New Methods for Multi-Loop Feynman Integrals

Andreas v. Manteuffel

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MULTI-LOOP FEYNMAN INTEGRALS

$$\boxed{I = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \, \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \qquad a_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.}}$$

linear dependencies:

- integration-by-parts (IBP) identities [Tkachov, Chetyrkin '81]
- systematic reduction to master integrals [Laporta '00]
- think of it as linear vector space with some arbitrary basis
- for explicite tensor reduction: combine with covariant decomposition

this talk: two solving methods and a reduction trick

1. differential equations in kinematic invariants
2. direct integration of Feynman parameters
3. integration-by-parts reduction via finite field sampling

Part I: Two-loop amplitudes for diboson production

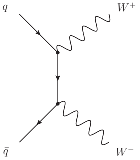
[Gehrmann, AvM, Tancredi, Weihs]

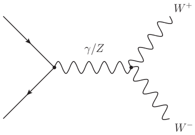
VECTOR BOSON PAIR PRODUCTION AT LHC

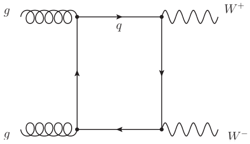
$pp \to VV' + X \to 4 \text{ leptons} + X$, where $VV' = ZZ$, W^+W^-, $\gamma^*\gamma^*$, $ZW^\pm$, $Z\gamma^*$, $W^\pm\gamma^*$

- sensitive to details of EWSB
- possible NP contributions at tree or loop level

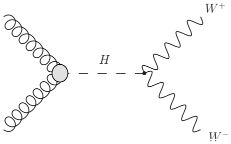
e.g. W^+W^- production:







important background to Higgs signals:



QCD full NNLO (equal masses):

- master integrals: [Gehrmann, Tancredi, Weihs '13; Gehrmann, AvM, Tancredi, Weihs '14]
- amplitudes: [Gehrmann, AvM, Tancredi (unpublished)]
- ZZ@NNLO [Cascioli,Gehrmann,Grazzini,Kallweit,Maierhöfer,AvM,Pozzorini,Rathlev,Tancredi,Weihs'14]
- WW@NNLO [Gehrmann, Grazzini, Kallweit, Maierhöfer, AvM, Pozzorini, Rathlev, Tancredi '14]

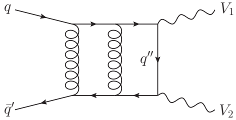
QCD full NNLO (different masses):

- master integrals: [Henn, Melnikov, Smirnov '14; Caola, Henn, Melnikov, Smirnov '14]; [Papadopoulos, Tammasini, Wever '14]; [Gehrmann, AvM, Tancredi '15]
- amplitudes $q\bar{q}' \to VV'$: [Caola, Henn, Melnikov, Smirnov '14]; [Gehrmann, AvM, Tancredi '15]
- amplitudes $gg \to VV'$: [Caola, Henn, Melnikov, Smirnov '15]; [AvM, Tancredi '15]
- $\gamma^*\gamma^*$@NNLO (partial): [Anastasiou, Cancino, Chavez, Duhr, Lazopoulos, Mistlberger, Müller '14]
- fully differential off-shell ZZ production [Grazzini, Rathlev, Kallweit '15]
- gg channel for W^+W^-, ZZ @NLO [Caola, Melnikov, Roentsch, Tancredi '15, '15]
- p_T spectra for W^+W^-, ZZ @NNLL+NNLO [Grazzini, Rathlev, Kallweit, Wiesemann '15]

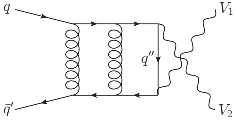
FEYNMAN DIAGRAMS (generated with Qgraf [Nogueira])

$q\bar{q}'$ channel (just non-zero classes shown):

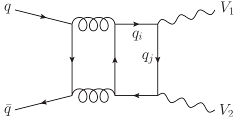
[A]



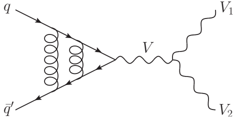
[B]



[C]

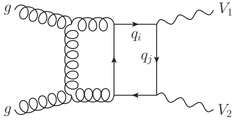


[F_V]

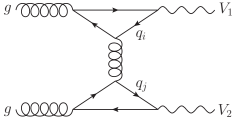


gg channel ([B] and [F_V] do not contribute):

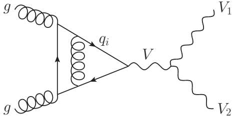
[A]



[B]

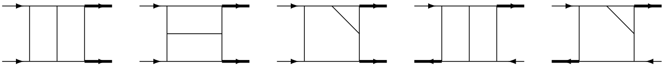


[F_V]

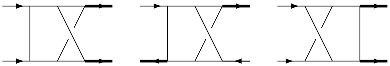


AMPLITUDE CALCULATION FOR $q\bar{q}' \to VV'$ AND $gg \to VV'$

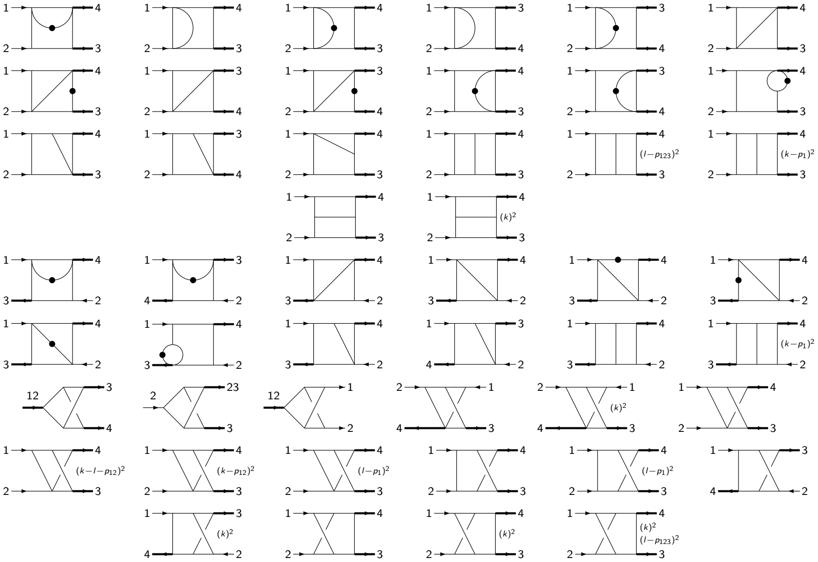
- helicity amplitudes: employ covariant decomposition + projectors onto scalar form factors
- shift finding + IBP reduction of multiloop integrals: `Reduze 2` [AvM, Studerus]
- end up with linear combination of 84 master integrals (w/ products, w/o crossings):
- planar top-level graphs



- non-planar top-level graphs



MASTER INTEGRALS FOR $q\bar{q}' \to VV'$ AND $gg \to VV'$



AN IMPROVED BASIS FOR DIFFERENTIAL EQUATIONS

- method by [Kotikov '91]; [Gehrmann, Remiddi '99], relies on IBP reduction
- system of diff. eqns for basis integrals wrt external invariants ($\epsilon = (4-d)/2$):

$$\frac{\partial}{\partial s_i}\vec{M}(\epsilon, s) = \bar{\mathbf{A}}^{(s_i)}(\epsilon, s)\vec{M}(\epsilon, s)$$

- in certain cases proper choice of basis achieves [Kotikov '10]; [Henn '13]:

$$\bar{\mathbf{A}}^{(s_i)}(\epsilon, s) = \epsilon\, \mathbf{A}^{(s_i)}(s)$$

such that

$$\boxed{\mathrm{d}\vec{M}(\epsilon, s) = \epsilon \sum\nolimits_n \mathbf{A}^{(n)}\, \mathrm{d}\ln l_n(s)\ \vec{M}(\epsilon, s)}$$

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such that

$$\boxed{\mathrm{d}\vec{M}(\epsilon, s) = \epsilon \textstyle\sum_n \mathbf{A}^{(n)}\, \mathrm{d}\ln l_n(s)\ \vec{M}(\epsilon, s)}$$

features:

- full decoupling after expansion in ϵ:

$$\vec{M} = \vec{M}^{(0)} + \epsilon\vec{M}^{(1)} + \dots$$
$$\mathrm{d}\vec{M}^{(k)}(s) = \sum_n \mathbf{A}^{(n)}\, \mathrm{d}\ln l_n(s)\ \vec{M}^{(k-1)}(s)$$

- every term of ϵ expansion: multiple polylogs of uniform weight
- applies also to phase space integrals [Höschele, Hoff, Ueda '14]; [AvM, Schabinger, Zhu '14], ...
- construction of canonical form: [Lee '14], [Tancredi '15]

Structure of result

vector of 111 master integrals in canonical basis with alphabet:

$$
\begin{aligned}
\{\bar{l}_1, \ldots, \bar{l}_{20}\} = \{&2, \bar{x}, 1+\bar{x}, 1-\bar{y}, \bar{y}, 1+\bar{y}, 1-\bar{x}\bar{y}, 1+\bar{x}\bar{y}, 1-\bar{z}, \bar{z},\\
&1+\bar{y}-2\bar{y}\bar{z}, 1-\bar{y}+2\bar{y}\bar{z}, 1+\bar{x}\bar{y}-2\bar{x}\bar{y}\bar{z}, 1-\bar{x}\bar{y}+2\bar{x}\bar{y}\bar{z},\\
&1+\bar{y}+\bar{x}\bar{y}+\bar{x}\bar{y}^2-2\bar{y}\bar{z}-2\bar{x}\bar{y}\bar{z}, 1+\bar{y}-\bar{x}\bar{y}-\bar{x}\bar{y}^2-2\bar{y}\bar{z}+2\bar{x}\bar{y}\bar{z},\\
&1-\bar{y}-\bar{x}\bar{y}+\bar{x}\bar{y}^2+2\bar{y}\bar{z}+2\bar{x}\bar{y}\bar{z}, 1-\bar{y}+\bar{x}\bar{y}-\bar{x}\bar{y}^2+2\bar{y}\bar{z}-2\bar{x}\bar{y}\bar{z},\\
&1-2\bar{y}-\bar{x}\bar{y}+\bar{y}^2+2\bar{x}\bar{y}^2-\bar{x}\bar{y}^3+4\bar{y}\bar{z}+2\bar{x}\bar{y}\bar{z}+2\bar{x}\bar{y}^3\bar{z},\\
&1-\bar{y}-2\bar{x}\bar{y}+2\bar{x}\bar{y}^2+\bar{x}^2\bar{y}^2-\bar{x}^2\bar{y}^3+2\bar{y}\bar{z}+4\bar{x}\bar{y}\bar{z}+2\bar{x}^2\bar{y}^3\bar{z}\}
\end{aligned}
$$

in parametrisation which rationalizes root of Källén function $\sqrt{s^2+p_3^4+p_4^4-2(s\,p_3^2+p_3^2\,p_4^2+p_4^2\,s)}$:

$$s=\bar{m}^2(1+\bar{x})^2, \quad t=-\bar{m}^2\bar{x}((1+\bar{y})(1+\bar{x}\bar{y})-2\bar{z}\bar{y}(1+\bar{x})), \quad p_3^2=\bar{m}^2\bar{x}^2(1-\bar{y}^2), \quad p_4^2=\bar{m}^2(1-\bar{x}^2\bar{y}^2)$$

integrated in terms of:

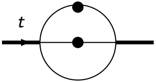
Multiple polylogarithms [Remiddi, Gehrmann]; [Goncharov]

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t),$$

- all difficult boundary values fixed by regularity
- checked against `SecDec 2` [Borowka, Carter, Heinrich '12]
- symbol and more [Brown '11], [Duhr '12], [Duhr, Gangl, Rhodes '11], [Vollinga, Weinzierl '04]

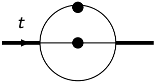
example result: (dots are squared propagators)

in traditional functional basis:

$-\epsilon^2 \bar{m}^{2\epsilon} t$ 

$$
\begin{aligned}
&= 1 \\
&\quad + \epsilon\big[- 2G(-1, \bar{y}) - 2G(0, \bar{x}) - 2G(-1/\bar{y}, \bar{x}) - 2G(((1 + \bar{y})(1 + \bar{x}\bar{y}))/(2(1 + \bar{x})\bar{y}), \bar{z})\big] \\
&\quad + \epsilon^2\big[4G(0, \bar{x})G(((1 + \bar{y})(1 + \bar{x}\bar{y}))/(2(1 + \bar{x})\bar{y}), \bar{z}) + 4G(-1/\bar{y}, \bar{x})G(((1 + \bar{y})(1 + \bar{x}\bar{y}))/(2(1 + \bar{x})\bar{y}), \bar{z}) \\
&\qquad + G(-1, \bar{y})(4G(0, \bar{x}) + 4G(-1/\bar{y}, \bar{x}) + 4G(((1 + \bar{y})(1 + \bar{x}\bar{y}))/(2(1 + \bar{x})\bar{y}), \bar{z})) + 4G(-1, -1, \bar{y}) \\
&\qquad + 4G(0, 0, \bar{x}) + 4G(0, -1/\bar{y}, \bar{x}) + 4G(-1/\bar{y}, 0, \bar{x}) + 4G(-1/\bar{y}, -1/\bar{y}, \bar{x}) \\
&\qquad + 4G(((1 + \bar{y})(1 + \bar{x}\bar{y}))/(2(1 + \bar{x})\bar{y}), ((1 + \bar{y})(1 + \bar{x}\bar{y}))/(2(1 + \bar{x})\bar{y}), \bar{z})\big] \\
&\quad + O(\epsilon^3)
\end{aligned}
$$

in optimized functional basis for numerical evaluation:

$-\epsilon^2 m^{2\epsilon} t$ 

$$
= 1 + \epsilon\big[- 2\ln(l_1) - 2\ln(l_5)\big] + \epsilon^2\big[2\ln^2(l_1) + 4\ln(l_1)\ln(l_5) + 2\ln^2(l_5)\big] + O(\epsilon^3)
$$

OPTIMISED FUNCTIONAL BASIS

choose real valued $\ln l_i$, $\mathrm{Li}_n(R_1)$, $\mathrm{Li}_{2,2}(R_1, R_2)$ with

$$|R_1| < 1, \qquad |R_1 R_2| < 1$$

where R_i are power products of letters (e.g. $-l_1, l_3, -l_8/(l_1 l_3), \ldots$)

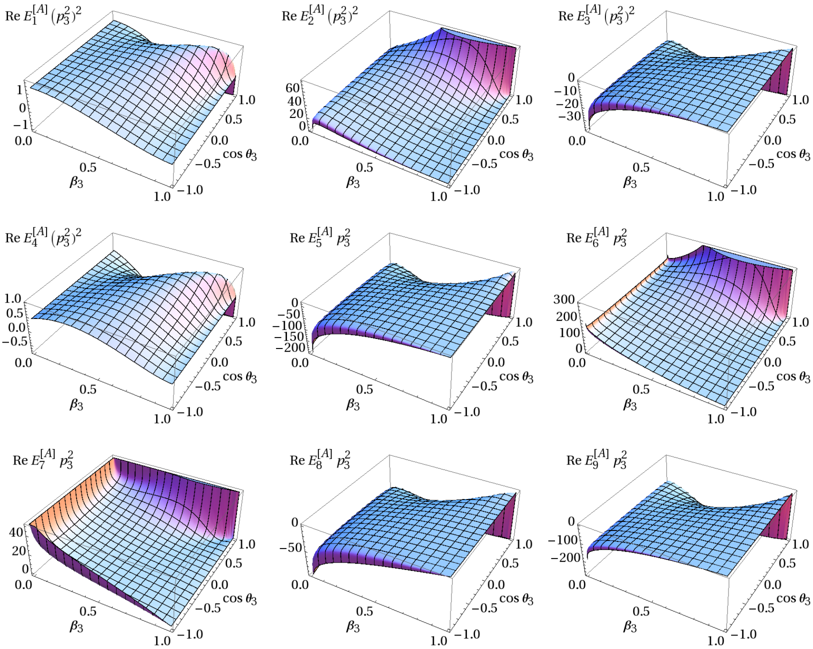
such that Li functions have convergent power series

$$\mathrm{Li}_n(R_1) = -\sum_{j_1=1}^{\infty} \frac{R_1^{j_1}}{j_1^n}, \qquad \mathrm{Li}_{2,2}(R_1, R_2) = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{R_1^{j_1}}{(j_1 + j_2)^2} \frac{(R_1 R_2)^{j_2}}{j_2^2}$$

features:

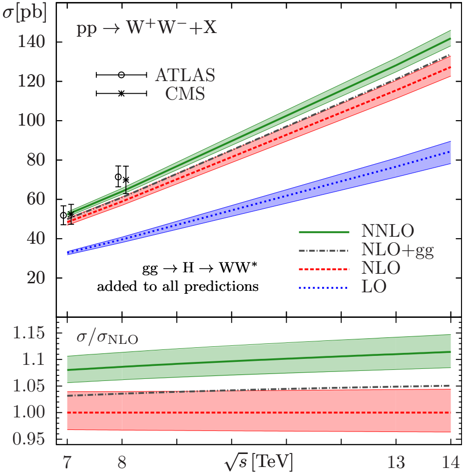
- symbol based rewriting [Goncharov, Spradlin, Vergu, Volovich '10]
- algorithmic argument construction [Duhr, Gangl, Rhodes '11]
- require absence of spurious letters
- fast and stable numerical evaluation:
  - **O(150ms)** full off-shell helicity amplitudes for $q\bar{q}' \to VV' \to 4$ leptons
  - O(35ms) equal mass interferences
  - orders of magnitude faster than traditional representation

helicity amplitudes for $q\bar{q}' \to VV'$ @ 2-loops [Gehrmann, AvM, Tancredi '15]



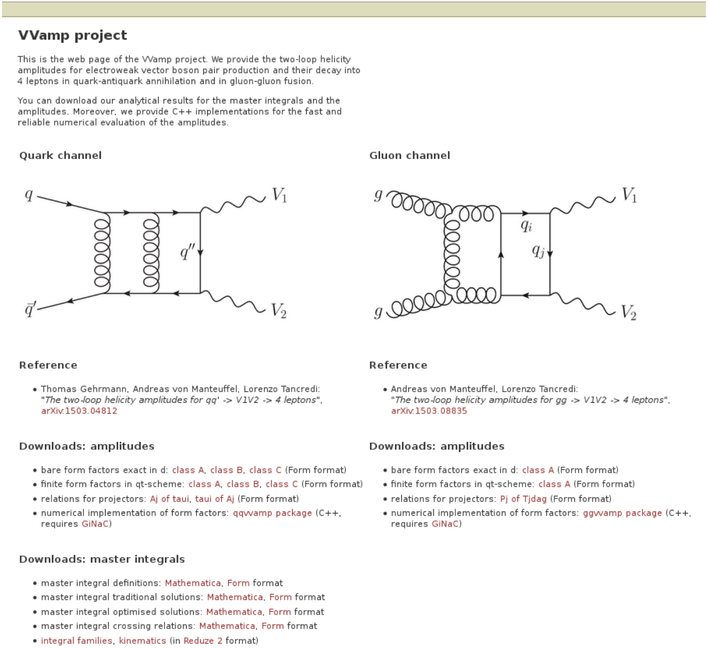
RESULT: W^+W^- PRODUCTION AT NNLO

NNLO prediction significantly reduces tension with data:



[Gehrmann, Grazzini, Kallweit, Maierhöfer, AvM, Pozzorini, Rathlev, Tancredi '14]

HTTPS://VVAMP.HEPFORGE.ORG



VVamp project

This is the web page of the VVamp project. We provide the two-loop helicity amplitudes for electroweak vector boson pair production and their decay into 4 leptons in quark-antiquark annihilation and in gluon-gluon fusion.

You can download our analytical results for the master integrals and the amplitudes. Moreover, we provide C++ implementations for the fast and reliable numerical evaluation of the amplitudes.

Quark channel

Reference

- Thomas Gehrmann, Andreas von Manteuffel, Lorenzo Tancredi: "*The two-loop helicity amplitudes for qq' -> V1V2 -> 4 leptons*", arXiv:1503.04812

Downloads: amplitudes

- bare form factors exact in d: class A, class B, class C (Form format)
- finite form factors in qt-scheme: class A, class B, class C (Form format)
- relations for projectors: Aj of taui, taui of Aj (Form format)
- numerical implementation of form factors: qqvvamp package (C++, requires GiNaC)

Gluon channel

Reference

- Andreas von Manteuffel, Lorenzo Tancredi: "*The two-loop helicity amplitudes for gg -> V1V2 -> 4 leptons*", arXiv:1503.08835

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- relations for projectors: Pj of Tjdag (Form format)
- numerical implementation of form factors: ggvvamp package (C++, requires GiNaC)

Downloads: master integrals

- master integral definitions: Mathematica, Form format
- master integral traditional solutions: Mathematica, Form format
- master integral optimised solutions: Mathematica, Form format
- master integral crossing relations: Mathematica, Form format
- integral families, kinematics (in Reduze 2 format)

Part II: A basis of finite Feynman integrals

[AvM, Panzer, Schabinger]

AN IMPROVED BASIS FOR FEYNMAN PARAMETERS

consider Feynman parameter representation of multi-loop integral

$$\boxed{I = N\left[\prod_{j=1}^{N}\int_0^\infty \mathrm{d}x_j x_j^{\nu_k-1}\right]\delta(1-x_N)\,\mathcal{U}^{\nu-(L+1)\frac{d}{2}}\,\mathcal{F}^{-\nu+L\frac{d}{2}}}$$

where

- $\nu = \sum_i \nu_i$, ν_i denotes propagator multiplicity
- $\mathcal{U}$ and $\mathcal{F}$ are Symanzik polynomials in x_i

problem:

- want expansion around $\epsilon = (4-d)/2$
- but have divergencies from x_i integrations
- can't directly expand in ϵ
- no straight-forward analytical or numerical integration

generic approaches to singularity resolution:

1. sector decomposition [Hepp '66, Binoth, Heinrich '00]
2. polynomial exponent raising [Bernstein '72, Tkachov '96, Passarino '00]
3. analytic regularisation [Panzer '14]
4. basis of finite Feynman integrals ("dims & dots") [AvM, Schabinger, Panzer '14]

SECTOR DECOMPOSITION

- very established method + codes
- but not always ideal: for example, calculate to $\mathcal{O}(\epsilon)$:

$$I(\epsilon) = \int_0^1 \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon} {}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

decompose into sectors: split at (arbitrary) $t = 1/2$, rescale, expand in plus distributions:

$$\begin{aligned} I_1(\epsilon) &= -\frac{1}{\epsilon} - 1 + \left(3 + \frac{1}{3}\pi^2 - 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right) \\ I_2(\epsilon) &= -\frac{1}{3\epsilon} + \frac{7}{3} + \left(-7 + \frac{1}{3}\pi^2 + 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right) . \end{aligned}$$

result:

$$I(\epsilon) = -\frac{4}{3\epsilon} + \frac{4}{3} + \left(-4 + \frac{2}{3}\pi^2\right)\epsilon + \mathcal{O}\left(\epsilon^2\right) .$$

split up of domain introduces spurious terms ln(2)

- can be worse: spurious order 5 polynomial denominators: [AvM, Schabinger, Zhu '13]
- destroys linear reducibility: no analytical integration a la [Brown '08; Panzer '14; Bogner '15]

ANALYTIC REGULARISATION [PANZER '14]

Euclidean integrals: all subdivergencies from integration boundaries

- check: rescale $x_j \to \lambda x_j$ or x_j/λ for some $j \in J$
- problematic scaling of integrand for $\lambda \to 0$ signals divergency
- convergence can be improved by regularising trafo based on partial integration: new integrand

  $$P' = -\frac{1}{\omega_J(P)} \frac{\partial}{\partial\lambda} \lambda^{-\deg_J(P)} P_{J_\lambda}\Big|_{\lambda\to 1} .$$

  iterate if necessary
- maps original integral to sum of dimensionally shifted integrals with higher powers of propagators (dots)

shortcomings:

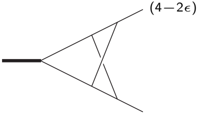
- proliferation of terms
- ambiguities
- spurious poles in ϵ

way out:

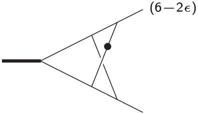
- consider full set of master integrals (basis)
- employ integration by parts (IBP) reductions

NEW PROPOSAL: MINIMAL DIMS & DOTS [AvM, Panzer, Schabinger '14]

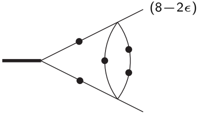
observation: always possible to decompose wrt basis of finite integrals



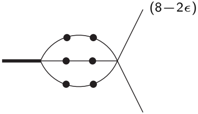
$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2q^2}$$



$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2q^2}$$



.

basis consists of standard Feynman integrals, but

- in shifted dimensions
- with additional dots (propagators taken to higher powers)
- old reg. shifts generated $\mathcal{O}(10\text{MB})$, here: 3 lines ! (more severe at higher loops)

existence of finite basis:

- proof along the lines of regularisation above, but pick one term on rhs

PRACTICAL ALGORITHM FOR BASIS CONSTRUCTION

given the existence proof, forget about previous construction and just do:

ALGORITHM: CONSTRUCTION OF FINITE BASIS

- systematic scan for finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

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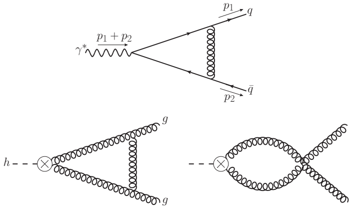
remarks:

- computationally expensive part shifted to IBP solver (`Fire`, `Reduze`, `LiteRed`)
- efficient, easy to automate (implemented in dev. version of `Reduze` 2)
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon

APPLICATION: MASSLESS FORM FACTORS

- massless quark and gluon form factors



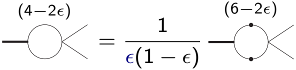
- purely virtual corrections to
  - Higgs production in gluon-fusion
  - Drell-Yan production

- simplest objects to study IR properties of QCD
  - cusp anomalous dimensions $1/\epsilon^2$: Casimir scaling ?
  - collinear anomalous dimensions $1/\epsilon$

- notation: $(p_1^2 + p_2)^2 = -1$

FORM FACTORS @ 1-LOOP

- consider one-loop quark and gluon form factors in massless QCD
- integral basis change to finite integrals

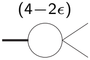


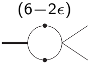
$$= \frac{1}{\epsilon(1-\epsilon)}$$

dot: squared propagator, subscript: space-time dimension

FORM FACTORS @ 1-LOOP

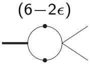
- consider one-loop quark and gluon form factors in massless QCD
- integral basis change to finite integrals



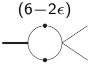
$= \frac{1}{\epsilon(1-\epsilon)}$


dot: squared propagator, subscript: space-time dimension

- form factors

$\mathcal{F}_1^q(\epsilon) = C_F \, \frac{1}{\epsilon^2} \, a_1$


$\qquad a_1 = \frac{-2+\epsilon-2\epsilon^2}{1-\epsilon}$

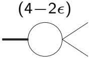
$\mathcal{F}_1^g(\epsilon) = C_A \, \frac{1}{\epsilon^2} \, b_1$


$, \qquad b_1 = \frac{-2(1-3\epsilon+2\epsilon^2+\epsilon^3)}{(1-\epsilon)^2}$

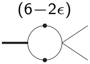
note: all divergencies explicit

FORM FACTORS @ 1-LOOP

- consider one-loop quark and gluon form factors in massless QCD
- integral basis change to finite integrals



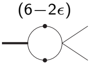
$$= \frac{1}{\epsilon(1-\epsilon)}$$



dot: squared propagator, subscript: space-time dimension

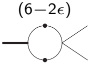
- form factors

$$\mathcal{F}_1^q(\epsilon) = C_F \, \frac{1}{\epsilon^2} \, a_1$$



$$a_1 = \frac{-2+\epsilon-2\epsilon^2}{1-\epsilon}$$

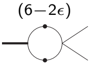
$$\mathcal{F}_1^g(\epsilon) = C_A \, \frac{1}{\epsilon^2} \, b_1$$



$$, \qquad b_1 = \frac{-2(1-3\epsilon+2\epsilon^2+\epsilon^3)}{(1-\epsilon)^2}$$

note: all divergencies explicit

- expansion in ϵ



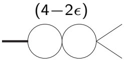
$$= 1 + \epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$a_1 = -2 - \epsilon - 3\epsilon^2 + \mathcal{O}(\epsilon^3)$$

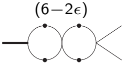
$$b_1 = -2 + 2\epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)$$

- Casimir scaling reflected by $a_1|_{\epsilon=0} = b_1|_{\epsilon=0}$

FORM FACTORS @ 2-LOOPS: TO FINITE BASIS



$$= \frac{1}{\epsilon^2}\frac{1}{(1-\epsilon)^2}$$



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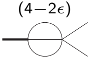
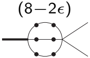


$$= \frac{1}{\epsilon}\frac{-4}{(2-\epsilon)^2(1-\epsilon)^2(1-2\epsilon)}$$



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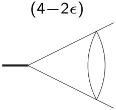
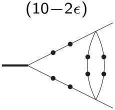


$$= \frac{1}{\epsilon^2}\frac{16(3-2\epsilon)(2-3\epsilon)}{(3-\epsilon)^2(2-\epsilon)^2(1-\epsilon)^3(1+2\epsilon)}$$



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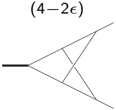
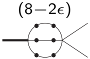
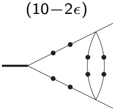


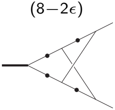
$$= \frac{1}{\epsilon^4}\frac{-4(2-3\epsilon)(14-81\epsilon+115\epsilon^2+14\epsilon^3-132\epsilon^4+72\epsilon^5)}{(2-\epsilon)^2(1-\epsilon)^2(1-2\epsilon)^2(2-\epsilon-2\epsilon^2)}$$



$$+ \frac{1}{\epsilon^4}\frac{-16(1+\epsilon)(3-2\epsilon)(2-3\epsilon)(10-61\epsilon+102\epsilon^2-44\epsilon^3-8\epsilon^4)}{(3-\epsilon)^2(2-\epsilon)^2(1-\epsilon)^3(1-2\epsilon)(1+2\epsilon)(2-\epsilon-2\epsilon^2)}$$



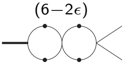
$$+ \frac{1}{\epsilon}\frac{4(3-4\epsilon)(1-4\epsilon)}{(2-\epsilon)(1-\epsilon)(1-2\epsilon)(2-\epsilon-2\epsilon^2)}$$



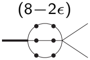
FORM FACTORS @ 2-LOOPS

quark form factor

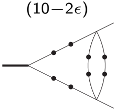
$$\mathcal{F}_2^q(\epsilon) = C_F^2 \Bigg\{ \frac{1}{\epsilon^4} \Bigg[c_1$$



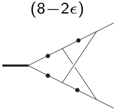
$$+ c_2$$



$$\Bigg] + \frac{1}{\epsilon^3} \Bigg[c_3$$

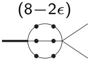


$$\Bigg] + \frac{1}{\epsilon} \Bigg[c_4$$

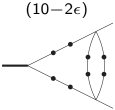


$$\Bigg] \Bigg\}$$

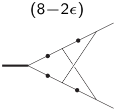
$$+ C_F C_A \Bigg\{ \frac{1}{\epsilon^4} \Bigg[c_5$$



$$+ c_6$$

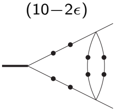


$$\Bigg] + \frac{1}{\epsilon} \Bigg[c_7$$



$$\Bigg] \Bigg\}$$

$$+ C_F N_f \Bigg\{ \frac{1}{\epsilon^3} \Bigg[c_8$$



$$\Bigg] \Bigg\}$$

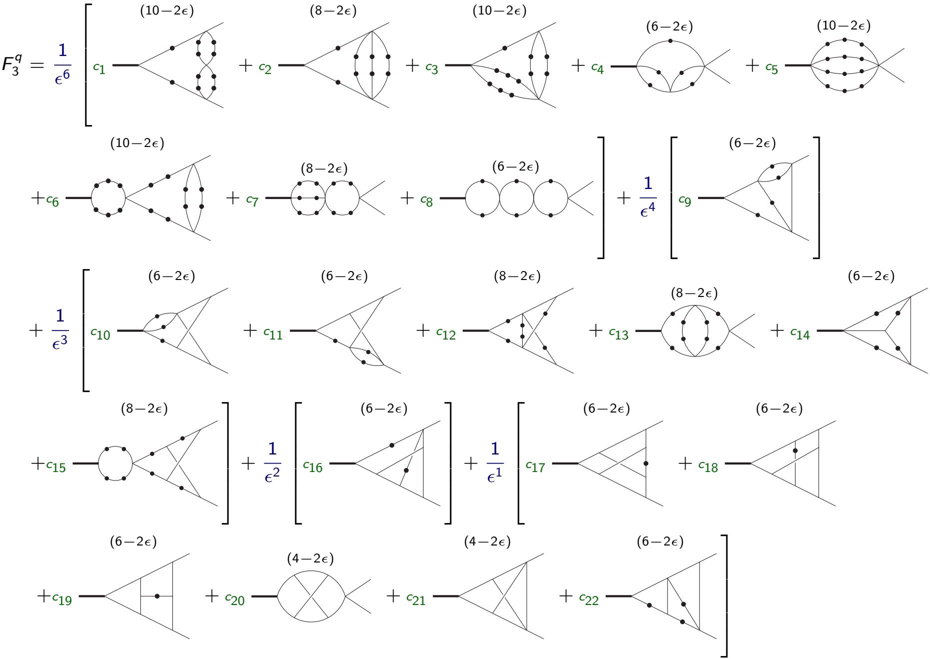
FORM FACTORS @ 3-LOOPS

- master integrals:
  - [Gehrmann, Heinrich, Huber, Studerus '06]
  - [Heinrich, Huber, Maître '07]
  - [Heinrich, Huber, Kosower, V. Smirnov '09]
  - [Lee, A. Smirnov, V. Smirnov '10]
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Lee, V. Smirnov '10] ⇐ the only complete weight 8
  - [Henn, A. Smirnov, V. Smirnov '14] (diff. eqns.)

- form factors @ 3-loops:
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]

- recalculation of all results via finite integrals:
  - [AvM, Panzer, Schabinger '15]
  - automated setup, fully analytical
  - `Qgraf` [Nogueira]:
    - Feynman diagrams
  - `Reduze 2` [AvM, Studerus]:
    - interferences
    - IBP reductions
    - finite integral finder
    - basis change with dimensional recurrences
  - `HyperInt` [Panzer]:
    - integration of ϵ expanded master integrals

QUARK FORM FACTOR @ 3-LOOPS [AvM, Panzer, Schabinger '15]



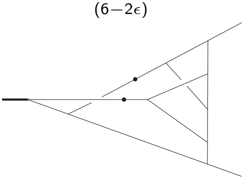
TOWARDS THE CUSP ANOMALOUS DIMENSION @ 4-LOOPS

- cusp anomalous dimension required for N^{3}LL resummation
- Casimir scaling ?
- reduced integrand for $\mathcal{N} = 4$: [Boels, Kniehl, Yang '15]
- QCD cusp anomalous dimension:

  in our basis: no contributions from most complicated topologies through to 3-loops !

  useful also at 4-loops because not all $\mathcal{O}(300)$ master integrals linearly reducible ?

- a non-planar 12-line topology @ 4-loops [AvM, Panzer, Schabinger '15]:



$$= \tfrac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \Big(24\zeta_2\zeta_3 + 20\zeta_5 - \tfrac{188}{105}\zeta_2^3 - 17\zeta_3^2 + 9\zeta_2^2\zeta_3 - 47\zeta_2\zeta_5 - 21\zeta_7 + \tfrac{6883}{2100}\zeta_2^4 + \tfrac{49}{2}\zeta_2\zeta_3^2 + \tfrac{1}{2}\zeta_3\zeta_5 - 9\zeta_{5,3}\Big)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

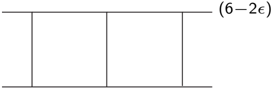
only shallow ϵ expansion needed

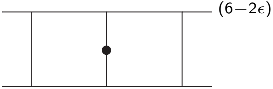
numerical result with `Fiesta` [A. Smirnov]: straight-forward confirmation to 4 digits

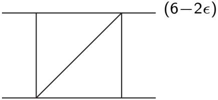
starts at weight 7, not expected to contribute to cusp anomalous dimension

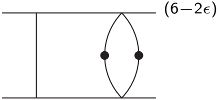
SCOPE OF THE METHOD

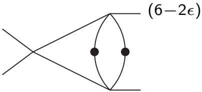
- "dims & dots method" general and automated
- e.g. basis of quasi-finite integrals for massless planar double boxes

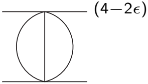
$b_1 =$


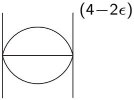
$b_2 =$


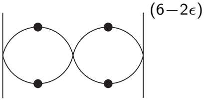
$b_3 =$


$b_4 =$


$b_5 =$


$b_6 =$


$b_7 =$


$b_8 =$


- works for integrals beyond multiple polylogarithms
- works for physical kinematics

NUMERICAL EVALUATIONS

advantages of (quasi-)finite basis:

- straight-forward to integrate numerically (in principle)
- no cancellation of spurious singularities (stability)
- no blow up in number of numerical integrations (speed, stability)
- very simple integrands also at high orders in ϵ (speed)

experiments with numerical evaluations:

- naive straight-forward implementation works already reasonably well
- convenient: employ existing sector decomposition programs
  - `Fiesta` [A. Smirnov]
  - `SecDec` [Borowka, Heinrich et al]
  - `sector_decomposition` [Bogner, Weinzierl]
- finite integrals: much faster & much more reliable

Part III: Integration-by-parts reductions via finite fields

[AvM, Schabinger]

REDUCTIONS USING INTEGRATION BY PARTS (IBP) IDENTITIES

- IBP reductions: express arbitrary integral for given problem via few basis integrals [Chetyrkin, Tkachov '81]
- integral and amplitude calculations discussed above all relied on IBP reductions
- reductions are typically the performance bottleneck
- example: massive 1-loop tadpole

$$\int d^d k \frac{1}{k^2 - m^2}$$

calculating IBP identity

$$\begin{aligned} 0 = & \int d^d k \frac{\partial}{\partial k_\mu} \left(k_\mu \frac{1}{k^2 - m^2} \right) \\ = & (d-2) \int d^d k \frac{1}{k^2 - m^2} - 2m^2 \int d^d k \frac{1}{(k^2 - m^2)^2} \end{aligned}$$

gives directly reduction of integral with additional numerator

$$\int d^d k \frac{1}{(k^2 - m^2)^2} = \frac{(d-2)}{2m^2} \int d^d k \frac{1}{k^2 - m^2}$$

general case: consider all loop and external momenta for IBP operators

SOLVING IBP SYSTEMS

- general case: recurrence relations not immediate
- relations for generic propagator exponents a_i ideal, but difficult
  - generic codes attempting this: [Smirnov, Smirnov]: `Fire`, [Lee]: `LiteRed`
  - recent ideas: [Ita '15], [Larsen, Zhan '15], [Chen, Liu, Xie, Zhang, Zhou '15], [Badger et al '12-'15], [Ueda, Vermaseren '15]
- in practice: relations for explicit propagator exponents a_i: "Laporta algorithm"
  - codes: `AIR` [Anastasiou], `FIRE` [Smirnov], `Reduze 1` [Studerus], `Reduze 2` [AvM, Studerus]

LAPORTA'S ALGORITHM

1. generate integration by parts identities (IBPs) for explicit integrals
2. results in sparse system of equations
3. solve linear system of equations

shortcomings of traditional system solving:

- coefficients in linear system of equations are multivariate rational functions
- Gaussian elimination: suffers from intermediate expression swell
- requires large number of auxiliary integrals and equations

IBP REDUCTIONS FROM FINITE FIELD SAMPLES

A NOVEL APPROACH TO IBPS [AvM, SCHABINGER '14]

1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field
2. solve finite field system
3. reconstruct rational solution from many such samples

advantages:

- no intermediate expression swell by construction
- massively parallisable
- techniques established in math literature + used in modern software

core algorithm:

EXTENDED EUCLIDEAN ALGORITHM (EEA)

1. begin with $(g_0, s_0, t_0) = (a, 1, 0)$ and $(g_1, s_1, t_1) = (b, 0, 1)$,
2. then repeat

$$\begin{aligned} q_i &= g_{i-1} \,\text{quotient}\, g_i \\ g_{i+1} &= g_{i-1} - q_i g_i \\ s_{i+1} &= s_{i-1} - q_i s_i \\ t_{i+1} &= t_{i-1} - q_i t_i \end{aligned}$$

3. until $g_{k+1} = 0$ for some k. at that point:

$$s_k a + t_k b = g_k = \text{GCD}(a, b)$$

- restrict first to linear systems with rational numbers coefficients
- use EEA to define inverse of integer b modulo m with $\text{GCD}(m, b) = 1$:

$$\begin{aligned} 1 &= s\,m + t\,b \\ \Rightarrow 1/b &:= t \mod m \end{aligned}$$

  this gives us a canonical homomorphism ϕ_m of $\mathbb{Z}$ onto $\mathbb{Z}_m$ with

$$\phi_m(a/b) = \phi_m(a)\phi_m(1/b)$$

- for large enough m, the map ϕ_m can be inverted !

given a finite field image of a/b modulo m for $m > 2\max(a^2, b^2)$, a unique rational reconstruction is possible:

RATIONAL RECONSTRUCTION [WANG '81; WANG, GUY, DAVENPORT '82]

to reconstruct a/b from its finite field image $u = a/b \mod m$:

- run EEA for u and m
- stop at first g_j with $|g_j| \leq \lfloor\sqrt{m/2}\rfloor$
- the unique solution is $a/b = g_j/t_j$

important details:

- since we don't know bound on m:
  veto $|t_j| > \lfloor\sqrt{m/2}\rfloor$ and $\mathrm{GCD}(t_j, g_j) \neq 1$ reconstructions, see *e.g.* [Monagan '04]
- construct large m with Chinese Remaindering:
  construct solution modulo $m = p_1 \cdots p_N$ from solutions modulo machine-sized primes p_i

FUNCTION RECONSTRUCTION

univariate rational function $\mathbb{Q}[d]$ reconstruction:

- works similar to the case $\mathbb{Q}$ since both $\mathbb{Q}$ and $\mathbb{Q}[d]$ are Euclidean domains
- Chinese remaindering becomes polynomial interpolation:

$$p_1 \cdots p_N \to (d - p_1) \cdots (d - p_N)$$

multivariate rational function $\mathbb{Q}[d, s, t, \ldots]$ reconstruction:

- by iteration

performance in practice:

- $\boxed{\text{in first IBP tests: much faster than Reduze 2}}$
- linear solver also useful for other applications

CONCLUSIONS

differential equations:

- powerful analytical method for multiscale integrals in QCD
- refinement via normal form basis (if applicable)
- essential: systematic treatment of multipole polylogs (symbol etc)
- optimize functional basis for result
- NNLO prediction for diboson production at LHC

basis of finite integrals:

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: quasi-finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations
- application: massless QCD form factors

reductions via finite field sampling:

- speeds up integration-by-parts reductions
- useful also in other contexts