Generalizations of polylogarithms for Feynman integrals

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based on joint work with Francis Brown (Oxford)
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Polylogarithms are generalizations of the logarithm function:

$$\mathrm{Li}_1(z) = -\ln(1-z) = \sum_{j=1}^{\infty} \frac{z^j}{j}$$

Classical polylogarithms (Leibniz):

$$\mathrm{Li}_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n}$$

Multiple polylogarithms in one variable:

$$\mathrm{Li}_{n_1,\ldots,n_r}(z) = \sum_{1 \le j_1 < \ldots < j_r} \frac{z^{j_r}}{j_1^{n_1} \ldots j_r^{n_r}}$$

Multiple polylogarithms in several variables (Goncharov):

$$\mathrm{Li}_{n_1,\ldots,n_r}(z_1,\ldots,z_r) = \sum_{1 \le j_1 < \ldots < j_r} \frac{z^{j_1} \ldots z_r^{j_r}}{j_1^{n_1} \ldots j_r^{n_r}}$$

$w = \sum_{i=1}^{r} n_i$ is called the *weight*.

Integral representations:

$$\frac{d}{dz}\mathrm{Li}_n(z) = \sum_{j=1}^{\infty} \frac{d}{dz}\frac{z^j}{j^n} = \frac{1}{z}\mathrm{Li}_{n-1}(z) \text{ for } n \geq 2$$

$$\begin{aligned}\mathrm{Li}_n(z) &= \int_0^z \frac{dx}{x}\mathrm{Li}_{n-1}(x) \\ &= \int_0^z \frac{dx_n}{x_n} \dots \int_0^{x_3} \frac{dx_2}{x_2} \underbrace{\int_0^{x_2} \frac{dx_1}{1-x_1}}_{-\ln(1-x_2)}\end{aligned}$$

For the functions $\mathrm{Li}_n(z)$ and $\mathrm{Li}_{n_1,\dots,n_r}(z)$ we only need the differential 1-forms $\omega_i \in \left\{ \frac{dx}{x}, \frac{dx}{1-x} \right\}$.

We denote such **iterated integrals** by

$$[\omega_r|\dots|\omega_2|\omega_1] = \int_0^z \omega_r(x_r)\dots\int_0^{x_3} \omega_2(x_2) \int_0^{x_2} \omega_1(x_1)$$

Example:

$$\mathrm{Li}_3(z) = \int_0^z \frac{dx_3}{x_3} \int_0^{x_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1-x_1} = \left[\frac{dx}{x} | \frac{dx}{x} | \frac{dx}{1-x}\right]$$

Extending the set of "building blocks":

- **Harmonic polylogarithms**: $\left\{\frac{dx}{x}, \frac{dx}{1-x}, \frac{dx}{1+x}\right\}$ (Remiddi, Vermaseren 1999)
- **Two-dimensional harmonic polylogarithms**: $\left\{\frac{dx}{x}, \frac{dx}{1-x}, \frac{dx}{x+y}, \frac{dx}{x+y-1}\right\}$ (Gehrmann, Remiddi 2001)
- **Hyperlogarithms**: $\left\{\frac{dx}{x}, \frac{dx}{x-y_i} | i = 1, ..., n\right\}$ (Poincare, Kummer, Lappo-Danilevsky)

Every multiple polylogarithm $\mathrm{Li}_{n_1,...,n_r}(z_1, ..., z_r)$ can be expressed in terms of hyperlogarithms. (Goncharov 2001)

Recent applications to **Feynman integrals**:

Computer programs by Panzer (`HyperInt`), Maitre (`HPL`), Vermaseren (in `FORM`), Vollinga, Weinzierl (in `GiNaC`), ...

and recent work by Duhr, Wissbrock, von Manteuffel, Schlotterer, Broedel, Stieberger,...

Observation:

Many Feynman integrals can be expressed in terms of **multiple polylogarithms**,

but **not all** of them.

Counter examples arise in electroweak physics (Bauberger et al 1994), in QCD and even in $\mathcal{N} = 4$ super Yang-Mills theory (Caron-Huot, Larsen 2012, Nandan, Paulos, Spradlin, Volovich 2013).

Outline:

- **Part 1:** MPL - a program for computations with multiple polylogarithms

- **Part 2:** The sunrise integral and elliptic polylogarithms

Part 1: MPL - a program for computations with multiple polylogarithms

(based on joint work with F. Brown)

An alternative to hyperlogarithms:

Instead of $\left\{\frac{dx}{x}, \frac{dx}{x-y_i} | i = 1, ..., n\right\}$ we use the differential forms

$$\Omega_n = \left\{ \frac{dx_1}{x_1}, ..., \frac{dx_n}{x_n}, \frac{d\left(\prod_{a\leq i\leq b} x_i\right)}{\prod_{a\leq i\leq b} x_i - 1} \text{ where } 1 \leq a \leq b \leq n \right\}$$

Example: For $n = 2$ variables:

$$\Omega_2 = \left\{ \frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \frac{dx_1}{x_1-1}, \frac{dx_2}{x_2-1}, \frac{x_1\,dx_2 + x_2\,dx_1}{x_1 x_2 - 1} \right\}$$

Not every ordering will provide a well-defined (i.e. homotopy invariant) function.

Chen (1977): $I = [\omega_1|...|\omega_r]$ has to satisfy satisfy the **integrability** condition

$$\sum_{i=1}^{m} [\omega_1|...|\omega_{i-1}|d\omega_i|\omega_{i+1}|...\omega_m] + \sum_{i=1}^{m-1} [\omega_1|...|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|...|\omega_m] = 0$$

Our construction uses an explicit **"symbol map"** algorithm (CB, Brown 2012, CB, Brown 2014) to take care of this.

We obtain a **vectorspace** $V(\Omega_n)$ **of iterated integrals** with the following properties (Brown '05):

- $V(\Omega_n)$ includes the multiple polylogarithms $\mathrm{Li}_{n_1,\ldots,n_r}(z_1,\ldots,z_r)$.
- Functional relations turn into algebraic identities (cf. literature on the "symbol").
- $V(\Omega_n)$ has an explicit basis.
- $V(\Omega_n)$ is **closed** under taking **primitives**.
- **Limits** at 0 and 1 are combinations of these functions with **multiple zeta values**.

$\Rightarrow$ Explicit **integration algorithms** based on these functions (CB, Brown 2014)

$\Rightarrow$ Implementation in a **Maple program** `MPL` (CB 2015)

MPL can compute integrals of the **cubical type**:

$$I = \int_0^1 dx_n \frac{q}{\left(\prod_j p_j^{a_j}\right)} f$$

where $f \in V(\Omega_n)$, q some polynomial, $a_j \in \mathbb{N}$ and all $p_j = 1 - \prod_{i=k}^n x_i$, $1 \leq k \leq n$

Example:

$$g = \frac{x_1^4 (1-x_1)^4 x_2^9 (1-x_2)^4 x_3^4 (1-x_3)^4}{(1-x_1 x_2)^5 (1-x_2 x_3)^5}$$

With `MPLCubicalIntegrate(g,x[3],3)` we compute

$$\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 g = -\frac{11424695}{144} + 66002\zeta(3).$$

An intermediate result can be obtained by `MPLCubicalIntegrate(g,x[3],2)`:

$$\int_0^1 dx_2 \int_0^1 dx_3 g = c_1 + c_2 \underbrace{\left[\frac{dx_1}{1-x_1}\right]}_{=-\ln(1-x_1)} + c_3 \underbrace{\left[\frac{dx_1}{1-x_1}\middle|\frac{dx_1}{1-x_1}\right]}_{=\frac{1}{2}\ln^2(1-x_1)} + c_4 \underbrace{\left[\frac{dx_1}{x_1}\middle|\frac{dx_1}{1-x_1}\right]}_{=\mathrm{Li}_2(x_1)}$$

Application to **Feynman integrals**:

D-dimensional, scalar L-loop integrals in Feynman parameters:

$$I(\Lambda) = \frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_0^\infty \ldots \int_0^\infty \left(\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1}\right) \delta(H) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{(\mathcal{F}(\Lambda))^{\nu - LD/2}},$$

N : # of edges, ν_i : integer propagator powers, Λ : kinematical invariants and masses, $H = 1 - \sum_{i \in S, S \subseteq \{1,\ldots,N\}} x_i$

Problem 1: UV and IR **divergences**
There are methods to **expand** in terms of finite integrals I_j :

$$I = \sum_{j=-2L}^{\infty} I_j \epsilon^j,$$

(Panzer 2014, v.Manteuffel, Panzer, Schabinger 2015, Binoth, Heinrich 2000)

Problem 2: The Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$ are more complicated than $1 - \prod_i x_i$
$\Rightarrow$ By systematic **changes of variables**, MPL maps the integrals I_j to the cubical type, and computes them.

These changes of variables exist under certain **conditions** to the polynomials $\mathcal{U}$ and $\mathcal{F}$.

Consider a generic integral

$$\int_0^\infty dx_N \ldots \int_0^\infty dx_1 \frac{\text{polynomial} \cdot \text{Iterated integral with "building blocks"} \frac{dP_i}{P_i}}{\prod_i P_i}$$

Condition: All P_i are **linear** in one of the variables x_k

⇒ map to **cubical type** and **integrate**

⇒ map back to Feynman parameters

⇒ The integrand depends on new polynomials P_i'

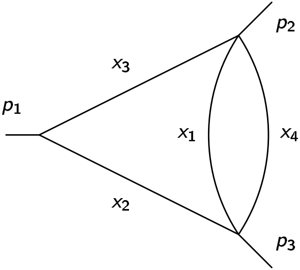
⇒ Repeat: Find next variable x_{k+1} in which all P_i' are **linear**...

If this iteration goes through, the integral is called **linearly reducible**.

An **algorithm** to check this condition was proposed by Brown (2008).

MPL **checks this condition** and returns possible orders of integration.
If two further technical conditions are satisfied, the integrations can be done with MPL.

Example: Massless two-loop triangle



Feynman integral: $I = \prod_{i=1}^{4} \int_0^\infty dx_i \delta(H) \mathcal{U}^{3\epsilon-2} \mathcal{F}^{-2\epsilon}$ omitting a trivial factor $\Gamma(2\epsilon)$

$\mathcal{U} = x_1 x_4 + (x_1 + x_4)(x_2 + x_3)$, $\mathcal{F} = -p_1^2 x_2 x_3 (x_1 + x_4) - p_2^2 x_1 x_3 x_4 - p_3^2 x_1 x_2 x_4$

Kinematical invariants: $\frac{p_2^2}{p_1^2} = (1 + x_5)(1 + x_6)$ and $\frac{p_3^2}{p_1^2} = x_5 x_6$

We consider the momentum space region where $x_5 > 0$, $x_6 > 0$.

With Panzer's (2014) method we expand: $I = \frac{1}{\epsilon} I_{-1} + I_0 + \epsilon I_1 + \mathcal{O}(\epsilon^2)$, Application of `MPLPolynomialReduction` to $\mathcal{U}$, $\mathcal{F}$ and of `MPLCheckOrder`

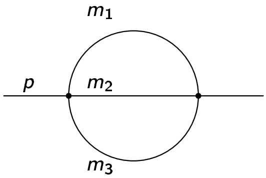
$\Rightarrow$ allowed order of integrations: x_1, x_4, x_3, x_2

With `MPLFeynmanIntegrate` we re-obtain (cf. Chavez, Duhr 2012):

$$
\begin{aligned}
I_{-1} &= 1, \\
I_0 &= 5, \\
I_1 &= \frac{1}{x_6 - x_5}\Bigg(2x_5(1+x_6)\left(\left[\frac{d(x_6)}{1+x_6}\right]\left[\frac{d(x_5)}{x_5}\right] + \left[\frac{d(x_5)}{x_5}\Big|\frac{d(x_5)}{1+x_5}\right]\left[\frac{d(x_5)}{1+x_5}\Big|\frac{d(x_5)}{x_5}\right]\right) \\
&\quad -2x_6(1+x_5)\left(\left[\frac{d(x_5)}{1+x_5}\right]\left[\frac{d(x_6)}{x_6}\right] + \left[\frac{d(x_6)}{x_6}\Big|\frac{d(x_6)}{1+x_6}\right] + \left[\frac{d(x_6)}{1+x_6}\Big|\frac{d(x_6)}{x_6}\right]\right)\Bigg) \\
&\quad -3\zeta(2) + 19.
\end{aligned}
$$

Part 2: The sunrise integral and elliptic polylogarithms

(joint work with L. Adams and S. Weinzierl)



We compute the massive sunrise integral

$$S(D,\ t) = \int \frac{d^D k_1 d^D k_2}{\left(i\pi^{D/2}\right)^2} \frac{1}{\left(-k_1^2 + m_1^2\right)\left(-k_2^2 + m_2^2\right)\left(-\left(p - k_1 - k_2\right)^2 + m_3^2\right)}$$

in $D = 2 - 2\epsilon$ and $D = 4 - 2\epsilon$ dimensions:

$$\begin{aligned} S(2-2\epsilon,\ t) &= S^{(0)}(2,\ t) + S^{(1)}(2,\ t)\epsilon + \mathcal{O}\left(\epsilon^2\right), \\ S(4-2\epsilon,\ t) &= S^{(-2)}(4,\ t)\epsilon^{-2} + S^{(-1)}(4,\ t)\epsilon^{-1} + S^{(0)}(4,\ t) + \mathcal{O}(\epsilon) \end{aligned}$$

where

$$\begin{aligned} t &= p^2 \leq 0, \\ 0 &< m_1 \leq m_2 \leq m_3 < m_1 + m_2. \end{aligned}$$

Feynman parameters
In $D = 2$ dimensions, the Feynman parametric version of the sunrise integral

$$\begin{aligned} S(2,\, t) &= \int_\sigma \frac{\omega}{\mathcal{F}}, \\ \omega &= x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2\,, \\ \sigma &= \left\{[x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0,\, i = 1,\, 2,\, 3\right\} \end{aligned}$$

involves the second Symanzik polynomial

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)(x_1 x_2 + x_2 x_3 + x_1 x_3)\,.$$

Remark: $\mathcal{F}$ **fails** the criterion of **linear reducibility**.

$\Rightarrow$ Direct iterated integration is **not** possible in the variables $x_1,\, x_2,\, x_3$

In $D=2$ dimensions:

Equal mass case: Second order differential equation (Broadhurst, Fleischer, Tarasov 1993);

Solutions Groote, Pivovarov 2000, Laporta, Remiddi 2004, Bloch, Vanhove 2013 ...

Arbitrary masses:
Caffo, Czyz, Laporta, Remiddi (1998): Coupled system of **four** equations of **first order**
Müller-Stach, Weinzierl, Zayadeh (2012): **One** differential equation of **second order**

$$\left(p_2(t)\frac{d^2}{dt^2}+p_1(t)\frac{d}{dt}+p_0(t)\right)S^{(0)}(2,\ t)=p_3(t)$$

$p_0(t),\ p_1(t),\ p_2(t)$: polynomials in t and the m_i^2 ; $p_3(t)$: also involving $\ln(m_i^2),\ i=1,\ 2,\ 3.$

Standard Ansatz:

$$S^{(0)}(2,\ t)=C_1\psi_1(t)+C_2\psi_2(t)+\int_0^t dt_1\,\frac{p_3(t_1)}{p_0(t_1)W(t_1)}\left(-\psi_1(t)\psi_2(t_1)+\psi_2(t)\psi_1(t_1)\right)$$

$\psi_1,\ \psi_2$: solutions of the homogeneous equation; $C_1,\ C_2$: constants; $W(t)$: Wronski determinant.

Underlying geometry:
Second Symanzik polynomial:

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)\left(x_1 x_2 + x_2 x_3 + x_1 x_3\right).$$

The variety $\mathcal{F} = 0$ intersects the integration domain at **three points**
$P_1 = [1:0:0],\ P_2 = [0:1:0],\ P_3 = [0:0:1]$.
Coosing one of these as **origin** defines an **elliptic curve**.

Transform to Weierstrass normal form $y^2 z - x^3 - g_2(t) x z^2 - g_3(t) z^3 = 0$.

For $z = 1$ define e_1, e_2, e_3 by $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$ with $e_1 + e_2 + e_3 = 0$.

$\Rightarrow$ Two **period integrals** of the elliptic curve are

$$\psi_1 = 2\int_{e_2}^{e_3} \frac{dx}{y} = \frac{4}{D^{\frac{1}{4}}} K(k), \quad \psi_2 = 2\int_{e_1}^{e_3} \frac{dx}{y} = \frac{4i}{D^{\frac{1}{4}}} K(k')$$

with the **complete elliptic integral of the first kind** $K(x) = \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-x^2t^2)}}$,
and modulus $k = \sqrt{\frac{e_3-e_2}{e_1-e_2}}$, $k' = \sqrt{1-k^2} = \sqrt{\frac{e_1-e_3}{e_1-e_3}}$.

The **period integrals** ψ_1, ψ_2 are **solutions** of the **homogeneous** differential equation.

The constants C_1, C_2 are determined from a simple property of ψ_1, ψ_2 and the limit of $S^{(0)}(2, t)$ at $t = 0$ (Davydychev, Tausk 1996).

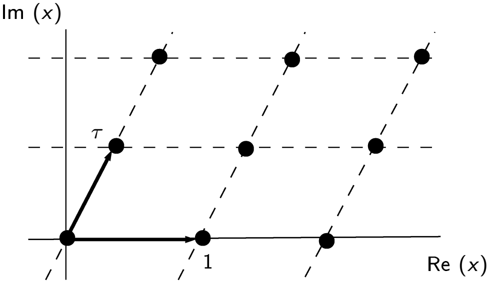
$\Rightarrow$ We obtain $S^{(0)}(2, t)$ as an **integral over** a combination of complete **elliptic integrals** of the first and second type (Adams, C.B., Weinzierl 2013).

Disadvantage: Elliptic integrals are well known in mathematics, but **integrals over elliptic integrals** are not. $\Longleftrightarrow$ No framework for iterated integrals.

Is there an alternative, "closer to" multiple polylogarithms?

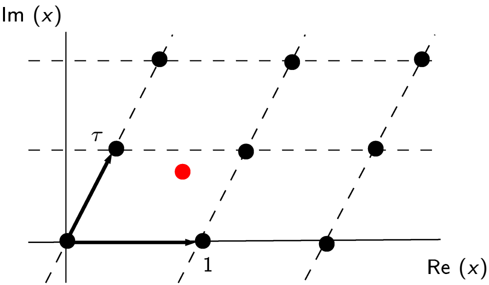
Important step by Bloch and Vanhove (2013) for the **equal mass case**:

New result in terms of an **elliptic dilogarithm**.



Consider the **lattice** $L = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$.

Elliptic functions f with respect to L: $f(x) = f(x + \lambda)$ for $\lambda \in L$.



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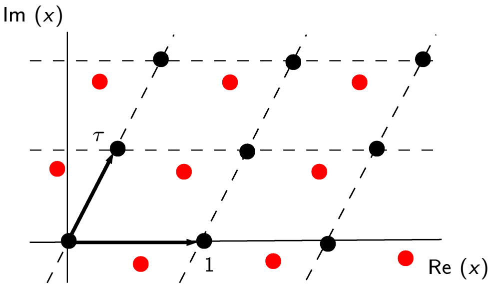

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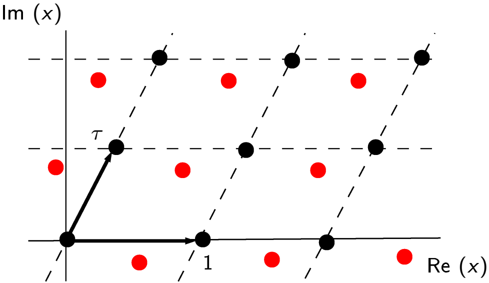

Consider the **lattice** $L = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$.

Elliptic functions f with respect to L: $f(x) = f(x + \lambda)$ for $\lambda \in L$.

Let $\tau = \frac{\psi_1}{\psi_2}$ with ψ_1, ψ_2 the **periods of an elliptic curve** E.

$\Rightarrow$ E is isomorphic to a cell of L. $\Rightarrow$Consider f as a function **defined on** E.

Change variables to $z = e^{2\pi i x} \in \mathbb{C}^\star$

$\Rightarrow$Ellipticity $f(x) = f(x+\lambda)$ means $\tilde{f}(z) = \tilde{f}(z \cdot q_\lambda)$, $q_\lambda \in e^{2\pi i \lambda}$ for $\lambda \in L$.

Particularly: $q = e^{2\pi i \tau}$.

Basic concept: For some function g **construct** an elliptic function of the type $f(z, q) = \sum_{n \in \mathbb{Z}} g(z \cdot q^n)$

E.g. Brown, Levin 2011 consider **elliptic polylogarithms** $\sum_{n \in \mathbb{Z}} u^n \mathrm{Li}_m(z \cdot q^n)$,

elliptic multiple polylogarithms and a framework of **iterated integrals**

(Also see previous definitions in Bloch 1977, Beilinson, Levin 1994, Levin 1997, Levin, Racinet 2007, ...)

Adams, C.B., Weinzierl 2014: Generalizing $\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}$ we define

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} y^k \mathrm{Li}_n(q^k x),$$

$$\mathrm{E}_{n;\,m}(x;\,y;\,q) =$$

$$\begin{cases} \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) - \mathrm{ELi}_{2;\,0}(x^{-1};\,y^{-1};\,q)\right) & ,\, n+m \text{ even}, \\ \frac{1}{2}\mathrm{Li}_2(x) + \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) + \mathrm{ELi}_{2;\,0}(x^{-1};\,y^{-1};\,q) & ,\, n+m \text{ odd}. \end{cases}$$

With this function, we obtain

$$S^{(0)}(2,\,t) = \frac{\psi_1(q)}{\pi}\sum_{i=1}^{3} \mathrm{E}_{2;\,0}(w_i(q);\,-1;\,-q) \text{ where } q = e^{\pi i \frac{\psi_1}{\psi_2}}.$$

The arguments $w_1,\,w_2,\,w_3$ are obtained from the intersection points $P_1,\,P_2,\,P_3$ by above transformations of the **elliptic curve**.

$\Rightarrow$ Every term in the result can be related to the underlying geometry.

Higher coefficients and $D = 4$ dimensions:
Tarasov's method (1996, 1997) relates coefficients of

$$S(2-2\epsilon,\ t) = S^{(0)}(2,\ t) + S^{(1)}(2,\ t)\epsilon + \mathcal{O}\left(\epsilon^2\right),$$
$$S(4-2\epsilon,\ t) = S^{(-2)}(4,\ t)\epsilon^{-2} + S^{(-1)}(4,\ t)\epsilon^{-1} + S^{(0)}(4,\ t) + \mathcal{O}(\epsilon)$$

- We consider **differential equations** for the higher coefficients of $S(2-2\epsilon,\ t)$.
- We solve the system explicitely to obtain $S^{(1)}(2,\ t)$.
  $\Rightarrow$This implies an explicit result for $S^{(0)}(4,\ t)$.(Adams, C.B., Weinzierl 2015, a)
- For the case $m_1 = m_2 = m_3$ we provide a method to compute **all coefficients** of $S(2-2\epsilon,\ t)$.
  This proves, that all of these terms belong to a certain class of elliptic generalizations of polylogarithms. (Adams, C.B., Weinzierl 2015, b)

As a further extension of

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk}$$

we define

$$\mathrm{ELi}_{n_1,...,n_l;m_1,...,m_l;2o_1,...,2o_{l-1}}(x_1,...,x_l;y_1,...,y_l;q)$$

$$= \sum_{j_1=1}^{\infty}...\sum_{j_l=1}^{\infty}\sum_{k_1=1}^{\infty}...\sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}}...\frac{x_l^{j_l}}{j_l^{n_l}}\frac{y_1^{k_1}}{k_1^{m_1}}..\frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1k_1+...+j_lk_l}}{\prod_{i=1}^{l-1}(j_ik_i+...+j_lk_l)}$$

For the case of $m_1 = m_2 = m_3$ and $D = 2$ **all ϵ-coefficients** $S^{(i)}$ in

$$S(2-2\epsilon,\ t) = \sum_{i=0}^{\infty} S^{(i)}(2,\ t)\epsilon^i$$

Conclusions:

- For Feynman integrals which can be computed with the help of **multiple polylogarithms**, powerful methods and programs are available today.
- The **Maple program MPL** is based on iterated integrals representing the multiple polylogarithms.

  It supports the computation of a certain class of Feynman integrals.
- For some Feynman integrals we have to go **beyond** multiple polylogarithms.
- Here the computation of the **massive sunrise integral** suggests a class of **elliptic generalizations** of polylogarithms.
- A further investigation of these functions and their relation with **elliptic iterated integrals** (Brown, Levin 2010, Broedel, Mafra, Matthes, Schlotterer 2014) will be interesting.

Differential equations

The sunrise integral $S(D, t)$ satisfies an inhomogeneous fourth-order differential equation (Caffo, Czyz, Laporta, Remiddi 1998) in t:

$$\left(P_4 \frac{d^4}{dt^4} + P_3 \frac{d^3}{dt^3} + P_2 \frac{d^2}{dt^2} + P_1 \frac{d^1}{dt^1} + P_0 \right) S\left(D,\, t\right) = c_{12}\, T_{12} + c_{13}\, T_{13} + c_{23}\, T_{23}$$

where T_{ij} are products of two tadpole integrals of propagators with masses m_i and m_j and where all P_k and c_{ij} are polynomials in $m_1^2,\, m_2^2,\, m_3^2,\, t,\, D$.

Each of the ϵ-coefficients $S^{(0)}(2,\, t)$, $S^{(1)}(2,\, t)$, $S^{(0)}(4,\, t)$ satisfies an inhomogeneous differential equation of second or higher order.

Remark: None of these differential operators **factorizes** completely into first order operators. If this would be the case, we could solve simply by iterated integration.