

Alternative method of Reduction of the Feynman Diagrams to a set of Master Integrals

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Abstract

We propose a new set of Master Integrals which can be used as a basis for multiloop calculation in any gauge massless field theory. In these theories we consider three-point Feynman diagrams with arbitrary number of loops. The corresponding multiloop integrals may be decomposed in terms of this set of the Master Integrals. We construct a new reduction procedure which can be applied to perform this decomposition.

Why do need this alternative method?

- It is most appropriate for determining the structure of the double ghost vertex in $\mathcal{N} = 4$ SYM
- The elements of the basis we propose may be represented in terms of triangle ladder diagrams in $d = 4 - 2\epsilon$
- Triangle ladders may be represented in terms of two-fold MB transforms
- The method is simple to program

The result is based on my previous papers. Co-authors

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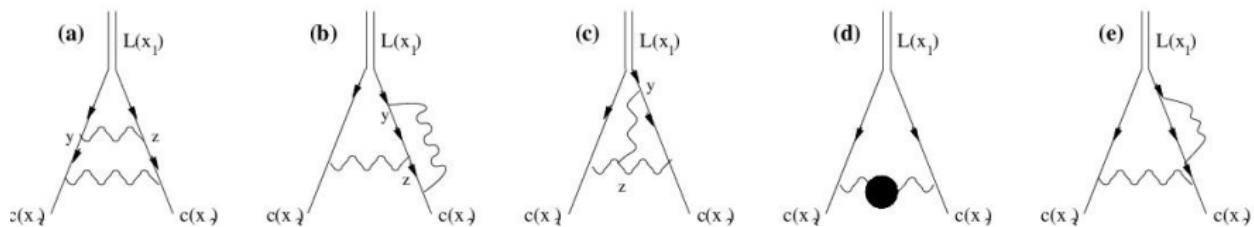
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-  G. Cvetič, I. Kondrashuk, A. Kotikov and I. Schmidt, "Towards the two-loop Lcc vertex in Landau gauge," *Int. J. Mod. Phys. A* **22** (2007) 1905 [[hep-th/0604112](#)].
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-  I. Kondrashuk and A. Kotikov, "Fourier transforms of UD integrals," in *Analysis and Mathematical Physics*, Birkhäuser Book Series Trends in Mathematics, edited by B. Gustafsson and A. Vasil'ev, (Birkhäuser, Basel, Switzerland, 2009), pp. 337 [[arXiv:0802.3468 \[hep-th\]](#)].
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-  P. Allendes, N. Guerrero, I. Kondrashuk and E. A. Notte Cuello, "New four-dimensional integrals by Mellin-Barnes transform," *J. Math. Phys.* **51** (2010) 052304 [[arXiv:0910.4805 \[hep-th\]](#)].
-  I. Kondrashuk and A. Vergara, "Transformations of triangle ladder diagrams," *JHEP* **1003** (2010) 051 [[arXiv:0911.1979 \[hep-th\]](#)].

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-  Pedro Allendes, Bernd Kniehl, Igor Kondrashuk, Eduardo A. Notte Cuello, Marko Rojas Medar, "Solution to Bethe-Salpeter equation via Mellin-Barnes transform", arXiv:1205.6257 [hep-th], Nuclear Physics B 870 (2013) 243-277
-  Ivan Gonzalez and Igor Kondrashuk, "Box ladders in non-integer dimension," arXiv:1210.2243 [hep-th], Theoretical and Mathematical Physics 177 (2013) 1515-1540
-  [Bernd Kniehl, Igor Kondrashuk, Eduardo A. Notte Cuello, Ivan Parra Ferrada, Marko Rojas Medar, "Two-fold Mellin-Barnes transforms of Usyukina-Davydychev functions", arXiv:1304.3004 [hep-th], Nuclear Physics B 876 (2013) 322-333

Double ghost vertex



- This vertex is finite in $\mathcal{N} = 4$ SYM in Landau gauge, it does not have poles in ϵ

Ladder diagrams

The diagram shows the equivalence between a ladder diagram and a Feynman diagram. On the left, a ladder diagram with three rungs is shown. It consists of two parallel horizontal lines representing external momenta p_1 and p_2 . Between them are three diagonal lines representing internal momenta $p_1 + r_1$, $p_1 + r_2$, and $p_1 + r_3$. The vertical distance between the rungs is labeled $r_1 - r_2$ and $r_2 - r_3$. On the right, a corresponding Feynman diagram is shown, consisting of two vertical lines representing external momenta p_1 and p_2 , connected by a single horizontal line representing the internal momentum $p_1 + r_s$. The vertical distance between the lines is labeled r_s .

$$C^{(n)}(p_1^2, p_2^2, p_3^2) \equiv \frac{1}{(p_3^2)^n} \phi^{(n)} \left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right)$$



V. V. Belokurov and N. I. Usyukina, "Calculation Of Ladder Diagrams In Arbitrary Order," J. Phys. A **16** (1983) 2811.

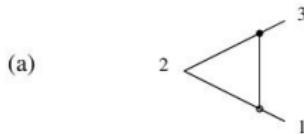


N. I. Usyukina and A. I. Davydychev, "An Approach to the evaluation of three and four point ladder diagrams," Phys. Lett. B **298** (1993) 363.



N. I. Usyukina and A. I. Davydychev, "Exact results for three and four point ladder diagrams with an arbitrary number of rungs," Phys. Lett. B **305** (1993) 136.

Fourier-invariance of UD functions

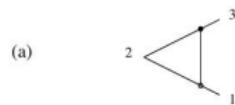


(b) $\partial_{(2)}^2 2$ = $(-4\pi^2) 2$

(c) $\partial_{(2)}^2 2$ = $(-4\pi^2) 2$

(d) $(\partial_{(2)}^2)^2 2$ = $(-4\pi^2)^2 2$

Fourier-invariance of UD functions



(b) $\partial_{\text{in}}^2 2 \triangle = (-4\pi^2) 2 \triangle$

(c) $\partial_{\text{in}}^2 2 \triangle = (-4\pi^2) 2 \triangle$

(d) $(\partial_{\text{in}}^2)^2 2 \triangle = (-4\pi^2)^2 2 \triangle$

(e) $(\partial_{\text{in}}^2)^2 2 \triangle = (-4\pi^2)^2 2 \triangle$

(f) $(\partial_{\text{in}}^2)^3 2 \triangle = (-4\pi^2)^3 2 \triangle$

Fourier-invariance of UD functions



(b) $\hat{d}_z^2 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array} = (-4\pi^1) \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array}$

(c) $\hat{d}_z^2 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array} = (-4\pi^1) \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array}$

(d) $(\hat{d}_z^2)^2 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array} = (-4\pi^1)^2 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array}$

(e) $(\hat{d}_z^2)^2 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array} = (-4\pi^1)^2 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array}$

(f) $(\hat{d}_z^2)^3 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array} = (-4\pi^1)^3 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array}$

(g) $(\hat{d}_z^2)^3 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array} = (-4\pi^1)^3 \cdot 2 \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array}$

Fourier-invariance of UD functions via MB transform

$$\frac{1}{[31]^2} \Phi^{(n)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right) = \frac{1}{(2\pi)^4} \int d^4 p_1 d^4 p_2 d^4 p_3 \delta(p_1 + p_2 + p_3) \times \\ \times e^{ip_2 x_2} e^{ip_1 x_1} e^{ip_3 x_3} \frac{1}{(p_2^2)^2} \Phi^{(n)} \left(\frac{p_1^2}{p_2^2}, \frac{p_3^2}{p_2^2} \right).$$

The explicit form of the function is given in Davydychev and Usyukina papers:

$$\Phi^{(n)}(x, y) = -\frac{1}{n! \lambda} \sum_{j=n}^{2n} \frac{(-1)^j j! \ln^{2n-j}(y/x)}{(j-n)!(2n-j)!} \left[\text{Li}_j \left(-\frac{1}{\rho x} \right) - \text{Li}_j(-\rho y) \right],$$

$$\rho = \frac{2}{1-x-y+\lambda}, \quad \lambda = \sqrt{(1-x-y)^2 - 4xy}.$$

Fourier-invariance of UD functions via MB transform

Mellin-Barnes transform for the ladder functions:

$$\Phi^{(n)}(x, y) = \oint dz_2 dz_3 x^{z_2} y^{z_3} \mathcal{M}^{(n)}(z_2, z_3)$$

$$\begin{aligned} \frac{1}{[31]^2} \Phi^{(n)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right) &= \frac{1}{(2\pi)^4} \int d^4 p_1 d^4 p_2 d^4 p_3 \delta(p_1 + p_2 + p_3) \times \\ &\quad \times e^{ip_2 x_2} e^{ip_1 x_1} e^{ip_3 x_3} \frac{1}{(p_2^2)^2} \Phi^{(n)} \left(\frac{p_1^2}{p_2^2}, \frac{p_3^2}{p_2^2} \right) = \\ &= \frac{1}{(2\pi)^8} \int d^4 p_1 d^4 p_2 d^4 p_3 d^4 x_5 e^{ip_2(x_2 - x_5)} e^{ip_1(x_1 - x_5)} e^{ip_3(x_3 - x_5)} \times \\ &\quad \times \frac{1}{(p_2^2)^2} \Phi^{(n)} \left(\frac{p_1^2}{p_2^2}, \frac{p_3^2}{p_2^2} \right) = \end{aligned}$$

Fourier-invariance of UD functions via MB transform

$$\begin{aligned} &= \frac{1}{(2\pi)^8} \int d^4 p_1 d^4 p_2 d^4 p_3 d^4 x_5 \oint dz_2 dz_3 \frac{e^{ip_2(x_2-x_5)} e^{ip_1(x_1-x_5)} e^{ip_3(x_3-x_5)}}{(p_2^2)^{2+z_2+z_3} (p_1^2)^{-z_2} (p_3^2)^{-z_3}} \times \\ &\quad \times \mathcal{M}^{(n)}(z_2, z_3) = \\ &= \frac{(4\pi)^6}{(2\pi)^8} \int d^4 x_5 \oint dz_2 dz_3 \frac{\Gamma(-z_2 - z_3)}{\Gamma(2 + z_2 + z_3)} \frac{\Gamma(2 + z_2)}{\Gamma(-z_2)} \frac{\Gamma(2 + z_3)}{\Gamma(-z_3)} \\ &\quad \times \frac{2^{2z_2+2z_3-2(2+z_2+z_3)} \mathcal{M}^{(n)}(z_2, z_3)}{(x_2 - x_5)^{-z_2-z_3} (x_1 - x_5)^{2+z_2} (x_3 - x_5)^{2+z_3}} = \\ &= \oint dz_2 dz_3 \frac{\mathcal{M}^{(n)}(z_2, z_3)}{[12]^{-z_3} [23]^{-z_2} [31]^{2+z_2+z_3}} \end{aligned}$$



P. Allendes, N. Guerrero, I. Kondrashuk and E. A. Notte Cuello, "New four-dimensional integrals by Mellin-Barnes transform," J. Math. Phys. **51** (2010) 052304 [arXiv:0910.4805 [hep-th]].

Loop reduction in d=4 dimensions

$$\begin{array}{c}
 \text{Diagram:} \\
 \begin{array}{c}
 \text{Top vertex: } 2 + \varepsilon_3 \\
 \text{Left edge: } 1 + \varepsilon_1 \\
 \text{Right edge: } 1 + \varepsilon_2 \\
 \text{Bottom edge: } 1 + \varepsilon_1 \\
 \text{Bottom-left edge: } 1 + \varepsilon_2 \\
 \text{Bottom-right edge: } 1 + \varepsilon_1 \\
 \text{Vertical edges: } 1 + \varepsilon_3, 1 + \varepsilon_3, 1 + \varepsilon_3
 \end{array}
 \end{array}
 = -\frac{1}{(1+\varepsilon_3)\varepsilon_3} J \left[
 \begin{array}{c}
 \text{Diagram:} \\
 \begin{array}{c}
 \text{Top vertex: } 1 + \varepsilon_1 \\
 \text{Left edge: } 1 \\
 \text{Right edge: } 1 - \varepsilon_1 \\
 \text{Bottom edge: } 1 \\
 \text{Bottom-left edge: } 1 \\
 \text{Bottom-right edge: } 1
 \end{array}
 \end{array}
 \right. \frac{J}{\varepsilon_2\varepsilon_3} \\
 + \begin{array}{c}
 \text{Diagram:} \\
 \begin{array}{c}
 \text{Top vertex: } 1 + \varepsilon_1 \\
 \text{Left edge: } 1 \\
 \text{Right edge: } 1 + \varepsilon_3 \\
 \text{Bottom edge: } 1 + \varepsilon_2 \\
 \text{Bottom-left edge: } 1 \\
 \text{Bottom-right edge: } 1 + \varepsilon_2
 \end{array}
 \end{array} \frac{1}{\varepsilon_1\varepsilon_2} + \begin{array}{c}
 \text{Diagram:} \\
 \begin{array}{c}
 \text{Top vertex: } 1 + \varepsilon_1 \\
 \text{Left edge: } 1 \\
 \text{Right edge: } 1 - \varepsilon_2 \\
 \text{Bottom edge: } 1 \\
 \text{Bottom-left edge: } 1 \\
 \text{Bottom-right edge: } 1 + \varepsilon_2
 \end{array}
 \end{array} \left. \frac{J}{\varepsilon_1\varepsilon_3} \right]$$

$$\begin{array}{c}
 \text{Diagram:} \\
 \begin{array}{c}
 \text{Top vertex: } 1 + \varepsilon_2 \\
 \text{Left edge: } 1 + \varepsilon_1 \\
 \text{Right edge: } 1 + \varepsilon_1 \\
 \text{Bottom edge: } 1 \\
 \text{Bottom-left edge: } 1 + \varepsilon_2 \\
 \text{Bottom-right edge: } 1 \\
 \text{Vertical edges: } 1 + \varepsilon_3, 1 + \varepsilon_3, 1 + \varepsilon_3
 \end{array}
 \end{array}
 = -\frac{1}{(1+\varepsilon_3)\varepsilon_3} J \left[
 \begin{array}{c}
 \text{Diagram:} \\
 \begin{array}{c}
 \text{Top vertex: } 1 + \varepsilon_1 \\
 \text{Left edge: } 1 - \varepsilon_1 \\
 \text{Right edge: } 1 \\
 \text{Bottom edge: } 1 \\
 \text{Bottom-left edge: } 1 \\
 \text{Bottom-right edge: } 1 + \varepsilon_1
 \end{array}
 \end{array}
 \right. \frac{J}{\varepsilon_2\varepsilon_3} \\
 + \begin{array}{c}
 \text{Diagram:} \\
 \begin{array}{c}
 \text{Top vertex: } 1 + \varepsilon_1 \\
 \text{Left edge: } 1 + \varepsilon_3 \\
 \text{Right edge: } 1 + \varepsilon_3 \\
 \text{Bottom edge: } 1 + \varepsilon_2 \\
 \text{Bottom-left edge: } 1 + \varepsilon_2 \\
 \text{Bottom-right edge: } 1 + \varepsilon_1
 \end{array}
 \end{array} \frac{J^{-1}}{\varepsilon_1\varepsilon_2} + \begin{array}{c}
 \text{Diagram:} \\
 \begin{array}{c}
 \text{Top vertex: } 1 + \varepsilon_2 \\
 \text{Left edge: } 1 - \varepsilon_2 \\
 \text{Right edge: } 1 - \varepsilon_2 \\
 \text{Bottom edge: } 1 \\
 \text{Bottom-left edge: } 1 + \varepsilon_2 \\
 \text{Bottom-right edge: } 1 + \varepsilon_2
 \end{array}
 \end{array} \left. \frac{J}{\varepsilon_1\varepsilon_3} \right]$$

Another form of the loop reduction in d=4 dimensions

$$\begin{array}{c} \text{Diagram 1: A ladder diagram with four horizontal legs and three vertical rungs. The top-left rung is labeled } 1 + \varepsilon_1, \text{ top-right } 1 + \varepsilon_2, \text{ middle-left } 1 + \varepsilon_3, \text{ middle-right } 1 + \varepsilon_3, \text{ bottom-left } 1 + \varepsilon_2, \text{ bottom-right } 1 + \varepsilon_1. \\ = \left[\begin{array}{c} \text{Diagram 2: A ladder diagram with four horizontal legs and three vertical rungs. The top-left rung is labeled } 1 + \varepsilon_1, \text{ top-right } 1 + \varepsilon_1, \text{ middle-left } 1 - \varepsilon_1, \text{ middle-right } 1 - \varepsilon_1, \text{ bottom-left } 1 + \varepsilon_1, \text{ bottom-right } 1. \\ \frac{J}{\varepsilon_2 \varepsilon_3} \end{array} \right] \\ + \begin{array}{c} \text{Diagram 3: A ladder diagram with four horizontal legs and three vertical rungs. The top-left rung is labeled } 1 + \varepsilon_1, \text{ top-right } 1 + \varepsilon_2, \text{ middle-left } 1 + \varepsilon_3, \text{ middle-right } 1 + \varepsilon_3, \text{ bottom-left } 1 + \varepsilon_1, \text{ bottom-right } 1 + \varepsilon_2. \\ \frac{1}{\varepsilon_1 \varepsilon_2} \end{array} \\ + \begin{array}{c} \text{Diagram 4: A ladder diagram with four horizontal legs and three vertical rungs. The top-left rung is labeled } 1 + \varepsilon_1, \text{ top-right } 1, \text{ middle-left } 1 - \varepsilon_2, \text{ middle-right } 1 - \varepsilon_2, \text{ bottom-left } 1, \text{ bottom-right } 1 + \varepsilon_2. \\ \frac{J}{\varepsilon_1 \varepsilon_3} \end{array} \end{array}$$



V. V. Belokurov and N. I. Usyukina, "Calculation Of Ladder Diagrams In Arbitrary Order," J. Phys. A **16** (1983) 2811.



Pedro Allendes, Bernd Kniehl, Igor Kondrashuk, Eduardo A. Notte Cuello, Marko Rojas Medar, "Solution to Bethe-Salpeter equation via Mellin-Barnes transform", arXiv:1205.6257 [hep-th], Nuclear Physics B 870 (2013) 243-277



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Bernd Kniehl, Igor Kondrashuk, Eduardo A. Notte Cuello, Ivan Parra Ferrada, Marko Rojas Medar, "Two-fold Mellin-Barnes transforms of Usyukina-Davydychev functions", arXiv:1304.3004 [hep-th], Nuclear Physics B 876 (2013) 322-333

Formulas to prove, IK's talks at ICMP 2015, Santiago

$$\oint_C dz_2 dz_3 D^{(u,v)}[1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3] \times \\ \times D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] = \\ J \left[\frac{D^{(u,v-\varepsilon_2)}[1 - \varepsilon_1]}{\varepsilon_2 \varepsilon_3} + \frac{D^{(u,v)}[1 + \varepsilon_3]}{\varepsilon_1 \varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1 - \varepsilon_2]}{\varepsilon_1 \varepsilon_3} \right],$$

$$D^{(z_2, z_3)}[\nu_1, \nu_2, \nu_3] = \frac{\Gamma(-z_2) \Gamma(-z_3) \Gamma(-z_2 - \nu_2 - \nu_3 + d/2)}{\prod_i \Gamma(\nu_i)} \\ \times \frac{\Gamma(-z_3 - \nu_1 - \nu_3 + d/2) \Gamma(z_2 + z_3 + \nu_3) \Gamma(\sum \nu_i - d/2 + z_3 + z_2)}{\Gamma(d - \sum \nu_i)},$$

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \quad \text{and} \quad D^{(u,v)}[1 + \nu] \equiv D^{(u,v)}[1, 1, 1 + \nu].$$

Barnes Lemmas, Smirnov's textbook

$$\oint_C dz \Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 - z) \Gamma(\lambda_4 - z) = \frac{\Gamma(\lambda_1 + \lambda_3) \Gamma(\lambda_1 + \lambda_4) \Gamma(\lambda_2 + \lambda_3) \Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)},$$

$$\oint_C dz \frac{\Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 + z) \Gamma(\lambda_4 - z) \Gamma(\lambda_5 - z)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + z)} = \frac{\Gamma(\lambda_1 + \lambda_4) \Gamma(\lambda_2 + \lambda_4) \Gamma(\lambda_3 + \lambda_4) \Gamma(\lambda_1 + \lambda_5) \Gamma(\lambda_2 + \lambda_5) \Gamma(\lambda_3 + \lambda_5)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5) \Gamma(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5) \Gamma(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)},$$

$$\oint_C dz \frac{\Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 - z) \Gamma(\lambda_4 - z)}{z} =$$

$$\frac{\Gamma(\lambda_1 + \lambda_3 - 1) \Gamma(\lambda_2 + \lambda_3) \Gamma(\lambda_2 + \lambda_4) \Gamma(\lambda_1 + \lambda_4 - 1)}{\Gamma(1 - \lambda_1) \Gamma(1 - \lambda_2)} \times$$

$$[\Gamma(1 - \lambda_1) \Gamma(1 - \lambda_2) - \Gamma(\lambda_3) \Gamma(\lambda_4)], \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2$$

The Integrant

$$\begin{aligned} & D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] = \\ & \frac{\Gamma(-z_2) \Gamma(-z_3) \Gamma(-z_2 + \varepsilon_2) \Gamma(-z_3 + \varepsilon_1) \Gamma(1 + z_2 + z_3)}{\Gamma(1 + \varepsilon_1) \Gamma(1 + \varepsilon_2) \Gamma(1 + \varepsilon_3)} \times \\ & \quad \times \Gamma(1 + z_2 + z_3 + \varepsilon_3), \\ & D^{(u, v)}[1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3] = \\ & \frac{\Gamma(-u) \Gamma(-v) \Gamma(-u + \varepsilon_1 + z_2) \Gamma(-v + \varepsilon_2 + z_3)}{\Gamma(1 + \varepsilon_1 - z_3) \Gamma(1 + \varepsilon_2 - z_2) \Gamma(1 + \varepsilon_3) \Gamma(1 + z_2 + z_3)} \times \\ & \quad \Gamma(1 + u + v + \varepsilon_3) \Gamma(1 - z_2 - z_3 + u + v), \\ & D^{(u, v)}[1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3] D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] \\ & = \frac{\Gamma(-u) \Gamma(-v) \Gamma(1 + u + v + \varepsilon_3)}{\Gamma(1 + \varepsilon_1) \Gamma(1 + \varepsilon_2) \Gamma^2(1 + \varepsilon_3)} \frac{1}{\varepsilon_1 - z_3} \frac{1}{\varepsilon_2 - z_2} \Gamma(-z_2) \Gamma(-z_3) \times \\ & \quad \times \Gamma(1 + z_2 + z_3 + \varepsilon_3) \Gamma(-u + \varepsilon_1 + z_2) \Gamma(-v + \varepsilon_2 + z_3) \times \\ & \quad \times \Gamma(1 - z_2 - z_3 + u + v). \end{aligned}$$

Simple trick, IK's talks at ICMP 2015, Santiago

$$\begin{aligned} & \frac{1}{z_3 - \varepsilon_1} \frac{1}{z_2 - \varepsilon_2} \Gamma(-z_2) \Gamma(-z_3) \Gamma(1 + z_2 + z_3 + \varepsilon_3) \Gamma(-u + \varepsilon_1 + z_2) \times \\ & \quad \times \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v) = \\ & = \frac{z_2 + z_2 + \varepsilon_3}{(z_3 - \varepsilon_1)(z_2 - \varepsilon_2)} \Gamma(-z_2) \Gamma(-z_3) \Gamma(z_2 + z_3 + \varepsilon_3) \Gamma(-u + \varepsilon_1 + z_2) \times \\ & \quad \times \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v) = \\ & = \left(\frac{1}{z_3 - \varepsilon_1} + \frac{1}{z_2 - \varepsilon_2} \right) \Gamma(-z_2) \Gamma(-z_3) \Gamma(z_2 + z_3 + \varepsilon_3) \Gamma(-u + \varepsilon_1 + z_2) \times \\ & \quad \times \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v). \end{aligned}$$

$$\oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(-z_2) \Gamma(-u + \varepsilon_1 + z_2) \oint_C dz_3 \Gamma(-z_3) \Gamma(z_2 + z_3 + \varepsilon_3) \times \\ \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v).$$

Application of the Barnes lemmas, IK's talks at ICMP 2015, Santiago

$$\begin{aligned} & \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(-z_2) \Gamma(-u + \varepsilon_1 + z_2) \oint_C dz_3 \Gamma(-z_3) \Gamma(z_2 + z_3 + \varepsilon_3) \times \\ & \quad \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v) = \\ & \quad \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(-z_2) \Gamma(-u + \varepsilon_1 + z_2) \times \\ & \quad \times \frac{\Gamma(z_2 + \varepsilon_3) \Gamma(-v + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + v) \Gamma(1 + \varepsilon_2 + u - z_2)}{\Gamma(1 + u - \varepsilon_1)} = \\ & = \frac{\Gamma(-v + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + v)}{\Gamma(1 + u - \varepsilon_1)} \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \times \\ & \quad \times \Gamma(-u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2). \end{aligned}$$

Another simple trick, IK's talks at ICMP 2015, Santiago

$$\begin{aligned} \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \Gamma(-u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2) = \\ \oint_C dz_2 \frac{1}{(z_2 - \varepsilon_2)(z_2 - u + \varepsilon_1)} \Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \Gamma(1 - u + \varepsilon_1 + z_2) \times \\ \times \Gamma(1 + \varepsilon_2 + u - z_2) = \\ -\frac{1}{u + \varepsilon_3} \oint_C dz_2 \left[\frac{1}{z_2 - \varepsilon_2} - \frac{1}{z_2 - u + \varepsilon_1} \right] \Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \times \\ \times \Gamma(1 - u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2) = \\ -\frac{1}{u + \varepsilon_3} \left[\oint_C dz_2 \frac{\Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \Gamma(1 - u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2)}{z_2 - \varepsilon_2} \right. \\ \left. - \oint_C dz_2 \frac{\Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \Gamma(1 - u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2)}{z_2 - u + \varepsilon_1} \right]. \end{aligned}$$

Shift of the contour, IK's talks at ICMP 2015, Santiago

$$\begin{aligned} & \oint_C dz_2 \frac{\Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \Gamma(1 - u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2)}{z_2 - \varepsilon_2} - \\ & - \oint_C dz_2 \frac{\Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \Gamma(1 - u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2)}{z_2 - u + \varepsilon_1} = \\ & \quad \oint_C dz_2 \frac{\Gamma(z_2 - \varepsilon_1) \Gamma(-z_2 - \varepsilon_2) \Gamma(1 - u - \varepsilon_3 + z_2) \Gamma(1 + u - z_2)}{z_2} - \\ & - \oint_C dz_2 \frac{\Gamma(z_2 + u + \varepsilon_3 - \varepsilon_1) \Gamma(-z_2 - u + \varepsilon_1) \Gamma(1 + z_2) \Gamma(1 - \varepsilon_3 - z_2)}{z_2}. \end{aligned}$$

Conjecture in $d = 4 - 2\epsilon$

$$\begin{array}{c}
 \text{Diagram 1:} \\
 \begin{array}{ccc}
 \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 2 \end{array} & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} \\
 \partial_{(2)}^2 & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} \\
 & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} &
 \end{array} \\
 = (-4\pi^2) & & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Diagram 2:} \\
 \begin{array}{ccc}
 \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} \\
 (\partial_{(2)}^2)^2 & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} \\
 & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} &
 \end{array} \\
 = (-4\pi^2)^2 & & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Diagram 3:} \\
 \begin{array}{ccc}
 \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} \\
 (\partial_{(2)}^2)^2 & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} \\
 & \begin{array}{c} 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array} &
 \end{array} \\
 = (-4\pi^2)^2 & & \begin{array}{c} 3 \\ | \\ 1-\epsilon \\ | \\ 1-\epsilon \\ | \\ 1 \end{array}
 \end{array}$$

Diagram with gluonic self-interaction in the limit $\epsilon \rightarrow 0$

$$\begin{aligned} & \left[\frac{13/2}{[12]^2[23]^2} + \frac{-33/2}{[12]^2[31]^2} + \frac{15/2}{[23]^2[31]^2} + \frac{11}{[12][23][31]^2} + \frac{-14}{[12][23]^2[31]} + \frac{10}{[12]^2[23][31]} \right] \\ & + \left[\frac{6}{[12][23]^2} + \frac{-6}{[12][31]^2} + \frac{[12]}{[23]^2[31]^2} + \frac{4[23]}{[12]^2[31]^2} + \frac{-2}{[12]^2[31]} + \frac{-5}{[23]^2[31]} + \frac{-2[31]}{[12]^2[23]^2} \right] J(1, 1, 1) \\ & + \left[\frac{-2}{[12]^2[23]^2} + \frac{6}{[12]^2[31]^2} + \frac{4}{[23]^2[31]^2} + \frac{1}{[12][23][31]^2} + \frac{-1}{[12][23]^2[31]} + \frac{-4}{[12]^2[23][31]} \right] \ln[12] \\ & + \left[\frac{-1/2}{[12]^2[23]^2} + \frac{-5}{[12]^2[31]^2} + \frac{-3/2}{[23]^2[31]^2} + \frac{-7/2}{[12][23][31]^2} + \frac{2}{[12][23]^2[31]} + \frac{7/2}{[12]^2[23][31]} \right] \ln[23] \\ & + \left[\frac{5/2}{[12]^2[23]^2} + \frac{-1}{[12]^2[31]^2} + \frac{-5/2}{[23]^2[31]^2} + \frac{5/2}{[12][23][31]^2} + \frac{-1}{[12][23]^2[31]} + \frac{1/2}{[12]^2[23][31]} \right] \ln[31]. \end{aligned}$$

$$[yz] = (y - z)^2, \quad [y1] = (y - x_1)^2$$



G. Cvetič and I. Kondrashuk, "Gluon self-interaction in the position space in Landau gauge," Int. J. Mod. Phys. A 23 (2008) 4145 [arXiv:0710.5762 [hep-th]].

Basis

Typical contribution is

$$\frac{(31)_\nu}{[31]^2} \int Dy \frac{(2y)_\sigma}{[2y]^2} \frac{(1y)_\lambda}{[1y]^2} \int Dz \left(\partial_\mu^{(z)} \Pi_{\rho\nu}(z3) \right) \Pi_{\rho\lambda}(zy) \Pi_{\mu\sigma}(z2),$$

the gluonic propagator is

$$\Pi_{\rho\lambda}(zy) = \frac{g_{\rho\lambda}}{[yz]^{1-\epsilon}} + 2(1-\epsilon) \frac{(yz)_\rho(yz)_\lambda}{[yz]^{2-\epsilon}}$$

Elements of the basis

$$\int Dy \int Dz \frac{1}{[31]^{1-\epsilon}} \frac{1}{[2y]^{1-\epsilon}} \frac{1}{[3z]^{1-\epsilon}} \frac{1}{[z2]^{1-\epsilon}} \partial_\mu^{(y)} \frac{1}{[1y]^{1-\epsilon}} \partial_\mu^{(y)} \frac{1}{[yz]^{1-\epsilon}}$$

Reduction of the basis to ladder diagrams

$$\begin{aligned} \int Dy Dz \frac{1}{[31]^{1-\epsilon}} \frac{1}{[2y]^{1-\epsilon}} \frac{1}{[3z]^{1-\epsilon}} \frac{1}{[z2]^{1-\epsilon}} \partial_\mu^{(y)} \frac{1}{[1y]^{1-\epsilon}} \partial_\mu^{(y)} \frac{1}{[yz]^{1-\epsilon}} = \\ \frac{1}{2} \int Dy Dz \frac{1}{[31]^{1-\epsilon}} \frac{1}{[2y]^{1-\epsilon}} \frac{1}{[3z]^{1-\epsilon}} \frac{1}{[z2]^{1-\epsilon}} \times \\ \times \left[\partial_{(y)}^2 \left(\frac{1}{[1y]^{1-\epsilon}} \frac{1}{[yz]^{1-\epsilon}} \right) - \left(\partial_{(y)}^2 \frac{1}{[1y]^{1-\epsilon}} \right) \frac{1}{[yz]^{1-\epsilon}} \right. \\ \left. - \frac{1}{[1y]^{1-\epsilon}} \left(\partial_{(y)}^2 \frac{1}{[yz]^{1-\epsilon}} \right) \right] \end{aligned}$$