdots$. If we equate the parentheses of the last expression to $\lambda_1, \lambda_2, \lambda_3, \ldots$ etc., clearly $\lambda_1 \geq \lambda_{i+1}$ and $\sum_i \lambda_i = n$. Thus $\lambda \equiv \{\lambda_i\}$ is a partition of n.

Theorem 5.2: The number of distinct Young diagrams for any given n is equal to the number of classes of S_n—which is, in turn, equal to the number of inequivalent irreducible representations of S_n. [Cf. Corollary to Theorem 3.7.]

Example: For S_3, the class $\{e\}$ corresponds to $\nu_1 = 3$, $\nu_2 = \nu_3 = 0$; the class $\{(12), (23), (31)\}$ to $\nu_1 = \nu_2 = 1$, $\nu_3 = 0$; and the class $\{(123), (321)\}$ to $\nu_1 = \nu_2 = 0$, $\nu_3 = 1$. The corresponding $(\lambda_1, \lambda_2, \lambda_3)$ are $(3, 0, 0)$, $(2, 1, 0)$, and $(1, 1, 1)$ respectively—as given before.

Definition 5.2 (Young Tableau, Normal Tableau, Standard Tableau): (i) A *Young Tableau* is obtained by filling the squares of a Young diagram with numbers $1, 2, \ldots, n$ in any order, each number being used only once;
(ii) A *normal Young tableau* is one in which the numbers $1, 2, \ldots, n$ appear in order from left to right and from the top row to the bottom row;
(iii) A *standard Young tableau* is one in which the numbers in each row appear increasing (not necessarily in strict order) to the right and those in each column appear increasing to the bottom.

Example: $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ are normal tableaux and $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$ are standard tableaux of S_4.

We shall refer to the normal Young tableau associated with a partition λ by the symbol Θ_λ. An arbitrary tableau can be obtained from the corresponding Θ_λ by applying an appropriate permutation p on the numbers $1, 2, \ldots, n$ in the boxes; hence it can be uniquely referred to as Θ_λ^p. Symbolically, we represent this relationship by: $\Theta_\lambda^p = p\,\Theta_\lambda$. It should be quite obvious that $q\,\Theta_\lambda^p = \Theta_\lambda^{qp}$.

5.3 Symmetrizers and Anti-Symmetrizers of Young Tableaux

To each Young tableau, one can define a primitive idempotent which generates an irreducible representation of S_n on the group algebra space. These idempotents are constructed from corresponding "symmetrizers" and "anti-symmetrizers", which, in turn, are built from "horizontal" and "vertical" permutations. We introduce them in logical order.

Definition 5.3 (Horizontal and Vertical Permutations): Given a Young tableau Θ_λ^p, we define *horizontal permutations* $\{h_\lambda^p\}$ which leave invariant the sets of numbers appearing in the same row of Θ_λ^p. Similarly, we define *vertical peermutations* $\{v_\lambda^p\}$ which leave invariant the sets of numbers appearing in the same column of Θ_λ^p.

It is easy to see that the cycles comprising a horizontal permutation h_λ^p must only contain numbers which appear in the same row of the corresponding Young diagram Θ_λ^p. Likewise, the cycles in a vertical permutation v_λ^p must only involve numbers in the same column.

Definition 5.4 (Symmetrizer, Anti-symmetrizer, Irreducible Symmetrizer): The *symmetrizer* s_λ^p, the *anti-symmetrizer* a_λ^p, and the *irreducible symmetrizer* e_λ^p associated with the Young tableau Θ_λ^p are defined as:

$$s_\lambda^p = \sum_h h_\lambda^p \qquad \text{(sum over all horizontal permutations);}$$

$$a_\lambda^p = \sum_v (-1)^{v_\lambda} v_\lambda^p \qquad \text{(sum over all vertical permutations);}$$

$$e_\lambda^p = \sum_{h,v} (-1)^{v_\lambda} h_\lambda^p v_\lambda^p \qquad \text{(sum over all } h_\lambda^p \text{ and } v_\lambda^p\text{).}$$

The irreducible symmetrizer e_λ^p will sometimes be called a *Young Symmetrizer*.
Example: We evaluate symmetrizers and anti-symmetrizers associated with the *normal* Young tableaux of the group S_3,

$\Theta_1 = \boxed{1|2|3}$: All p are h_λ; only e is a v_λ. Hence $s_1 = \sum_p p = s$ (symmetrizers of the full group); $a_1 = e$; and $e_1 = es = s$.

$\Theta_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$: Now, e, (12) are h_λ; e, (31) are v_λ. Hence $s_2 = e + (12)$; $a_2 = e - (31)$; and $e_2 = s_2 a_2 = e + (12) - (31) - (321)$.

$\Theta_3 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$: Only e is a h_λ, all p are v_λ. Hence $s_3 = e$; $a_3 = \sum_p (-1)^p p = a$; and $e_3 = ea = a$.

Similarly, for the only remaining standard tableau,

$\Theta_2^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$: The h_λ are $e, (31)$; the v_λ are $e, (12)$; hence $s_2^{(23)} = e + (31)$; $a_2^{(23)} = e - (12)$; and $e_2^{(23)} = e + (31) - (12) - (123)$.

Based on this example, we make a series of observations which will prove to be useful for subsequent development.

(i) For each tableau Θ_λ, the horizontal permutations $\{h_\lambda\}$ and the vertical permutations $\{v_\lambda\}$ each form a subgroup of S_n;

(ii) Since s_λ and a_λ are the (total) symmetrizer and anti-symmetrizer of the respective subgroups, they clearly satisfy the relations, $s_\lambda h_\lambda = h_\lambda s_\lambda = s_\lambda$, $a_\lambda v_\lambda = v_\lambda a_\lambda = (-1)^{v_\lambda} a_\lambda$; $s_\lambda s_\lambda = n_\lambda s_\lambda$; $a_\lambda a_\lambda = n_\lambda a_\lambda$ (where $n_\lambda = \lambda_1 ! \lambda_2 ! \cdots \lambda_n !$). Thus s_λ and a_λ are idempotents [cf. Appendix III]. They are, however, in general, not primitive idempotents.

(iii) e_λ are primitive idempotents. This is evident for $e_1 = s$ and $e_3 = a$. We leave it as an exercise [cf. Problem 5.3] for the reader to verify by explicit calculation that e_2 and $e_2^{(23)}$ are also primitive idempotents.

(iv) We already know that e_1 and e_3 generate the two inequivalent one-dimensional representations of S_3. Likewise, e_2 generates the two-dimensional irreducible representation of S_3 mentioned in Chap. 3. [See Table 3.3 and Problem 3.5.] We can verify that right multiplication by e_2 on $p \in S_3$ generates a two-dimensional subspace of the group algebra. Indeed,

$$
\begin{aligned}
ee_2 &= e_2 \qquad\qquad (12)e_2 = e_2 \\
(23)e_2 &= (23) + (321) - (123) - (12) \equiv r_2 \\
(31)e_2 &= (31) + (123) - e - (23) = -e_2 - r_2 \\
(123)e_2 &= (123) + (31) - (23) - e = -e_2 - r_2 \\
(321)e_2 &= (321) + (23) - (12) - (123) = r_2
\end{aligned}
$$

and the subspace (or left ideal) is spanned by e_2 and r_2. (The choice of the basis vectors, of course, is arbitrary.) We conclude that the symmetrizers of the normal Young tableaux generate all irreducible representations of the group.

(v) It is equally straightforward to verify that $e_2^{(23)}$ also generates a two-dimensional irreducible representation. It is, by necessity, *equivalent* to the one described above (as S_3 has only one such representation). We note, however, the invariant subspace (i.e. left ideal) generated by $e_2^{(23)}$ is distinct from the previous one. It is spanned by $e_2^{(23)}$ and $r_2^{(23)} = (123) + (23) - (31) - (321)$; it does not overlap with any of the left ideals generated by the other tableaux.

(vi) The four left ideals generated by the idempotents of the four standard Young tableaux e_1, e_2, $e_2^{(23)}$, and e_3 together span the whole group algebra space $\tilde{S}_3$. In other words, $\tilde{S}_3$ is the direct sum of these (non-overlapping) left ideals. The identity element has the following decomposition:

$$
e = \frac{1}{6} e_1 + \frac{1}{3} e_2 + \frac{1}{3} e_2^{(23)} + \frac{1}{6} e_3 \quad . \tag{5.3-1}
$$

Thus, the regular representation of S_3 is fully reduced by using the irreducible symmetrizers associated with the standard Young tableaux.

5.4 Irreducible Representations of S_n

Based on the experience with the S_3 example discussed above, we now develop the central theorems of the theory of irreducible representations of S_n. The symmetrizers and anti-symmetrizers associated with Young tableaux form the basis to construct primitive idempotents according to the general method described in Appendix III. To avoid obscuring the essential results by too many technical details, we relegate to Appendix IV a number of lemmas whose sole use to us is to help prove the following theorems. To simplify the notation, a superscript p to Θ_λ, s_λ, a_λ, and e_λ is omitted in most of the following discussions: all results apply to an arbitrary Young tableau although the explicit notation may suggest a normal tableau.

Theorem 5.3: The symmetrizers associated with a Young tableau Θ_λ have the following properties:

(5.4-1) $$s_\lambda r\, a_\lambda = \xi e_\lambda \qquad \text{for every } r \in \tilde{\mathrm{S}}_n$$

(5.4-2) $$e_\lambda{}^2 = \eta\, e_\lambda$$

where ξ and η are two ordinary numbers. Furthermore, $\eta \neq 0$ hence e_λ is essentially idempotent.

Proof: (i) Let h_λ (v_λ) be an arbitrary horizontal (vertical) permutation associated with Θ_λ; it follows from simple identities established in Lemma IV.2 that,

$$h_\lambda (s_\lambda r\, a_\lambda) v_\lambda = (h_\lambda s_\lambda) r (a_\lambda v_\lambda) = (s_\lambda r\, v_\lambda)(-1)^{v_\lambda} \quad .$$

Lemma IV.5 then ensures that $s_\lambda r\, a_\lambda = \xi\, e_\lambda$.
(ii) Since $e_\lambda{}^2 = (s_\lambda a_\lambda)(s_\lambda a_\lambda) = s_\lambda (a_\lambda s_\lambda) a_\lambda$, the first part of this theorem guarantees that the right-hand side is proportional to e_λ. Hence $e_\lambda{}^2 = \eta\, e_\lambda$.
(iii) To check whether $\eta = 0$, we need to examine the proof of Lemma IV.5 in some detail. As applied to the case at hand, the factor η is equal to the coefficient of the identity term in the expansion of $e_\lambda{}^2 = \sum h_\lambda v_\lambda h_\lambda' v_\lambda' (-1)^{v_\lambda} (-1)^{v'_\lambda}$ in terms of group elements. But since e is the common element of the two subgroups $\{v_\lambda\}$ and $\{h_\lambda\}$, it appears at least once in the expansion. Furthermore, if it occurs more than once in the above sum, the contribution is always positive. Therefore the relevant coefficient is always a non-vanishing positive integer, and e_λ is essentially idempotent. QED

Theorem 5.4: The "irreducible symmetrizer" e_λ associated with the Young tableau Θ_λ is a primitive idempotent. It generates an irreducible representation of S_n on the group algebra space.

Proof: We already know e_λ is an idempotent. Making use of Theorem 5.3 above, we easily establish

$$e_\lambda r\, e_\lambda = s_\lambda (a_\lambda r\, s_\lambda) a_\lambda = \xi\, e_\lambda$$

for all $r \in \tilde{\mathrm{S}}_n$. By the criterion of Theorem III.3, e_λ is a primitive idempotent. QED

We remind the reader that in the previous section we have seen several examples of these irreducible symmetrizers at work for the group S_3: e_1 and e_3 for the one-dimensional representations, e_2 and $e_2^{(23)}$ for two equivalent two-dimensional representations.

Theorem 5.5: The irreducible representations generated by e_λ and $e_\lambda{}^p$, $p \in \mathrm{S}_n$, are equivalent.

Proof: This result can be established by applying Theorem III.4. We simply note: $e_\lambda{}^p = p\, e_\lambda\, p^{-1}$, hence $e_\lambda{}^p\, p\, e_\lambda = p\, e_\lambda\, p^{-1}\, p\, e_\lambda = \eta\, p\, e_\lambda$ which is non-vanishing. QED

For an example of this result, we again recall the two-dimensional representations generated by e_2 and $e_2^{(23)}$ for S_3.

Theorem 5.6: Two irreducible symmetrizers e_λ and e_μ generate inequivalent (irreducible) representations if the corresponding Young diagrams are different (i.e. if $\lambda \neq \mu$).

Proof: With no loss of generality, assume $\lambda > \mu$. Let p be any element of S_n, then

$$e_\mu p e_\lambda = e_\mu (p e_\lambda p^{-1}) p = (e_\mu e_\lambda^p) p = 0$$

where the last equality is a consequence of Lemma IV.6. It follows that,

$$e_\mu r e_\lambda = 0 \qquad \text{for all } r \in \tilde{S}_n$$

as r is a linear combination of $p \in S_n$. By theorem III.4, the two primitive idempotents e_μ and e_λ generate inequivalent representations. QED

We have seen, in the case of S_3, that e_1, e_2, and e_3—which are associated with distinct Young diagrams—do generate inequivalent representations.

Corollary: If $\lambda \neq \mu$, then $e_\lambda^p e_\mu^q = 0$, for all $p, q \in S_n$.

When $\lambda < \mu$, the result is proved in Lemma IV.6. When $\lambda > \mu$, the proof is left as an exercise [Problem 5.4].

Theorem 5.7 (Irreducible Representations of S_n): The irreducible symmetrizers $\{e_\lambda\}$ associated with the normal Young tableaux $\{\Theta_\lambda\}$ generate all the inequivalent irreducible representations of S_n.

Proof: This very important result is an obvious consequence of the following observations:

(i) The number of inequivalent irreducible representations of S_n is equal to the number of Young diagrams [Theorem 5.2];
(ii) There is one e_λ associated with each Young diagram (corresponding to the normal tableau); and
(iii) Every e_λ generates an inequivalent irreducible representation [Theorem 5.6]. QED

To conclude, we state without proof the theorem governing the complete decomposition of the regular representation of S_n. [Boerner, Miller]

Theorem 5.8 (Decomposition of the Regular Representation of S_n): (i) The left ideals generated by the idempotents associated with distinct *standard* Young tableaux are linearly independent; (ii) the direct sum of the left ideals generated by all standard tableaux spans the whole group algebra space $\tilde{S}_n$.

5.5 Symmetry Classes of Tensors

An important application of the Young-tableau method and the irreducible representations of S_n concerns the construction and the classification of irreducible tensors in physics and in mathematics.

Let V_m be a m-dimensional vector space, and $\{g\}$ be the set of invertible linear transformations on V_m. With respect to the law of multiplication for linear transformations, $\{g\}$ forms a group commonly called the *general linear group* GL(m, C). In this chapter, we shall refer to this group simply as G_m. Given any basis

$\{|i\rangle, i = 1,2,\ldots,m\}$ on V_m, a natural matrix representation of G_m is obtained by:

(5.5-1) $$g\,|i\rangle = |j\rangle\, g^j{}_i$$

where $(g^j{}_i)$ are elements of an invertible $m \times m$ matrix (i.e. det $g \neq 0$).

Definition 5.5 (Tensor Space): The direct product space $V_m \times V_m \times \cdots \times V_m$ involving n factors of V shall be referred to as the *tensor space* and denoted by V^n_m.

Given a basis $\{|i\rangle\}$ on V_m, a natural basis for V^n_m is obtained in the form:

(5.5-2) $$|i_1 i_2 \cdots i_n\rangle = |i_1\rangle \cdot |i_2\rangle \cdots |i_n\rangle \quad .$$

When no confusion is likely to arise, we shall refer to this basis simply as $\{|i\rangle_n\}$. An arbitrary element x of the tensor space V^n_m has the decomposition,

(5.5-3) $$|x\rangle = |i_1\, i_2 \cdots i_n\rangle\, x^{i_1 i_2 \cdots i_n}$$

where $\{x^{i_1 i_2 \cdots i_n}\} \equiv \{x^{\{i\}}\}$ are the *tensor components* of x. The above equation shall often be abbreviated as:

(5.5-4) $$|x\rangle = |i\rangle_n\, x^{\{i\}} \quad .$$

Elements of the group G_m (defined on V_m) induce the following linear transformations on the tensor space V^n_m.

(5.5-5) $$g|i\rangle_n = |j\rangle_n\, D(g)^{\{j\}}{}_{\{i\}}$$

where

(5.5-6) $$D(g)^{\{j\}}{}_{\{i\}} = g^{j_1}{}_{i_1}\, g^{j_2}{}_{i_2} \ldots g^{j_n}{}_{i_n}$$

for all $g \in G_m$. It can easily be verified that $\{D(g)\}$ forms a $(n \cdot m)$-dimensional representation of G_m, and that for any $|x\rangle \in V^n_m$,

(5.5-7) $$g\,|x\rangle = |x_g\rangle = |j\rangle_n x_g{}^{\{j\}}$$

where

(5.5-8) $$x_g{}^{\{j\}} = D(g)^{\{j\}}{}_{\{i\}}\, x^{\{i\}}$$

where the simplified notation of Eqs. (5.5-4)–(5.5-6) is used.

Independently, the symmetric group S_n also has a natural realization on the tensor space V^n_m. In particular, consider the mapping $p \in S_n \rightarrow p =$ linear transformation on V^n_m defined by,

(5.5-9) $$p\,|x\rangle = |x_p\rangle,$$

where $|x\rangle, |x_p\rangle \in V^n_m$ and

(5.5-10) $$x_p{}^{i_1 i_2 \cdots i_n} = x^{i_{p1} i_{p2} \ldots i_{pn}} \quad .$$

In terms of the basis vectors $\{|i\rangle_n\}$, the action of p goes as

(5.5-11) $$p\,|i_1\, i_2 \cdots i_n\rangle = |i_{p_1^{-1}}\, i_{p_2^{-1}} \cdots i_{p_n^{-1}}\rangle = |i_{p^{-1}}\rangle_n \quad .$$

Therefore, if we write

(5.5-12) $$p\,|i\rangle_n = |j\rangle_n\, D(p)^{\{j\}}{}_{\{i\}}$$

then

(5.5-13)
$$D(p)^{\{j\}}{}_{\{i\}} = \delta^{j_1}_{i_{p1}^{-1}} \cdots \delta^{j_n}_{i_{pn}^{-1}} = \delta^{j_{p1}}_{i_1} \cdots \delta^{j_{pn}}_{i_n} \quad .$$

The last equality involves permuting the n δ-factors by p. The reader should verify that Eq. (5.5-9), or equivalently (5.5-11), does provide a representation of the group S_n. [Problem 5.5]

Both the representations discussed above, $D[G_m]$ for the linear group and $D[S_n]$ for the symmetric group, are in general reducible. As S_n is a finite group, we know from the theorems of Chap. 3 that $D[S_n]$ can be decomposed into irreducible representations. In fact, we shall see shortly that the irreducible symmetrizers associated with Young tableaux provide an effective method to achieve this decomposition. On the other hand, the group G_m is an infinite group. A general reducible representation of G_m is not guaranteed to be fully decomposible. We shall demonstrate, however, that the reduction of the tensor space V_m^n by Young symmetrizers from the $\tilde{S}_n$ algebra leads naturally to a full decomposition of $D[G_m]$. This interesting and useful result is a consequence of the fact that linear transformations on V_m^n representing $\{g \in G_m\}$ and $\{p \in S_n\}$ commute with each other, and each type of operator constitutes essentially the "maximal set" which has this property.

It is useful to bear in mind the above observation as we begin to establish the relevant theory. The underlying principle behind these results is very similar to, and in fact is a generalization of, the familiar facts that: (i) a complete set of commuting operators on a vector space share common eigenvectors; and (ii) a decomposition of reducible subspaces with respect to some subset of the commuting operators often leads naturally to diagonalization of the remaining operator(s). We have made use of this principle to diagonalize the Hamiltonian for a general one-dimensional lattice by taking advantage of the discrete translational symmetry group. Similarly, as is often done in the solution to physical problems involving spherical symmetry, the Hamiltonian is diagonalized by decomposing first with respect to angular momentum operators.

Lemma 5.1: The representation matrices $D(G_m)$, Eq. (5.5-6), and $D(S_n)$, Eq. (5.5-13) satisfy the following symmetry relation:

(5.5-14)
$$D^{\{j\}}{}_{\{i\}} = D^{\{j_q\}}{}_{\{i_q\}}$$

where $\{i_q\} = (i_{q_1} i_{q_2} \cdots i_{q_n})$ and $q = \begin{pmatrix} 1 & 2 & \cdots n \\ q_1 & q_2 & q_n \end{pmatrix} \in S$.

Linear transformations on V_m^n satisfying this condition are said to be *symmetry-preserving*.

Proof: The equality follows from the product form of the matrix elements, Eqs. (5.5-6) and (5.5-13). The value of the products clearly does not depend on the order in which the n-factors are placed. Permuting the n-factors by an arbitrary element (q) of S_n results in the simultaneous reshuffling of the superscripts and the subscripts by the same permutation. QED

Theorem 5.9: The two sets of matrices $\{D(p), p \in S_n\}$ and $\{D(g), g \in G_m\}$ commute with each other.

Proof: Consider the action of pg and gp on the basis vectors in turn:

(i) $pg\,|i\rangle_n = p\,|j\rangle_n D(g)^{\{j\}}{}_{\{i\}} = |j_{p^{-1}}\rangle_n D(g)^{\{j\}}{}_{\{i\}} = |j\rangle_n D(g)^{\{j_p\}}{}_{\{i\}}$;

(ii) $gp\,|i\rangle_n = g\,|i_{p^{-1}}\rangle_n = |j\rangle_n D(g)^{\{j\}}{}_{\{i_{p^{-1}}\}} = |j\rangle_n D(g)^{\{j_p\}}{}_{\{i\}}$.

In the last step of (i) we made use of the fact that $\{j\}$ are dummy summation indices, hence can be labelled in any convenient way. In the last step of (ii) we invoked Lemma 5.1. The equality of the right-hand sides of (i) and (ii) establishes the theorem. QED

Example 1: Consider second rank tensors ($n = 2$) in 2-dimensional space ($m = 2$). the basis vectors will be denoted by $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$. Since the group S_2 has only two elements, and the identity element leads to trivial results, we need only to consider $p = (12) \in S_2$ and its interplay with elements of G_2. It is quite straightforward to see that,

$$pg\,|\pm\pm\rangle = p\,|ik\rangle g^i{}_\pm g^k{}_\pm = |ki\rangle g^k{}_\pm g^i{}_\pm = gp\,|\pm\pm\rangle$$
$$pg\,|\pm\mp\rangle = p\,|ik\rangle g^i{}_\pm g^k{}_\mp = |ki\rangle g^k{}_\mp g^i{}_\pm = g\,|\mp\pm\rangle = gp\,|\pm\mp\rangle.$$

These equalities hold for any element $g \in G_2$.

We shall now decompose the tensor space V^n_m into irreducible subspaces with respect to S_n and G_m, utilizing the irreducible symmetrizers associated with various Young tableaux of S_n. As before, let Θ^p_λ be a particular Young tableau and e^p_λ be the irreducible symmetrizer and L_λ be the left ideal generated by e_λ. The main results will be: (i) a subspace consisting of tensors of the form $r|\alpha\rangle$ for fixed $|\alpha\rangle \in V^n_m$ and arbitrary $r \in L_\lambda$ is irreducibly invariant under S_n; (ii) a subspace consisting of tensors of the form $e^p_\lambda|\alpha\rangle$ for arbitrary $\alpha \in V^n_m$ and fixed Θ^p_λ are irreducibly invariant under G_m; and (iii) the tensor space V^n_m can be decomposed in such a way that the basis vectors are of the "factorized" form $|\lambda, \alpha, a\rangle$ where λ denotes a *symmetry class* specified by a Young diagram, α labels the various irreducible invariant subspaces under S_n, and "a" labels the various irreducible invariant subspaces under G_m.

Definition 5.6 (Tensors of Symmetry Θ^p_λ and Tensors of Symmetry Class λ): To each Young tableau Θ^p_λ we associate *tensors of the symmetry* Θ^p_λ consisting of $\{e^p_\lambda|\alpha\rangle;\ |\alpha\rangle \in V^n_m\}$. For a given Young diagram characterized by λ, the set of tensors $\{re_\lambda|\alpha\rangle,\ r \in \tilde{S}_n,\ \alpha \in V^n_m\}$ is said to belong to the *symmetry class* λ.

We first consider the subspace $T_\lambda(\alpha)$ consisting of tensors $\{re_\lambda|\alpha\rangle,\ r \in \tilde{S}_n\}$ for a given $|\alpha\rangle$.

Theorem 5.10 (i) $T_\lambda(\alpha)$ is an irreducible invariant subspace with respect to S_n; (ii) if $T_\lambda(\alpha)$ is not empty, then the realization of S_n on $T_\lambda(\alpha)$ coincides with the irreducible representation generated by e_λ on the group algebra $\tilde{S}_n$.

Proof: (i) Let $|x\rangle \in T_\lambda(\alpha)$, then by definition,

$$|x\rangle = re_\lambda|\alpha\rangle \qquad \text{for some } r \in \tilde{S}_n$$

hence,

$$p|x\rangle = pre_\lambda|\alpha\rangle \in T_\lambda(\alpha) \qquad \text{for all } p \in S_n \quad .$$

This means $T_\lambda(\alpha)$ is invariant under S_n.

(ii) Since $T_\lambda(\alpha)$ is not empty, we know $e_\lambda|\alpha\rangle \neq 0$. Let $\{r_i e_\lambda\}$ be a basis of L_λ, then $\{r_i e_\lambda|\alpha\rangle\}$ form a basis of $T_\lambda(\alpha)$. Hence, if

$$p|r_i e_\lambda\rangle = |pr_i e_\lambda\rangle = |r_j e_\lambda\rangle D(p)^j{}_i \qquad \text{on } \tilde{S}_n$$

then,

$$pr_i e_\lambda|\alpha\rangle = r_j e_\lambda|\alpha\rangle D(p)^j{}_i \qquad \text{on } T_\lambda(\alpha)$$

for all $p \in S_n$. Hence the invariant subspace is irreducible, and the representation matrices on $T_\lambda(\alpha)$ coincide with those on $\tilde{S}_n$. QED

Let $\Theta_{\lambda=s} = \boxed{}\boxed{}\boxed{\cdots}\boxed{}$, then $e_s = \sum_p p/n!$ is the total symmetrizer. Since $re_s = e_s$ for all $r \in \tilde{S}_n$, the left ideal L_s is one-dimensional. Correspondingly, for any given element $|\alpha\rangle$ in the tensor space V_m^n, the irreducible subspace $T_s(\alpha)$ consists of all multiples of $e_s|\alpha\rangle$. These are *totally symmetric tensors*, as it is straightforward to verify:

$$e_s|\alpha\rangle\, n! = \sum_p p\,|i\rangle_n \alpha^{\{i\}} = \sum_p |i_{p^{-1}}\rangle_n \alpha^{\{i\}} = |i\rangle_n \sum_p \alpha^{\{i_p\}} \quad ; \tag{5.5-15}$$

hence the components are totally symmetric in the n-indices. The realization of S_n on $T_\lambda(\alpha)$ is the one-dimensional identity representation because all permutations leave a totally symmetric tensor unchanged.

Example 2: Consider third rank tensors ($n = 3$) in two dimensions ($m = 2$). Four distinct totally symmetric tensors can be generated by starting with different elements of $V_2^{n=3}$:

$$\begin{array}{lll}
\text{(i)} \ |\alpha\rangle = |+++\rangle & e_s|\alpha\rangle = |+++\rangle & \equiv |s,1,1\rangle \\
\text{(ii)} \ |\alpha\rangle = |++-\rangle & e_s|\alpha\rangle = [|++-\rangle + |+-+\rangle + |-++\rangle]/3 & \equiv |s,2,1\rangle \\
\text{(iii)} \ |\alpha\rangle = |--+\rangle & e_s|\alpha\rangle = [|--+\rangle + |-+-\rangle + |+--\rangle]/3 & \equiv |s,3,1\rangle \\
\text{(iv)} \ |\alpha\rangle = |---\rangle & e_s|\alpha\rangle = |---\rangle & \equiv |s,4,1\rangle \ .
\end{array}$$

In the last column, we introduced the labelling scheme for these irreducible tensors which was mentioned in the paragraph preceding Definition 5.6. This classification is used extensively in the following discussions. Each of the above totally symmetric tensors is invariant under all permutations of the S_3 group. Together, they represent all totally symmetric tensors that can be constructed in V_2^3; they are tensors of the symmetry class s, where s represents the Young tableau with one single row. We shall denote the subspace of tensors of the symmetry class s by T'_s.

Can we similarly generate totally anti-symmetric tensors in V_m^n? We leave as an exercise [Problem 5.6] for the reader to show that they exist only if $m \geq n$. The total anti-symmetrizer is $e_a = \sum_p (-1)^p p/n!$. Since $pe_a = (-1)^p e_a$, both L_a and $T_a(\alpha)$ are one-dimensional, and the realization of S_n on $T_a(\alpha)$ corresponds to the one-dimensional representation $p \to (-1)^p$.

Example 3: There is one and only one independent totally anti-symmetric tensor of rank n in n-dimensional space, usually denoted by ε. In two dimensions, its components are $\varepsilon^{12} = -\varepsilon^{21} = 1$, $\varepsilon^{11} = \varepsilon^{22} = 0$. In three dimensions, the components are $\varepsilon^{ijk} = \pm 1$ according to whether (ijk) is an even or odd permutation of (123); else, if any two indices are equal, then $\varepsilon^{ijk} = 0$.

Example 4: Consider second rank tensors ($n = 2$) in m-dimensions ($m \geq 2$),

$$e_s|ii\rangle = |ii\rangle \qquad i = 1, 2, \ldots, m$$
$$e_s|ij\rangle = [|ij\rangle + |ji\rangle]/2 \qquad i \neq j \quad .$$

There are $m(m-1)/2$ distinct anti-symmetric tensors, as

$$e_a|ii\rangle = 0$$
$$e_a|ij\rangle = [|ij\rangle - |ji\rangle]/2 \qquad i \neq j \quad .$$

Let us now turn to tensors with *mixed symmetry.*

Example 5: We return to 3rd rank tensors in 2-dimensions [cf. Example 2]. Consider tensors with symmetry associated with the normal young tableau $\Theta_{\lambda=m}$ and irreducible symmetrizer e_m, where

$$\Theta_m = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad e_m = [e + (12)][e - (31)]/4 \quad .$$

It is straightforward to show that two independent irreducible invariant subspaces of tensors with mixed symmetry can be generated.

(i) By choosing $|\alpha\rangle = |++-\rangle$, we obtain:

$$\begin{aligned} e_m|\alpha\rangle &= [e + (12)][|++-\rangle - |-++\rangle]/4 \\ &= [2|++-\rangle - |-++\rangle - |+-+\rangle]/4 \equiv |m, 1, 1\rangle \\ (23)e_m|\alpha\rangle &= (23)[2|++-\rangle - |-++\rangle - |+-+\rangle]/4 \\ &= [2|+-+\rangle - |-++\rangle - |++-\rangle]/4 \equiv |m, 1, 2\rangle \quad ; \end{aligned}$$

and, for any $r \in \tilde{S}_3$, $re_m|\alpha\rangle$ is a linear combination of the above two tensors. These two mixed tensors form a basis for $T_{\lambda=m}(1)$.

(ii) By choosing $|\alpha\rangle = |--+\rangle$, we similarly obtain:

$$\begin{aligned} e_m|\alpha\rangle &= [2|--+\rangle - |+--\rangle - |-+-\rangle]/4 \equiv |m, 2, 1\rangle \\ (23)e_m|\alpha\rangle &= [2|-+-\rangle - |+--\rangle - |--+\rangle]/4 \equiv |m, 2, 2\rangle \end{aligned}$$

as the basis for another irreducible invariant subspace of tensors with mixed symmetry $T_{\lambda=m}(2)$.

The realization of the group S_3 on either $T_m(1)$ or $T_m(2)$ corresponds to the 2-dimensional irreducible representation discussed in Sec. 5.2 and described earlier in Chap. 3 [cf. Table 3.3].

The two tensors of mixed symmetry $|m, i, 1\rangle, i = 1, 2$ (first ones of the two sets given above), are two linearly independent tensors of the form $e_m|\alpha\rangle$ with $|\alpha\rangle$ ranging over V_m^n. [Problem 5.8] They are tensors of the symmetry Θ_m. We call the subspace spanned by these vectors $T'_m(1)$. $T'_m(1)$ is an invariant subspace under G_2 since

$$g\, e_m|\alpha\rangle = e_m\, g|\alpha\rangle \in T'_m(1)$$

for all $|\alpha\rangle \in V_m^n$. One can also show that this invariant subspace is irreducible under G_2. [Problem 5.8]

Similarly, the two tensors $|m, i, 2\rangle, i = 1, 2$ (second ones of the two sets) are two linearly independent tensors of the form $e_m^{(23)}|\alpha\rangle$—as can easily be verified by noting that $(23)e_m = e_m^{(23)}(23)$. They are tensors of the symmetry $\Theta_m^{(23)}$. We denote the subspace spanned by these tensors by $T'_m(2)$. $T'_m(2)$ is also invariant under group transformations of G_2, and it is irreducible. Together, the two sets $\{T'_m(a), a = 1, 2\}$ comprise tensors of the symmetry class m, where m denotes the Young diagram (frame) associated with the normal tableau Θ_m. For the sake of economy of indices, we shall use "α" in place of the label "i" from now on; it is understood that the range of this label is equal to the number of independent tensors that can be generated by $e_\lambda|\alpha\rangle$ with $|\alpha\rangle \in V_m^n$.

We note that for the 8-dimensional tensor space V_2^3, the use of Young symmetrizers (in Examples 2 and 5) leads to the complete decomposition into irreducible tensors $|\lambda, \alpha, a\rangle$ where λ $(=s, m)$ characterizes the symmetry class (Young diagram); "α" labels the distinct (but equivalent) sets of tensors $T_\lambda(\alpha)$ invariant under S_n; and "a" labels the basis elements within each set $T_\lambda(\alpha)$, it is associated with distinct symmetries (tableaux) in the same symmetry class. We have 4 totally symmetric tensors (Example 2) and 2 sets of 2 linearly independent mixed symmetry tensors. The latter can be classified either as belonging to two invariant subspaces under S_3 $\{T_m(\alpha), \alpha = 1, 2\}$, or as belonging to two invariant subspaces under G_2 $\{T'_m(a), a = 1, 2\}$. The latter comprise of tensors of two distinct symmetries associated with Θ_m and $\Theta_m^{(23)}$.

Bearing in mind these results for V_2^3, we return to the general case.

Theorem 5.11: (i) Two tensor subspaces irreducibly invariant with respect to S_n and belonging to the same symmetry class either overlap completely or they are disjoint; (ii) Two irreducible invariant tensor subspaces corresponding to two distinct symmetry classes are necessarily disjoint.

Proof: (i) Let $T_\lambda(\alpha)$ and $T_\lambda(\beta)$ be two invariant subspaces generated by the same irreducible symmetrizer e_λ. Either they are disjoint or they have at least one non-zero element in common. In the latter case, there are $s, s' \in \tilde{S}_n$ such that

$$s e_\lambda |\alpha\rangle = s' e_\lambda |\beta\rangle \quad .$$

This implies, $rse_\lambda|\alpha\rangle = rs'e_\lambda|\beta\rangle$ for all $r \in \tilde{S}_n$. When r ranges over all $\tilde{S}_n$, so do rs and rs'. Therefore, the left-hand side of the last equation ranges over $T_\lambda(\alpha)$ and the right-hand side ranges over $T_\lambda(\beta)$, hence the two invariant subspaces coincide.

(ii) Given any two subspaces $T_\lambda(\alpha)$ and $T_\mu(\beta)$ invariant under S_n; their intersection is also an invariant subspace. If $T_\lambda(\alpha)$ and $T_\mu(\beta)$ are irreducible, then either the intersection is the null set or it must coincide with both $T_\lambda(\alpha)$ and $T_\mu(\beta)$. If λ and μ correspond to different symmetry classes, then the second possibility is ruled out. Hence $T_\lambda(\alpha)$ and $T_\mu(\beta)$ have no elements in common if $\lambda \neq \mu$. QED

These general results permit the complete decomposition of the tensor space V_m^n into irreducible subspace $T_\lambda(\alpha)$ invariant under S_n. As explained when working on the the example of V_2^3, we shall use α as the label for distinct subspaces corresponding to the same symmetry class λ. The decomposition can be expressed as

$$V_m^n = \sum_{\lambda\oplus} \sum_{\alpha\oplus} T_\lambda(\alpha) \quad . \tag{5.5-16}$$

The basis elements of the tensors in the various symmetry classes are denoted by $|\lambda, \alpha, a\rangle$ where a ranges from 1 to the dimension of $T_\lambda(\alpha)$. We can choose these bases in such a way that the representation matrices for S_n on $T_\lambda(\alpha)$ are identical for all α associated with the same λ, or

$$p\,|\lambda, \alpha, a\rangle = |\lambda, \alpha, b\rangle\, D_\lambda(p)^b{}_a \tag{5.5-17}$$

independently of α.

The central result of this section will be that the decomposition of V^n_m according to the symmetry classes of S_n, as described above, automatically provides a complete decomposition with respect to the general linear group G_m as well. We have already seen how this worked out in the case of V^3_2.

Theorem 5.12: If $g \in G_m$ and $\{|\lambda, \alpha, a\rangle\}$ is the set of basis tensors generated according to the above procedure, then the subspaces $T'_\lambda(a)$ spanned by $\{|\lambda, \alpha, a\rangle\}$ with fixed λ and a are invariant with respect to G_m, and the representations of G_m on $T'_\lambda(a)$ are independent of a: i.e.

$$g\,|\lambda, \alpha, a\rangle = |\lambda, \beta, a\rangle\, D_\lambda(g)^\beta{}_\alpha \quad . \tag{5.5-18}$$

Proof: (i) Given $r\, e_\lambda\,|\alpha\rangle \in T_\lambda(\alpha)$ and $g \in G_m$, we have

$$g\,(re_\lambda)\,|\alpha\rangle = (re_\lambda)\,g\,|\alpha\rangle \in T_\lambda(g\alpha) \quad .$$

Hence, the operations of the linear group do not change the symmetry class of the tensor, or

$$g\,|\lambda, \alpha, a\rangle = |\lambda, \beta, b\rangle\, D_\lambda(g)^{\beta b}{}_{\alpha a} \quad ;$$

(ii) We now show that $D_\lambda(g)$ is diagonal in the indices (b, a). To this end, we note, for $g \in G_m$ and $p \in S_n$,

$$g\,p\,|\lambda, \alpha, a\rangle = g\,|\lambda, \alpha, c\rangle\, D_\lambda(p)^c{}_a = |\lambda, \beta, b\rangle\, D_\lambda(g)^{\beta b}{}_{\alpha c}\, D_\lambda(p)^c{}_a \quad ;$$

and

$$p\,g\,|\lambda, \alpha, a\rangle = p\,|\lambda, \beta, c\rangle\, D_\lambda(g)^{\beta c}{}_{\alpha a} = |\lambda, \beta, b\rangle\, D_\lambda(p)^b{}_c\, D_\lambda(g)^{\beta c}{}_{\alpha a} \quad .$$

Since $g\,p = p\,g$ (Theorem 5.9), the two product matrices on the right-hand sides can be equated to each other. For clarity, let us designate quantities in square brackets as matrices in the space of Latin indices, and suppress these indices. We obtain

$$[D_\lambda(g)^\beta{}_\alpha][D_\lambda(p)] = [D_\lambda(p)][D_\lambda(g)^\beta{}_\alpha] \quad . \tag{5.5-19}$$

For given g, this equation holds for all $p \in S_n$. According to Schur's Lemma, the matrix $D_\lambda(g)^{\beta b}{}_{\alpha a}$ must be proportional to the unit matrix in the Latin indices. QED

Theorem 5.13 (Irreducible Representations of G_m): The representations of G_m on the subspace $T'_\lambda(a)$ of V^n_m as described above are irreducible representations.

Proof: Even though the complete proof involves some technical details [Miller], the basic idea behind it is rather easy to understand: since G_m is, so to speak, the most general group of transformations which commutes with S_n on V^n_m, on the subspace