Computational tools for multiloop calculations and their application to the Higgs boson production cross section

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1 TopoID

2 $N^3LO$ Higgs production: $qq'$-channel
(Small) Introduction

- LHC at 14 TeV $\Rightarrow$ precision measurements
- Extraction of SM parameters $\Rightarrow$ need accurate theory predictions
  (3-loop Higgs production, 4-loop MS-OS relation, (4-loop $g-2$), 4-loop cusp anomalous dimension, 5-loop $\beta$-function . . . )
- Usually: Evaluation of Feynman diagrams
  - Feynman rules $\rightarrow$ Feynman diagrams
    $\Rightarrow$ Factorial growth
  - Color-, Lorentz-, Dirac-algebra
    (FORM, FeynCalc, Tracer, HEPMath, . . .)
  - Scalar integrals (tensorial)
    (tensor reduction or projector)
  - Classification in “topologies”/“families”
    $\Rightarrow$ Automated in TopoID, ( . . . ?)
  - Reduction to “master integrals”
    $\Rightarrow$ “Manually solve” the IBPs or Laporta’s approach
    (MINCER, MATAD, . . . ; LiteRed; AIR, FIRE, Reduze, . . . )
- Rise in complexity
  - Reduction
    $\leftarrow$ Diagrammatic/“topologic”
    $\leftarrow$ Algebraic
    Systematic “solving” of the IBPs, reconstruction of coefficients from sampling over prime field [von Manteuffel, Schabinger; ’14], improved Laporta, . . . ?
  - (Master integrals $\leftarrow$ new function classes)
(Small) Introduction

Generic topology at NLO:

\[ d_1 = m_H^2 + k_1^2 \]

\[ d_2 = (p_1 + p_2 + k_1)^2 = -s + 2p_1 \cdot k_1 + 2p_2 \cdot k_1 + k_1^2 \]

\[ d_3 = (p_2 + k_1)^2 = 2p_2 \cdot k_1 + k_1^2 \]

\[ d_4 = (p_1 + k_1)^2 = 2p_1 \cdot k_1 + k_1^2 \]

Integration-by-parts relations (IBPs):

\[ 0 = \int d_{k_1}^D \frac{\partial}{\partial k_1^\mu} \{ k_1^\mu, p_1^\mu, p_2^\mu \} \frac{1}{d_1^{a_1} d_2^{a_2} d_3^{a_3} d_4^{a_4}} \]

\[ \Rightarrow \text{ Linear relations among integrals from a family} \]

Reduction: express a large set of integrals in terms of few “master integrals”
TopoID
Topology IDentification

Idea: Generic, process independent Mathematica package

- Feynman diagrams $\rightarrow$ reduced result (Laporta not included)

- Topology construction
  (identification, minimal sets; partial fractioning; factorization, ...)

- Handle properties
  (completeness, linear dependence; subtopologies, scalelessness, symmetries;
   graphs, unitarity cuts, ...)

- FORM code generator
  (diagram mapping, topology processing, integral reduction, ...)

- Master integral identification
  (base changes, non-trivial relations, ...)

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FORM code generator
(diagram mapping, topology processing, integral reduction, ...)

Master integral identification
(base changes, non-trivial relations, ...)

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Bring the polynomial $P$ with $m$ terms into unique form $\hat{P}$ by renaming the $n$ variables $\{x_j\}$:

1. Convert $P$ into $m \times (n + 1)$ matrix $M^{(0)}$
   (row: term, 1st column: coefficient, remaining columns: powers of $\{x_j\}$)
2. Start with considering the above $M^{(0)}$ and the 2nd column ($k = 1$)
3. Compute for all considered matrices $M^{(k),\sigma}$ all transpositions of columns $k$ and $k + 1, \ldots$ (and collect permutations $\sigma$)
4. Sort rows in each matrix lexicographically by the first $k$ columns
5. Extract in columns $k$ the lexicographically largest vector
6. Keep only matrices with this maximal vector;
   If $k < n - 1$: $k \rightarrow k + 1$ and goto Step 3
7. Each remaining matrix encodes the same unique $\hat{P}_\sigma$ and a permutation of variables $\sigma$
1. Convert $P$ into $m \times (n + 1)$ matrix $M^{(0)}$
   (row: term, 1st column: coefficient, remaining columns: powers of $\{x_j\}$)

2. Start with considering the above $M^{(0)}$ and the 2nd column ($k = 1$)

$$P = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2 \rightarrow M^{(0)} = \begin{pmatrix}
1 & 2 & 0 & 0 \\
2 & 1 & 1 & 0 \\
1 & 0 & 2 & 0 \\
1 & 0 & 0 & 2
\end{pmatrix} \quad \text{(Step 1)}$$

$$S^{(1)} = \{ M^{(0)(123)} = M^{(0)} \} , \quad k = 1 \quad \text{(Step 2)}$$
3 Compute for all considered matrices $M^{(k),\sigma}$ all transpositions of columns $k$ and $k + 1, \ldots$ (and collect permutations $\sigma$)

\[ S'(1) : \quad M'(1)_{(123)} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}, \quad M'(1)_{(213)} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}, \quad M'(1)_{(321)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \]  

(Step 3-1)
Sort rows in each matrix lexicographically by the first $k$ columns

$$S''(1) : \quad M''(1)(123) = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}, \quad M''(1)(213) = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix},$$

$$M''(1)(321) = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

(Step 4-1)
5. Extract in columns $k$ the lexicographically largest vector

6. Keep only matrices with this maximal vector;
   If $k < n - 1$: $k \rightarrow k + 1$ and goto Step 3

$$\hat{M}''^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$  \hspace{1cm} \text{(Step 5-1)}$$

$$S^{(2)} = \left\{ M''^{(1)(123)}, M''^{(1)(213)} \right\}, \quad k = 2$$  \hspace{1cm} \text{(Step 6-1)}$$
Compute for all considered matrices $M^{(k),\sigma}$ all transpositions of columns $k$ and $k + 1, \ldots$ (and collect permutations $\sigma$)

\[ S^{'}(2) : M^{(2)(123)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}, \quad M^{(2)(132)} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \]
\[ M^{(2)(213)} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}, \quad M^{(2)(231)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \] (Step 3-2)
Sort rows in each matrix lexicographically by the first $k$ columns.

$S''^{(2)} : M''^{(2)(123)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$, $M''^{(2)(132)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$,

$M''^{(2)(213)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$, $M''^{(2)(231)} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$

(Step 4-2)
5. Extract in columns $k$ the lexicographically largest vector.

6. Keep only matrices with this maximal vector;
   If $k < n - 1$: $k \rightarrow k + 1$ and goto Step 3

\[
\hat{M}'''(2) = \begin{pmatrix}
0 \\
2 \\
0 \\
1
\end{pmatrix}
\]  
(Step 5-2)

\[
S^{(2)} = \{ M'''(2)^{(123)}, M'''(2)^{(213)} \}
\]  
(Step 6-2)
Each remaining matrix encodes the same unique $\hat{P}_\sigma$ and a permutation of variables $\sigma$

$$\hat{P} = P = x_3^2 + x_2^2 + x_1^2 + 2x_1x_2, \quad \hat{\sigma} = \{(123), (213)\}$$  \hspace{1cm} (Step 7)
\[ \hat{P} = P = x_3^2 + x_2^2 + x_1^2 + 2x_1x_2, \quad \hat{\sigma} = \{(123), (213)\} \]  

(Step 7)

- \( P \) already in canonical form; two permutations \( \{(123), (213)\} \)
- \( (213) \) denotes \( (x_1, x_2, x_3) \rightarrow (x_2, x_1, x_3) \); symmetry of \( P \) under \( x_1 \leftrightarrow x_2 \)

Application to Feynman integrals

- Use \( U + F \) from the Feynman representation
- Unique identifier \( \hat{U} + \hat{F} \); independent of momentum space representation
- Returned permutations: symmetries of Feynman integrals

⇒ Many useful applications
Minimal set for NNLO Higgs production:

Note: Sufficient for all 2946 diagrams
Non-trivial relation for NNLO Higgs production:

\[
\begin{align*}
\text{Diagram 1} & \quad = \quad \text{Diagram 2} + (\ldots) \quad \times \quad (\ldots) + (\ldots)
\end{align*}
\]

- Cross-topology relations; not from Laporta reduction
- Simplify calculation
- Useful cross-checks
Minimal set for NNLO Higgs production:
TopoID
Topology “merging”

3-loop massless propagators:

\[ q_1 = k_1, \quad q_4 = p - k_1 - k_2, \quad q_7 = k_1 + k_2 + k_3, \]
\[ q_2 = p - k_1, \quad q_5 = k_3, \quad q_8 = k_1 + k_2, \]
\[ q_3 = k_2, \quad q_6 = p - k_1 - k_2 - k_3, \quad q_9 = k_1 + k_3. \]

- 1 external, 3 internal momenta
- \( \Rightarrow \) 9 scalar products
- 3 incomplete topologies with 8 lines
- Identify “greatest common subtopology”
- Find “supertopology” with these 3 different graphs as subtopologies
Partial fractioning for NLO Higgs production:

\[ p_2 \rightarrow d_2 d_3 d_4 p_1 \]

Via Gröbner basis:

\[
\begin{align*}
    d_4 & \rightarrow -m_H^2 + s + d_1 + d_2 - d_3 \\
    \frac{d_3}{d_4} & \rightarrow \frac{1}{d_4} \left( -m_H^2 + s + d_1 + d_2 - d_4 \right) \\
    \frac{d_2}{d_3 d_4} & \rightarrow \frac{1}{d_3 d_4} \left( m_H^2 - s - d_1 + d_3 + d_4 \right) \\
    \frac{d_1}{d_2 d_3 d_4} & \rightarrow \frac{1}{d_2 d_3 d_4} \left( m_H^2 - s - d_2 + d_3 + d_4 \right) \\
    \frac{1}{d_1 d_2 d_3 d_4} & \rightarrow \frac{1}{m_H^2 - s} \frac{1}{d_1 d_2 d_3 d_4} (d_1 + d_2 - d_3 - d_4)
\end{align*}
\]
N$^3$LO Higgs production: $qq'$-channel

Motivation and introduction

- Higgs production at the LHC dominated by gluon fusion
- After [Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger; '14, '15]: 2.2% corrections, 3% uncertainty at N$^3$LO
  $\Rightarrow$ Cross-check

- Loop-induced process, dominated by top quark mass
- Effective field theory with top quark integrated out:

\[ \mathcal{L}_{Y, eff} = -\frac{H}{v} C_1 \mathcal{O}_1 \quad \text{and} \quad \mathcal{O}_1 = \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \]

- Reduced numbers of scales and loops:
  single dimensionless variable $x = m_H^2/s$ (soft: $x \to 1$)
- Finite matching coefficient $C_1$ needed to 4-loop

[Chetyrkin, Kniehl, Steinhauser; '98] [Schröder, Steinhauser; '06] [Chetyrkin, Kühn, Sturm; '06]
**$N^3LO$ Higgs production: $qq'$-channel**

**Status**

- **LO calculation (exact)**
  
  [Ellis et al.; '76] [Wilczek et al.; '77] [Georgi et al.; '78] [Rizzo; '80]

- **NLO (exact)**
  
  [Dawson; '91] [Djouadi, Spira, Zerwas; '91]

- **NNLO (EFT) $\Rightarrow$ soft expansion to 3rd order valid to $O(1\%)$**
  
  [Harlander, Kilgore; '02] [Anastasiou, Melnikov; '02] [Ravindran, Smith, van Neerven; '03]

- **NNLO $O(1/M_t^2)$ corrections $\approx$ NNLO $+1\%$**
  
  [Pak, Rogal, Steinhauser; '09-'11] [Harlander, Mantler, Marzani, Ozeren; '09-'10]

- **$N^3LO$ IR counterterms**

  - 3-loop splitting functions
    
    [Moch, Vermaseren, Vogt; '02]

  - NNLO master integrals to higher orders in $\epsilon$
    
    [Pak, Rogal, Steinhauser; '11] [Anastasiou, Bühler, Duhr, Herzog; '12]

  - Cross sections and convolution integrals
    
    [Höschele, JH, Pak, Steinhauser, Ueda; '12, '13] [Bühler, Lazopoulos; '13]

- **$N^3LO$ scale variation $\Rightarrow O(2\% - 8\%)$**
  
  [Bühler, Lazopoulos; '13]
\(N^3\text{LO} \) Higgs production: \(qq'\)-channel

**Status**

- **\(N^3\text{LO} \) corrections**
  - \(VV^2\) and \(V^3\) – 3-loop gluon form factor
    - [Baikov, Chetyrkin, Smirnov\(^2\), Steinhauser; '09] [Gehrmann, Glover, Huber, Ikizlerli, Studerus; '09]
  - \(VRV\) exact in \(x\)
    - [Anastasiou, Duhr, Dulat, Herzog, Mistlberger; '13] [Kilgore; '14]
  - \(V^2R\) exact in \(x\)
    - [Dulat, Mistlberger; '14] [Duhr, Gehrmann, Jaquier; '14]
  - \(VR^2\) expansion in \(x \rightarrow 1\)
    - [Anastasiou, Duhr, Dulat, Furlan, Gehrmann; '14] [Li, von Manteuffel, Schabinger, Zhu; '14], [Anastasiou, Duhr, Dulat, Furlan, Herzog, Mistlberger; '15]
  - \(R^3\) expansion in \(x \rightarrow 1\)
    - [Anastasiou, Duhr, Dulat, Mistlberger; '13]
  - 37 terms in the \(x \rightarrow 1\) expansion
    - [Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger; '14, '15]

\(\Rightarrow\) Sufficient for phenomenology

- \(qq'\)-channel exact in \(x\) (\(VR^2\), \(R^3\))
  - [Höschele, JH, Ueda; '14] [Anzai, Hasselhuhn, Höschele, JH, Kilgore, Steinhauser, Ueda; '15]

\(\Rightarrow\) Independent cross-check

- **Many different resummations**
**N^3LO Higgs production: qq'-channel**

**Status**

- **N^3LO corrections**
  - **VV^2 and V^3** – 3-loop gluon form factor
    - [Baikov, Chetyrkin, Smirnov^2, Steinhauser; ’09] [Gehrmann, Glover, Huber, Ikizlerli, Studerus; ’09]
  - **VRV exact in x**
    - [Anastasiou, Duhr, Dulat, Herzog, Mistlberger; ’13] [Kilgore; ’14]
  - **V^2R exact in x**
    - [Dulat, Mistlberger; ’14] [Duhr, Gehrmann, Jaquier; ’14]
  - **VRV^2 expansion in x → 1**
    - [Anastasiou, Duhr, Dulat, Furlan, Gehrmann; ’14] [Li, von Manteuffel, Schabinger, Zhu; ’14],
    [Anastasiou, Duhr, Dulat, Furlan, Herzog, Mistlberger; ’15]
  - **R^3 expansion in x → 1**
    - [Anastasiou, Duhr, Dulat, Mistlberger; ’13]
  - **37 terms in the x → 1 expansion**
    - [Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger; ’14, ’15]

⇒ **Sufficient for phenomenology**

- **qq'-channel exact in x (VR^2, R^3)**
  - [Höschele, JH, Ueda; ’14] [Anzai, Hasselhuhn, Höschele, JH, Kilgore, Steinhauser, Ueda; ’15]

⇒ **Independent cross-check**

- **Many different resummations**


N^3LO Higgs production: \(qq'-\)channel

Status

- N^3LO corrections
  - \(VV^2\) and \(V^3\) – 3-loop gluon form factor
    - [Baikov, Chetyrkin, Smirnov^2, Steinhauser; '09] [Gehrmann, Glover, Huber, Ikizlerli, Studerus; '09]
  - \(VRV\) exact in \(x\)
    - [Anastasiou, Duhr, Dulat, Herzog, Mistlberger; '13] [Kilgore; '14]
  - \(V^2R\) exact in \(x\)
    - [Dulat, Mistlberger; '14] [Duhr, Gehrmann, Jaquier; '14]
  - \(VR^2\) expansion in \(x \to 1\)
    - [Anastasiou, Duhr, Dulat, Furlan, Gehrmann; '14] [Li, von Manteuffel, Schabinger, Zhu; '14],
    - [Anastasiou, Duhr, Dulat, Furlan, Herzog, Mistlberger; '15]
  - \(R^3\) expansion in \(x \to 1\)
    - [Anastasiou, Duhr, Dulat, Mistlberger; '13]
  - 37 terms in the \(x \to 1\) expansion
    - [Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger; '14, '15]

⇒ Sufficient for phenomenology

- \(qq'-\)channel exact in \(x\) (\(VR^2\), \(R^3\))
  - [Höschele, JH, Ueda; '14] [Anzai, Hasselhuhn, Höschele, JH, Kilgore, Steinhauser, Ueda; '15]

⇒ Independent cross-check

- Many different resummations
\textbf{N}^{3}\text{LO Higgs production: } qq'-\text{channel}

\textbf{Generalities}

1. Reduction of integrals with full $x$-dependence
   \[\rightarrow \text{Only contributing cuts}\]

2. Construct differential equations for master integrals
   \[\rightarrow \text{Canonical basis (in general: coupled system)}\]

3. Soft limit $x \rightarrow 1$ as boundary condition
   \[\rightarrow \text{Leading term using Mellin-Barnes representation}\]

\textit{Canonical differential equations:}

\[\frac{d}{dx} m_i(x, \epsilon) = \epsilon A_{ij}(x) m_j(x, \epsilon) \quad \text{with} \quad d = 4 - 2 \epsilon\]

[Henn et al.; '13-\ldots]

- $\epsilon$- and $x$-dependence factorize
- Solve order-by-order in $\epsilon$
- $A_{ij}$: alphabet of appearing functions

\textbf{E.g. Harmonic Polylogarithms (HPLs):}

\[H_{\vec{w}}(x) = \int_0^x dx' \ f_{w_1}(x') H_{\vec{w}_{n-1}}(x') \quad \text{and} \quad f_{0}(x) = \frac{1}{x}, \ f_{\pm 1}(x) = \frac{1}{1 \mp x}\]
Optical theorem and Cutkosky’s rules

Higher-order corrections:

Virtual  More loop integrations

Real  Additional final state particles (different phase spaces)

Optical theorem:

\[ \sigma(i \rightarrow f) \sim \sum_f \int d\Pi_f \ |M(i \rightarrow f)|^2 \sim \text{Disc } M(i \rightarrow i) \]

Cutkosky’s rules: Consider only valid diagrammatic cuts for Disc

- Two connectivity components
- Separate in- and outgoing momenta (s-channel)
- Contribute to the process (1 or 2 Higgs, 0 to 3 parton lines)
Optical theorem and Cutkosky’s rules

E.g. to NNLO:

$$\int d\Pi_1 \left[ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \right] + \ldots + \left[ \begin{array}{c} \text{Diagram 4} \\ \text{Diagram 5} \end{array} \right] + \ldots$$

$$\int d\Pi_2 \left[ \begin{array}{c} \text{Diagram 6} \\ \text{Diagram 7} \end{array} \right] + \text{Diagram 8} + \ldots$$

$$\int d\Pi_3 \left[ \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right] + \ldots + \text{Diagram 11} + \ldots$$

$$= \left[ \begin{array}{c} \text{Diagram 12} \\ \text{Diagram 13} \end{array} \right] + \left[ \begin{array}{c} \text{Diagram 14} \\ \text{Diagram 15} \end{array} \right] + \ldots + \left[ \begin{array}{c} \text{Diagram 16} \\ \text{Diagram 17} \end{array} \right] + \ldots + \ldots$$
Optical theorem and Cutkosky’s rules

Optical theorem:

\[ \sigma(i \rightarrow f) \sim \sum_f \int d\Pi_f |\mathcal{M}(i \rightarrow f)|^2 \sim \text{Disc} \mathcal{M}(i \rightarrow i) \]

Pros

- Forward scattering \( \Rightarrow \) simplified kinematics
- Common treatment of loop and phase space integrals
- Calculation of Disc only for master integrals

Cons

- More loops and diagrams
- Only total cross section (naively)

Approach first used in [Anastasiou, Melnikov; '02]
Optical theorem and Cutkosky’s rules

Handling cut-diagrams:

Filter diagrams

- Fast graph-based algorithm build into Perl script to process QGRAF output
- $N^3\text{LO}$ Higgs: $860\,118 \rightarrow 174\,938$
- NNLO Higgs pair (SV): $17\,667\,600 \rightarrow 42\,252$

Assist reduction

- Build also into TopoID $\Rightarrow$ pass to Laporta reduction
- Typically: only $O(10\%)$ of subtopologies (or relations)
N$^3$LO Higgs production: $qq'$-channel
Calculation: toolchain

Reduction

1. Generate Feynman diagrams
   QGRAF [Nogueira; '93]

2. Select diagrams with specific cuts
   filter [JH, Pak; (unpublished)]

3. Map diagrams to topologies (← graph information)
   exp [Harlander, Seidensticker, Steinhauser; '98]
   reg [Pak; (unpublished)]

4. Reduction to scalar integrals (← generic topologies)
   FORM [Kuipers, Ueda, Vermaseren, Vollinga; '13]

5. Reduction to master integrals (← basic topologies)
   rows [JH, Pak; (unpublished)]
   FIRE [Smirnov]

6. Minimal basis of master integrals
   TopoID [JH, Pak; (unpublished)]

220 diagrams, e.g.
$N^3LO$ Higgs production: $qq'$-channel

Calculation: 17 topologies with 3- and 4-particle cuts
$N^3LO$ Higgs production: $qq'$-channel

Calculation: e.g. “sea snake” topology
N$^3$LO Higgs production: $qq'$-channel

Results: functions beyond HPLs

- Laporta: 332 master integrals; TopoID: 108 and cancellation of $\xi$
- Some Feynman integrals generate functions with alphabet beyond HPLs:
  \[ \left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1+4x}, \frac{1}{x\sqrt{1+4x}} \right\} \]
- Traced back to common subtopology:

![Diagram](image)

- Numerical implementation in Mathematica:
  - Change alphabet to \( \left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1+4x}, \frac{1}{x\left(\frac{1}{\sqrt{1+4x}} - 1\right)} \right\} \)
  - Use series expansions $x \to 0$ and $x \to 1 \Rightarrow 10$ digits in 1 second
- Note: $x \to (1-x)/x^2 \Rightarrow \text{“Cyclotomic Polylogarithms”}$
  - Representation as “Goncharov Polylogarithms” (linear denominators)
  - Letters of 6th roots of unity; here only:
  \[ \left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{(-1)^{3/4} - x} \right\} \]
Conclusion

**TopoID:**
- Generic, process independent Mathematica package for multiloop calculations; especially for many topologies
- Until now two applications:
  - $N^3$LO Higgs production and NNLO Higgs pair production
- Works also for 5-loop propagators
- Publication soon . . .

$qq'$-channel in $N^3$LO Higgs production:
- New iterated integrals beyond HPLs appear
- Agreement of leading logarithms with
  - [Anastasiou, Duhr, Dulat, Furlan, Herzog, Mistlberger; '15]
- Full calculation underway . . .
NLO and NNLO Higgs pair production

Motivation

Higgs Potential in the Standard Model:

\[ V(H) = \frac{1}{2} m_H^2 H^2 + \lambda v H^3 + \frac{1}{4} \lambda H^4, \quad \lambda^{\text{SM}} = \frac{m_H^2}{2v^2} \approx 0.13, \quad v: \text{Higgs vev.} \]

- Verify mechanism of spontaneous symmetry breaking in the SM
- Measure the Higgs self-coupling \( \Rightarrow \) sensitive process

\[ \sigma^{\text{LO}}(gg \rightarrow HH)(fb) \]

\[ \sqrt{s} \ (\text{GeV}) \]

\[ 0 \quad 0.5 \quad 1 \quad 1.5 \]

\[ 200 \quad 600 \quad 1000 \quad 1400 \]

(a) \sim \quad (b) \sim \quad (c) \sim
NLO and NNLO Higgs pair production

Theory status

Prospects for the LHC @ 14 TeV:

- $b\bar{b}\gamma\gamma$-channel, $600 \, fb^{-1}$: $\lambda \neq 0$  
  [Baur, Plehn, Rainwater; '04]

- $b\bar{b}\gamma\gamma$, $b\bar{b}\tau^+\tau^-$-channels: “promising”; 
  $b\bar{b}W^+W^-$-channel: “not promising”  
  [Baglio, Djouadi, Gröber, Mühlleitner, Quevillon, Spira; '13]

- $600 \, fb^{-1}$: $\lambda > 0$; $3000 \, fb^{-1}$: $\lambda^{+30\%}_{-20\%}$ (ratio with Higgs cross section)  
  [Goertz, Papaefstathiou, Yang, Zurita; '13]

- And many others, e.g.:
  [Dolan, Englert, Spannowsky; '12] [Papaefstathiou, Yang, Zurita; '13]
  [Barr, Dolan, Englert, Spannowsky; '13] [Barger, Everett, Jackson, Shaughnessy; '14]
  [Englert, Krauss, Spannowsky, Thompson; '15] [...]

- Until now: Higgs pair production not observed in $b\bar{b}b\bar{b}$- and 
  $b\bar{b}\gamma\gamma$-channels (as expected in the SM)  
  [ATLAS; '15] [CMS; '15]

⇒ Wait for HL-LHC
NLO and NNLO Higgs pair production

Theory status

**Known since long:**
- LO result with exact $M_t$ dependence
  [Glover, van der Bij; '88] [Plehn, Spira, Zerwas; '98]
- NLO result in $M_t \to \infty$ limit
  \[ \sigma_H \approx 20^{\text{LO}} \text{ fb} + 20^{\text{NLO}, M_t \to \infty} \text{ fb} \]
  for \( \sqrt{s_H} = 14 \text{ TeV}, \mu = 2m_H \)

**More recently:**
- NLO + NNLL ($M_t \to \infty$) \( \approx \) NLO +20%
  [Shao, Li, Li, Wang; '13]
- NNLO w/ or w/o soft-virtual approx. ($M_t \to \infty$) \( \approx \) NLO +20%
  [de Florian, Mazzitelli; '13]
- \( \mathcal{O}(1/M_t^8) \) corrections at NLO \( \approx \) NLO +10%
  [Grigo, JH, Melnikov, Steinhauser; '13]
- NLO real exact in $M_t$, NLO virt. for $M_t \to \infty$ \( \approx \) NLO −10%
  [Maltoni, Vryonidou, Zaro; '14]
- Cross-check of virtual NNLO corrs.; NNLO matching coefficient for \( ggHH \)-coupling \( \approx \) NNLO +1%
  [Grigo, Melnikov, Steinhauser; '14]
- Improved \( \mathcal{O}(1/M_t^{12}) \) NLO, \( \mathcal{O}(1/M_t^4) \) NNLO soft-virt. corrections
  [Grigo, JH, Steinhauser; 15']
NLO and NNLO Higgs pair production

Generalities

- Operate on full-theory diagrams at NLO and NNLO
- Virt. corrs. in two independent calculations:
  - amplitude (differential; 2-/3-loop)
  - forward scattering (total; 4-/5-loop)
- Real corrs.: only via forward scattering at NLO
- Perform expansion for $M_t \rightarrow \infty$; improve upon effective theory results for NLO [Dawson, Dittmaier, Spira; '98], NNLO [de Florian, Mazzitelli; '13]
- Laporta reduction to master integrals for the “soft” subdiagrams
- Remaining “hard” massive tadpoles via MATAD
- Master integrals as series around $\sqrt{s} = 2m_H$ (not in this talk)
Differential factorization

Factorization of the LO result exact in $M_t$ for:

Total cross section

$$\sigma^{(i)} = \Delta^{(i)} \sigma_{\text{exact}}^{(0)} = \frac{\sigma_{\text{exact}}^{(0)}}{\sigma_{\exp}^{(0)}} \int_{4m_H^2}^{s} dQ^2 \frac{d\sigma_{\exp}^{(i)}}{dQ^2}$$

with $\Delta^{(i)} = \frac{\sigma_{\exp}^{(i)}}{\sigma_{\exp}^{(0)}}$, $\sigma_{\exp}^{(i)} = \sum_{n=0}^{N} c_n^{(i)} \rho^n$, $\rho = \frac{m_H^2}{M_t^2}$

Differential cross section

$$\sigma^{(i)} = \int_{4m_H^2}^{s} dQ^2 \frac{\left(\frac{d\sigma_{\text{exact}}^{(0)}}{dQ^2}\right)}{\left(\frac{d\sigma_{\exp}^{(0)}}{dQ^2}\right)} \frac{d\sigma_{\exp}^{(i)}}{dQ^2}$$

"Cure" the invalidity of the $M_t \rightarrow \infty$ expansion for the large-$Q^2$ region

- Virt. corrs. via amplitude: access to $Q^2$-dependence $\sim \delta(s - Q^2)$
- Real corrs. via optical theorem (naively): only total cross section

$\Rightarrow$ Use the soft-virtual approximation [de Florian, Mazzitelli; '12]
Soft-virtual approximation

- Split $\sigma$ into its contributions (works also for $d\sigma/d Q^2$):

$$\sigma = \sigma^{\text{virt+ren}} + \sigma^{\text{real+split}} = \text{finite}$$

$$= \Sigma_{\text{div}} + \Sigma_{\text{fin}} + \Sigma_{\text{soft}} + \Sigma_{\text{hard}} = \Sigma_{\text{SV}=\text{finite}} + \Sigma_{\text{H}=\text{finite}}$$

- $\Sigma_{\text{div}}$ universal for color-less final state [de Florian, Mazzitelli; '12]

- **Compute $\sigma^{\text{virt+ren}}$ as $\rho$-expansion**

- Solve $\sigma^{\text{virt+ren}} = \Sigma_{\text{div}} + \Sigma_{\text{fin}}$ for $\Sigma_{\text{fin}}$

- $\Sigma_{\text{div}}$ and $\Sigma_{\text{soft}}$ (soft coll. counterterms + soft real corrs.)

  $\sim$ exact $\sigma^{\text{LO}}$ (include $M_t$ effects)

$$Q^2 \frac{d\sigma}{dQ^2} = \sigma^{\text{LO}} z G(z) \quad \text{with} \quad z = \frac{Q^2}{s}, \quad G(z) = G_{\text{SV}}(z) + G_{\text{H}}(z)$$

$$\sigma_{(SV)} = \int_{1-\delta}^{1} dz \sigma^{\text{LO}}(zs) G_{(SV)}(z) \quad \text{with} \quad \delta = 1 - \frac{4m_H^2}{s}$$

- $G_{SV}(z)$ constructed from $\sigma_{\text{fin}}^{(i)}$ and $\sigma^{\text{LO}}$ only

  [de Florian, Mazzitelli; '12] [Grigo, JH, Steinhauser; '15]
Asymptotic expansion

- Expand at integrand level for all contributing regions
  \[ \equiv \text{series expansion in analytic result} \]
- Hierarchy: \( M_t^2 \gg s, m_H^2 \Rightarrow \text{series in } \rho = m_H^2/M_t^2 \)
- Effectively reduce number of loops and scales

- Here: regions correspond to subgraphs (in general more than one)
- Hard mass expansion: subgraphs must contain all heavy lines

Example: NLO real with one region

\[ \begin{align*}
\{ M_t^2, m_H^2, s \} & \quad \rightarrow \quad \{ M_t^2 \} \times \{ m_H^2, s \} \times \{ M_t^2 \}
\end{align*} \]
Asymptotic expansion

- Expand at integrand level for all contributing regions
  ≡ series expansion in analytic result
- Hierarchy: $M_t^2 \gg s, m_H^2 \Rightarrow$ series in $\rho = m_H^2 / M_t^2$
- Effectively reduce number of loops and scales

- Here: regions correspond to subgraphs (in general more than one)
- Hard mass expansion: subgraphs must contain all heavy lines

Example: NLO virt. with two regions

\[
\{ M_t^2, m_H^2, s \} \rightarrow \{ M_t^2 \} \times \{ m_H^2, s \} \times \{ M_t^2 \}
\]

\[
\{ M_t^2 \} \times \{ m_H^2, s \} \times \{ M_t^2 \}
\]
NLO and NNLO Higgs pair production
Calculation (via optical theorem)

LO topology:

Virt. NLO topologies:

Real NLO topologies:
Note:

- Different regions in asymptotic expansion $\Rightarrow$ different loop-orders
- Here: multiplied with 1- to 3-loop massive tadpoles
NLO and NNLO Higgs pair production
Calculation (via optical theorem)

Virt. and real LO-NLO master integrals:

Virt. NNLO master integrals (in addition):
NLO and NNLO Higgs pair production
Splitting in soft-virtual and real contributions at NLO

- Partonic NLO correction; total factorization

Using $\mu = 2m_H$ (also in the following)

- Different behavior for higher orders in $\rho$ expansion:
  - $\text{SV}$ increasing
  - $H$ decreasing (flat for $\sqrt{s} \gtrsim 400$ GeV)

$\Rightarrow$ SV numerically dominant
NLO and NNLO Higgs pair production

Total vs. differential factorization (DF) at NLO

- DF applied only to SV part; H treated via total factorization (i.e. identical)

Maxima of DF curves at lower $\sqrt{s}$; smaller cross sections

⇒ Improvement of convergence:

difference of $\rho^0$ and corrections to $\rho^6$ (for $\sqrt{s} = 400$ GeV):

0.25 fb vs. 0.05 fb

- (Partonic K-factor: behavior at top quark pair threshold not washed out)
Technical upper cut on $\sqrt{s}$ (good proxy to $Q^2$):

$$\sigma_H(s_H, s_{\text{cut}}) = \int_{4m_H^2/s_H}^1 d\tau \left( \frac{d\mathcal{L}_{gg}}{d\tau} \right)(\tau) \sigma(\tau s_H) \theta(s_{\text{cut}} - \tau S_H)$$

$\sqrt{s_{\text{cut}}} \rightarrow \infty$: total cross section for 14 TeV

Spread of $\rho$-orders $\Rightarrow \pm 10\%$ uncertainty of EFT at NLO due to $M_t$
Lessons from NLO for NNLO:

- SV approximation contracted for $z \to 1$;
  $G_{SV}(z)$ can be replaced by $f(z) G_{SV}(z)$ with $f(1) = 1$

  Splitting into SV and H not unique

- Tune $f(z)$ at NLO such that $\sigma \approx \Sigma_{SV}$
  $\Rightarrow f(z) = z$ accurate to 2%

- Replace RGE logarithms ($\sqrt{s} \approx Q^2$ in the soft limit):
  $\Rightarrow \log(\mu^2/s) \to \log(\mu^2/Q^2)$

Discrepancy to [Maltoni, Vryonidou, Zaro; '14]:

- Real corrs.: treated exactly; Virt. corrs.: EFT result
- Claim: $-10\%$ correction at NLO

But: Dominant positive shift from virtual $1/M_t$-corrections (c.f. backup slide)
NLO and NNLO Higgs pair production

NNLO SV corrections

- EFT result plus $\rho$- and $\rho^2$-terms
- Peaks at smaller $\sqrt{s}$
- Same pattern of $\rho$-corrections

Conv. up to $\sqrt{s_{\text{cut}}} \approx 400$ GeV
- $\rho$- and $\rho^2$-corrections: $\pm 2.5\%$

$M_t$-uncertainty at NNLO: 5%

(= NNLO corrs. $\approx 20\%$)
NLO and NNLO Higgs pair production

NNLO SV K-factor

- Strong raise close to threshold $\iff$ steeper NNLO correction
- For total cross section: $K_{H}^{\text{NNLO}} \approx 1.7 - 1.8$
Backup: NLO and NNLO Higgs pair production

Partonic NLO K-factor

Note: Behavior around top quark pair threshold not washed out.
Backup: NLO and NNLO Higgs pair production
Splitting into real and virtual corrections at NLO

Note: R and V separately divergent; only finite contributions shown
Backup: NLO and NNLO Higgs pair production

Total vs. differential factorization without hard contributions at NLO
Backup: NLO and NNLO Higgs pair production
Partonic NNLO cross section for different scales choices and \( f(z) \)

Note: \( f(z) = z \) (better proxy) leads to higher values