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Geometrical splitting and reduction of Feynman diagrams

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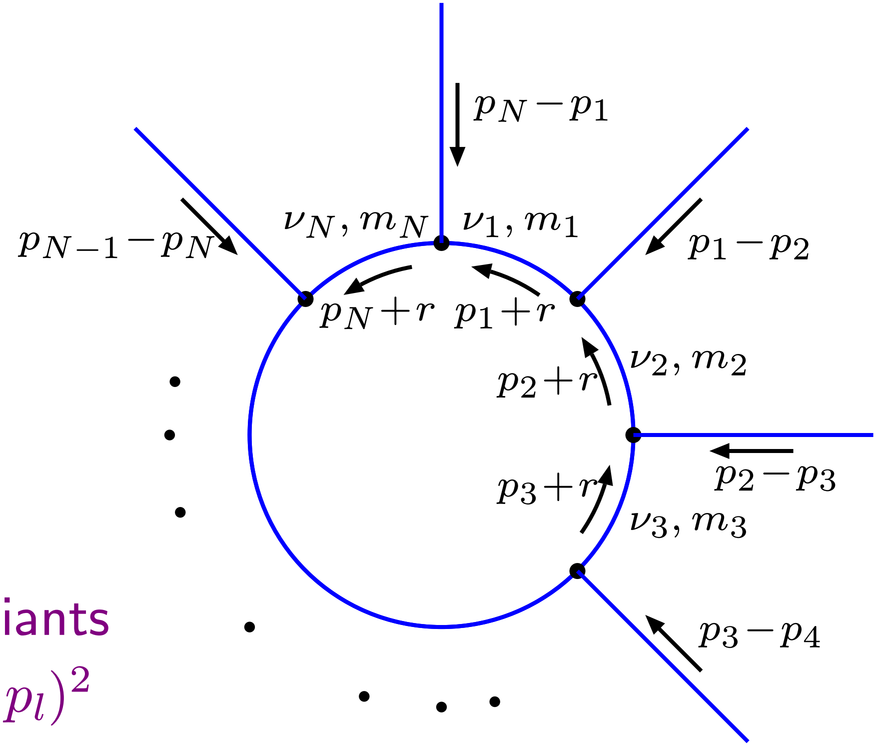
partly based on work with **R. Delbourgo** and **M. Yu. Kalmykov**

ACAT-2016

One-loop N-point function $J^{(N)}(n;\nu_1,\ldots,\nu_N)$



Depends on

$\frac{1}{2}N(N-1)$ invariants

$k_{jl}^2=(p_j-p_l)^2$

and N masses m_i

$$J^{(N)}(n;\nu_1,\ldots,\nu_N)\equiv\int\frac{\mathrm{d}^n k}{\left[(p_1+k)^2-m_1^2\right]^{\nu_1}\cdots\left[(p_N+k)^2-m_N^2\right]^{\nu_N}}$$

Geometrical Approach

The idea is to use geometrical description not only when analyzing the singularities (thresholds, etc.), but also when *calculating* dimensionally-regulated Feynman integrals. In particular, it may be used to predict types of functions (and their arguments) appearing in higher orders of ε-expansion.
Such geometrical approach was developed and summarized in

A.I.D. and R. Delbourgo, J. Math. Phys. **39** (1998) 4299.

Examples include explicit results for arbitrary dimension n and/or *all* terms of the ε-expansion for

- the one-loop two-point function with arbitrary masses,
- two-loop vacuum diagrams with arbitrary masses,
- the three-point function with arbitrary momenta and masses

A.I.D., Phys. Rev. **D61** (2000) 087701;

A.I.D. and M.Yu. Kalmykov, Nucl. Phys. B (PS) **89** (2000) 283; Nucl. Phys. **B605** (2001) 266

A.I.D., AIHENP-99 Proceedings (hep-th/9908032); Nucl.Instr.Meth. **A559** (2006) 293

Feynman parameters

Parametric representation for the one-loop N-point function in n dimensions:

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-n}\pi^{n/2}\,\Gamma\left(N-\frac{n}{2}\right)\int\limits_0^1 \ldots \int\limits_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i - 1\right)}{\left[\sum\limits_{j<l}\sum \alpha_j\alpha_l k_{jl}^2 - \sum \alpha_i m_i^2\right]^{N-n/2}}$$

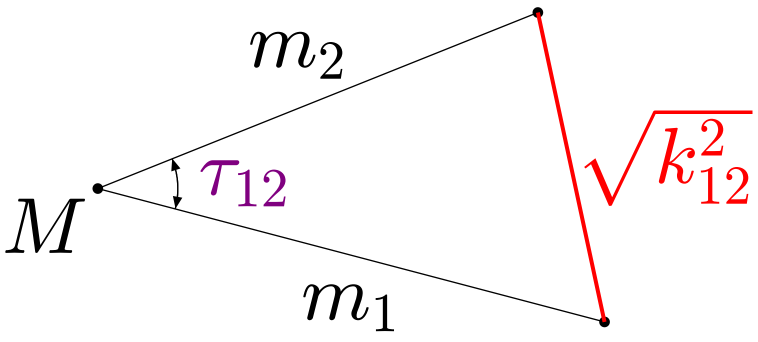
By using $\sum\alpha_i = 1$ we can make the quadratic form homogeneous in α_i:

$$\left[\sum\limits_{j<l}\sum \alpha_j\alpha_l k_{jl}^2 - \left(\sum\alpha_i\right)\left(\sum\alpha_i m_i^2\right)\right] \Rightarrow -\left[\sum\alpha_i^2 m_i^2 + 2\sum\limits_{j<l}\sum \alpha_j\alpha_l m_j m_l c_{jl}\right],$$

$$c_{jl} \equiv \frac{m_j^2+m_l^2-k_{jl}^2}{2m_jm_l}, \qquad c_{jl} = \cos\tau_{jl} = \begin{cases} 1, & k_{jl}^2 = (m_j-m_l)^2 \quad \text{pseudothreshold} \\ -1, & k_{jl}^2 = (m_j+m_l)^2 \quad \text{threshold} \end{cases}$$

Direct geometrical interpretation: when $-1 \le c_{jl} \le 1$ (i.e., angles τ_{jl} are real)

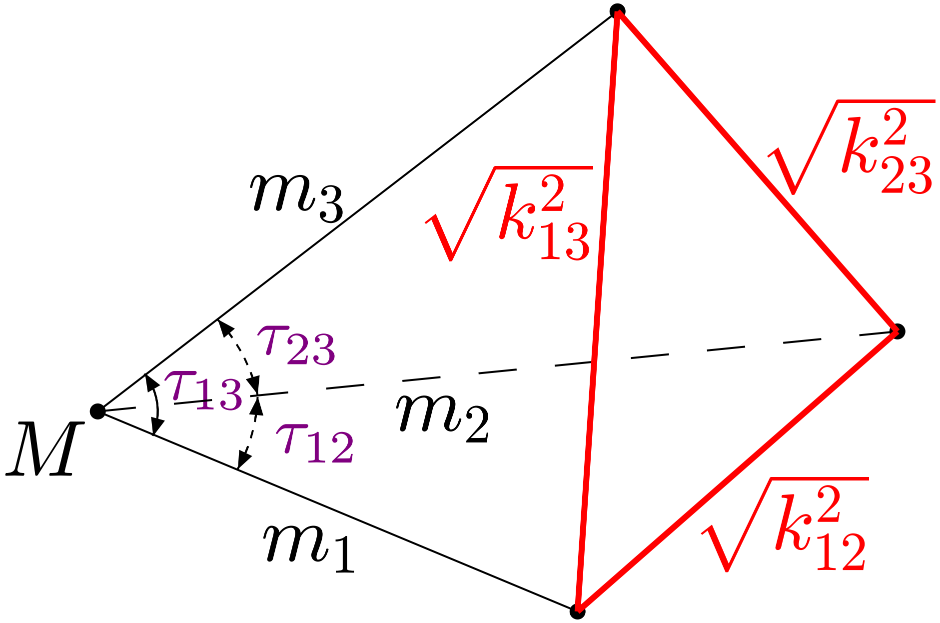
Two-point function: the basic triangle



$$\cos\tau_{12} = c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}$$

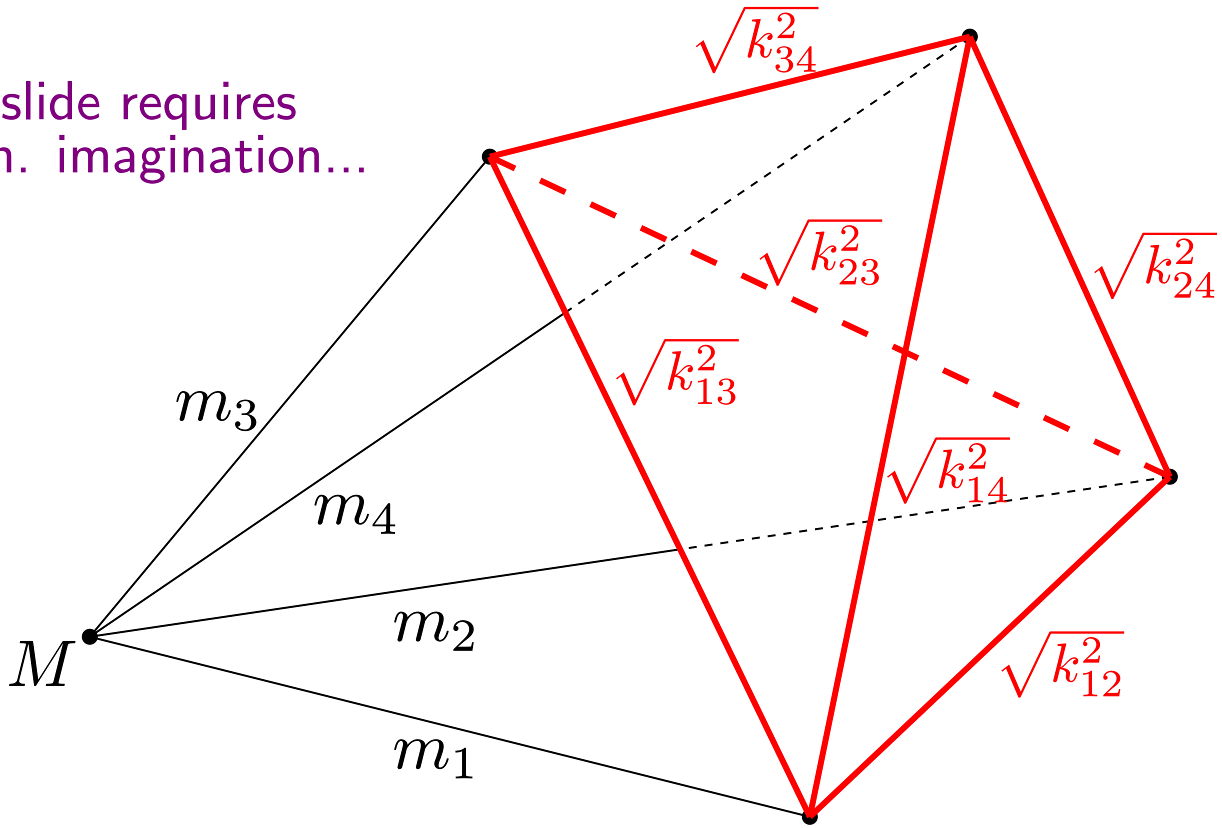
$$c_{12} = \cos\tau_{12} = \begin{cases} 1, & k_{12}^2 = (m_1 - m_2)^2 \quad \text{pseudothreshold} \quad (\tau_{12} = 0) \\ -1, & k_{12}^2 = (m_1 + m_2)^2 \quad \text{threshold} \quad (\tau_{12} = \pi) \end{cases}$$

Three-point function: the basic tetrahedron



The basic simplex for $N = 4$

This slide requires
4-dim. imagination...

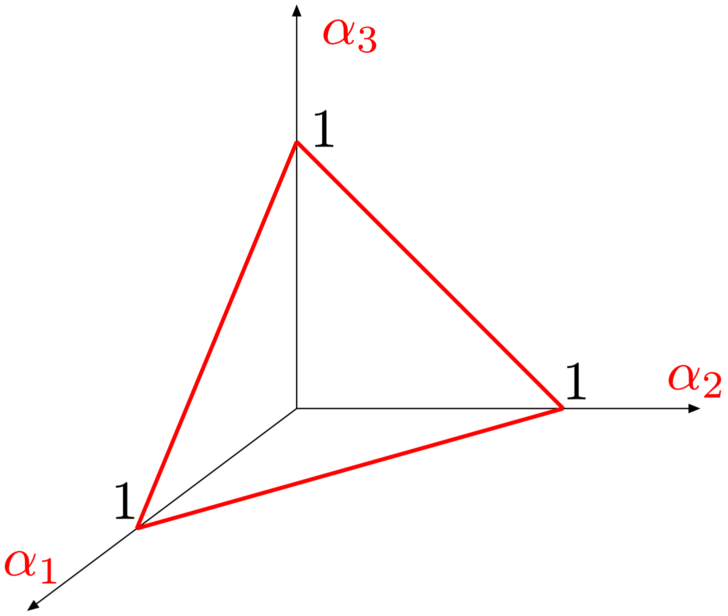


$$D^{(N)} = \det \|c_{jl}\| \, , \qquad \Lambda^{(N)} = \det \|(k_{jN} \cdot k_{lN})\| \, ,$$

$$V^{(N)} = \frac{(\Pi m_i)}{N!}\sqrt{D^{(N)}}, \qquad \overline{V}_0^{(N-1)} = \frac{1}{(N-1)!}\sqrt{\Lambda^{(N)}}, \qquad m_0 = (\Pi m_i)\sqrt{\frac{D^{(N)}}{\Lambda^{(N)}}}$$

Feynman parameters: limits of integration

$$\int\limits_0^1 \ldots \int\limits_0^1 \left(\prod \mathsf{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right) \{\ldots\} = \int\limits_0^\infty \ldots \int\limits_0^\infty \left(\prod \mathsf{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right) \{\ldots\}$$



Feynman parameters: substitutions

Using linear and *quadratic* substitutions of α variables, we arrive at

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\Gamma\left(N-\frac{n}{2}\right)\left(\Pi f_i\right)\int\limits_0^\infty\ldots\int\limits_0^\infty\frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\alpha^T\|C\|\alpha-1\right)}{\left(\sum\alpha_i f_i\right)^{n-N}}$$

Modified matrix: $C_{jl} = \left(\sqrt{F_j^{(N)}}c_{jl}\sqrt{F_l^{(N)}}\right)$, with $F_i^{(N)} = \frac{\partial}{\partial m_i^2}\left(m_i^2 D^{(N)}\right)$

obeying $\sum\limits_{l=1}^{N} c_{jl}F_l^{(N)}\frac{1}{m_l} = D^{(N)}\frac{1}{m_j} \Rightarrow \sum\limits_{l=1}^{N} C_{jl}\frac{\sqrt{F_l^{(N)}}}{m_l} = D^{(N)}\frac{\sqrt{F_j^{(N)}}}{m_j} \Rightarrow$

Eigenvector: $f_i = \frac{\sqrt{F_i^{(N)}}}{m_i}$, Eigenvalue: $D^{(N)} = \det\|c_{jl}\|$ (Gram determinant)

Feynman parameters: diagonalization

Whenever a quadratic form occurs, an obvious idea is to *diagonalize* it: "rotate" variables $\alpha_i \to \beta_i$ so that $\alpha^T \|C\| \alpha = \sum_{i=1}^{N} \lambda_i \beta_i^2$
One of the β's (say β_N) is directed along f_i, so that $\lambda_N = D^{(N)}$ and denominator $(\sum \alpha_i f_i)$ is proportional to β_N.

Assume (for a moment) that all $\lambda_i > 0$ and rescale $\beta_i = \frac{\gamma_i}{\sqrt{\lambda_i}} \Rightarrow$

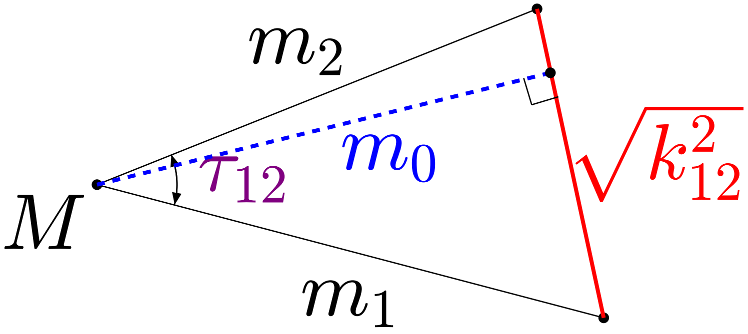
$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\;\int_{\Omega^{(N)}}\!\!\ldots\int \frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\,\delta\left(\sum \gamma_i^2 - 1\right)$$

Remarkably: the same N-dim. solid angle $\Omega^{(N)}$ as in the *basic simplex*!

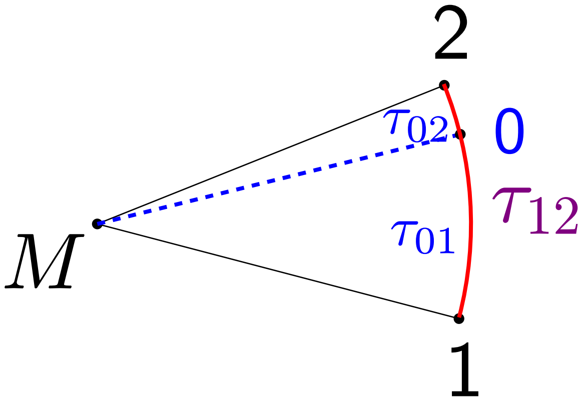
Special case: $N = n$ ($N = 2$ in 2d, $N = 3$ in 3d, $N = 4$ in 4d, etc.)

If some of λ_i are negative – *hyperbolic* surface (instead of *spherical*)
↔ analytical continuation!

Two-point function, geometrical approach



the basic triangle

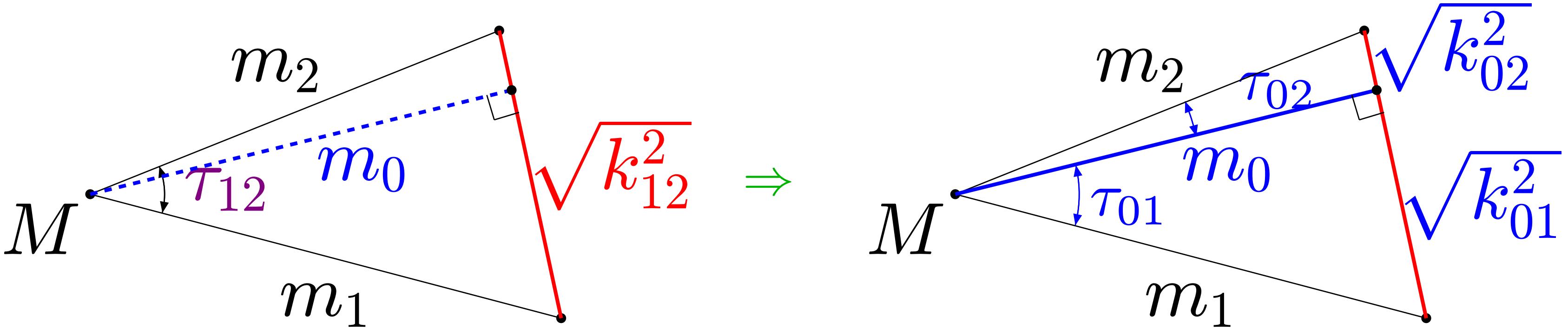


the arc τ_{12}

$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2\tau_{12}, \quad \Lambda^{(2)} = k_{12}^2,$$

$$m_0 = m_1m_2\sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1m_2\sin\tau_{12}}{\sqrt{k_{12}^2}}, \qquad \cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}.$$

Two-point function, splitting the basic triangle



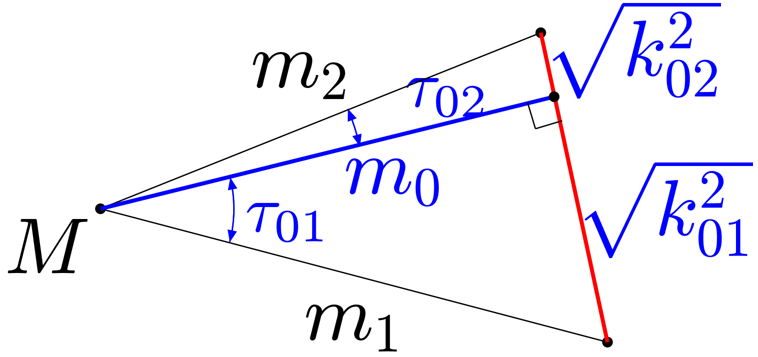
$$k_{01}^2 = \frac{(k_{12}^2 + m_1^2 - m_2^2)^2}{4k_{12}^2}, \qquad k_{02}^2 = \frac{(k_{12}^2 - m_1^2 + m_2^2)^2}{4k_{12}^2}$$

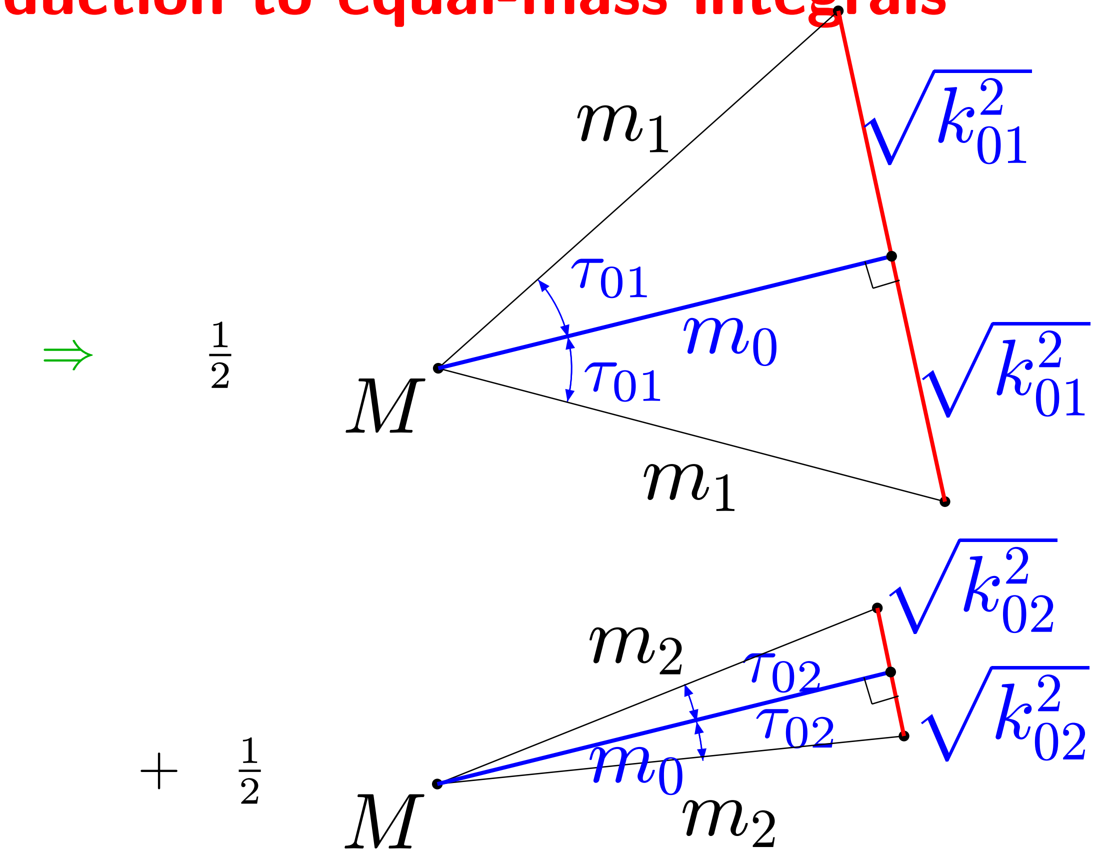
$$\begin{aligned} J^{(2)}\left(n; 1, 1 | k_{12}^2; m_1, m_2\right) \quad = \quad & \frac{1}{2k_{12}^2} \Big\{ (k_{12}^2 + m_1^2 - m_2^2) J^{(2)}\left(n; 1, 1 | k_{01}^2; m_1, m_0\right) \\ & + (k_{12}^2 - m_1^2 + m_2^2) J^{(2)}\left(n; 1, 1 | k_{02}^2; m_2, m_0\right) \Big\} \end{aligned}$$

This is an example of a functional relation between integrals with different momenta and massses, similar to those described in

O.V. Tarasov, Phys.Lett. **B670** (2008) 67

Two-point function, reduction to equal-mass integrals





⇒ equal-mass functions

$$J^{(2)}\left(n;1,1|k_{12}^2;m_1,m_2\right) = \frac{1}{4k_{12}^2}\Big\{(k_{12}^2+m_1^2-m_2^2)J^{(2)}\left(n;1,1|4k_{01}^2;m_1,m_1\right)$$

$$+(k_{12}^2-m_1^2+m_2^2)J^{(2)}\left(n;1,1|4k_{02}^2;m_2,m_2\right)\Big\}$$

with $k_{01}^2 = \frac{(k_{12}^2+m_1^2-m_2^2)^2}{4k_{12}^2}$, $k_{02}^2 = \frac{(k_{12}^2-m_1^2+m_2^2)^2}{4k_{12}^2}$

Two-point function in arbitrary dimension

$$J^{(2)}(n;1,1|k_{12}^2;m_1,m_2) = \mathrm{i}\pi^{n/2}\Gamma\left(\frac{4-n}{2}\right)\,\frac{m_0^{n-3}}{\sqrt{k_{12}^2}}\left\{\Omega_1^{(2;n)}+\Omega_2^{(2;n)}\right\}$$

with

$$\Omega_i^{(2;n)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos^{n-2}\theta} = \tan\tau_{0i}\;\; (\cos\tau_{0i})^{4-n}\;\; {}_2F_1\left(\begin{array}{c} 1, (4-n)/2 \\ 3/2 \end{array}\Big|\sin^2\tau_{0i}\right)$$

$$c_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \quad D^{(2)} = 1-c_{12}^2 = \sin^2\tau_{12}, \quad m_0 = m_1m_2\sqrt{\frac{D^{(2)}}{k_{12}^2}},$$

$$\cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01}+\tau_{02} = \tau_{12}.$$

Two-point function: number of dimensionless variables

in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$:

$$3-1(\text{dimension})=2$$

in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$:

$$3-1(\text{relation}: k_{01}^2=m_1^2-m_0^2)-1(\text{dimension})=1$$

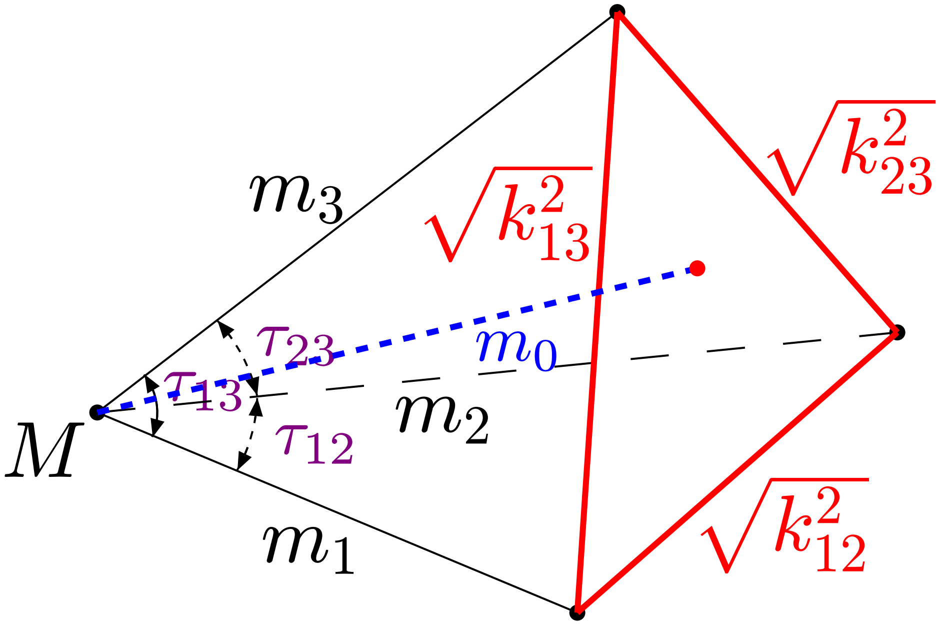
in $J^{(2)}\left(n;1,1\middle|4k_{01}^2;m_1,m_1\right)$

$$3-1(\text{relation}: \text{equal masses})-1(\text{dimension})=1$$

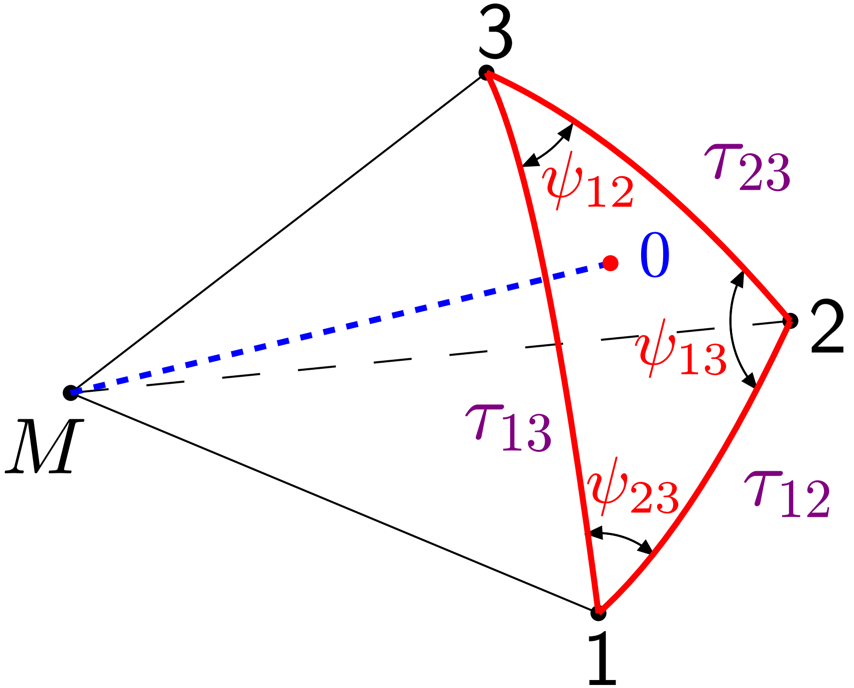
Here

$$k_{01}^2=\frac{(k_{12}^2+m_1^2-m_2^2)^2}{4k_{12}^2}$$

Three-point function: geometrical approach



the basic tetrahedron



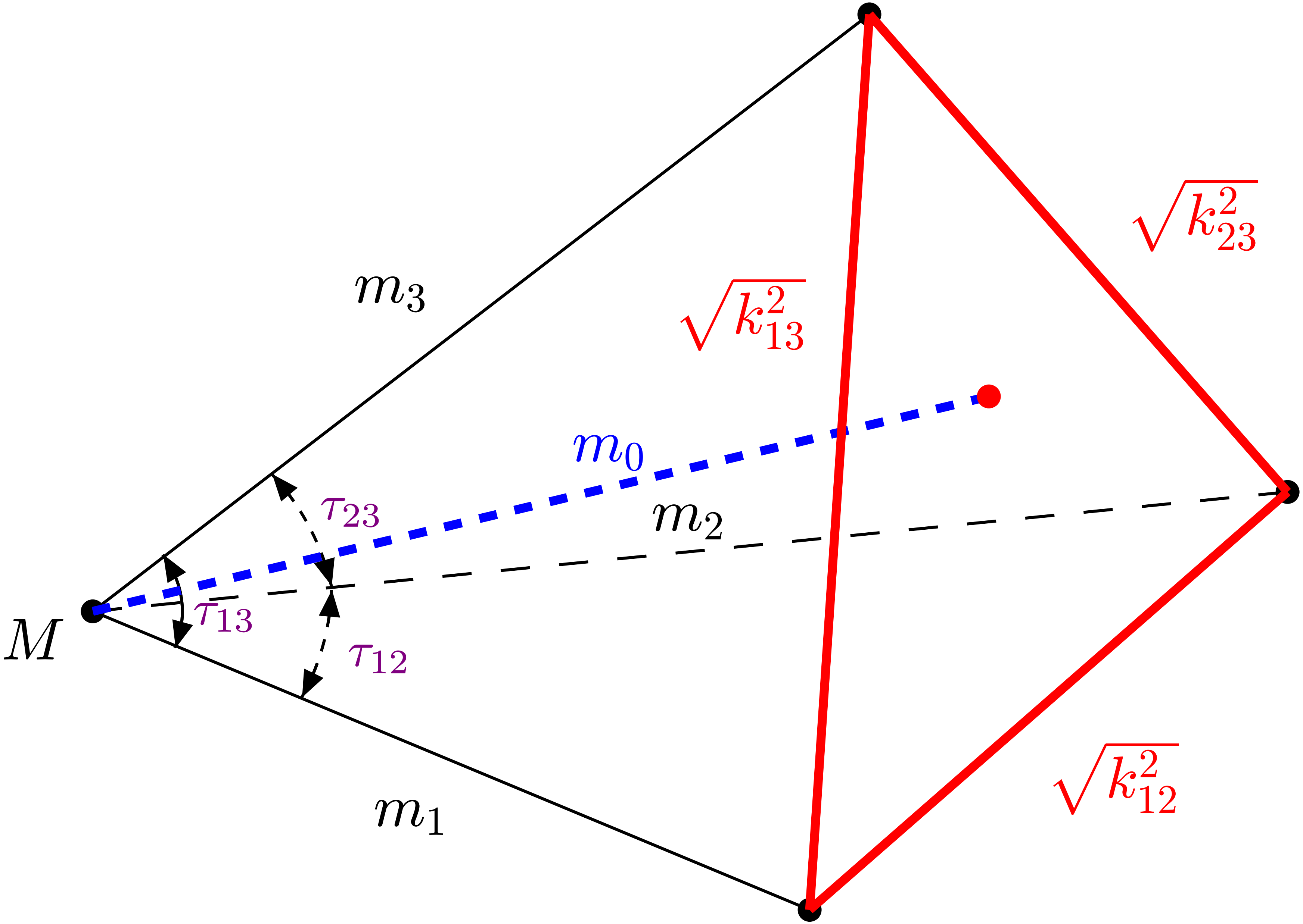
the solid angle

Special case $n = 3 \Rightarrow$ the area of spherical triangle ("spherical excess"):

$$\Omega^{(3;3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi \ .$$

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Three-point function: the basic tetrahedron



Three-point function: an example of non-symmetric splitting

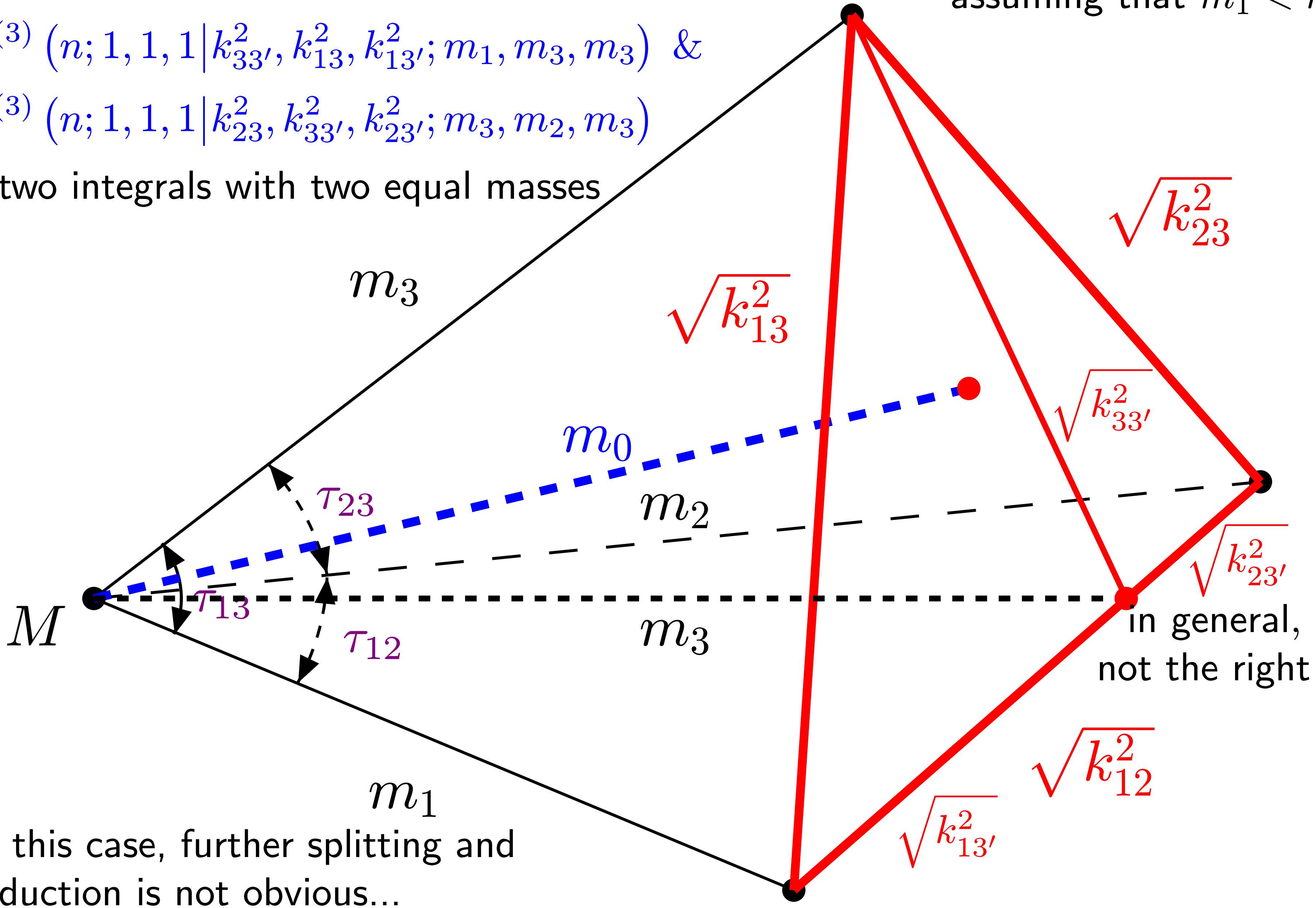
$J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$

$\Rightarrow J^{(3)}\left(n;1,1,1\middle|k_{33'}^2,k_{13}^2,k_{13'}^2;m_1,m_3,m_3\right)$ &

$J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{33'}^2,k_{23'}^2;m_3,m_2,m_3\right)$

– two integrals with two equal masses

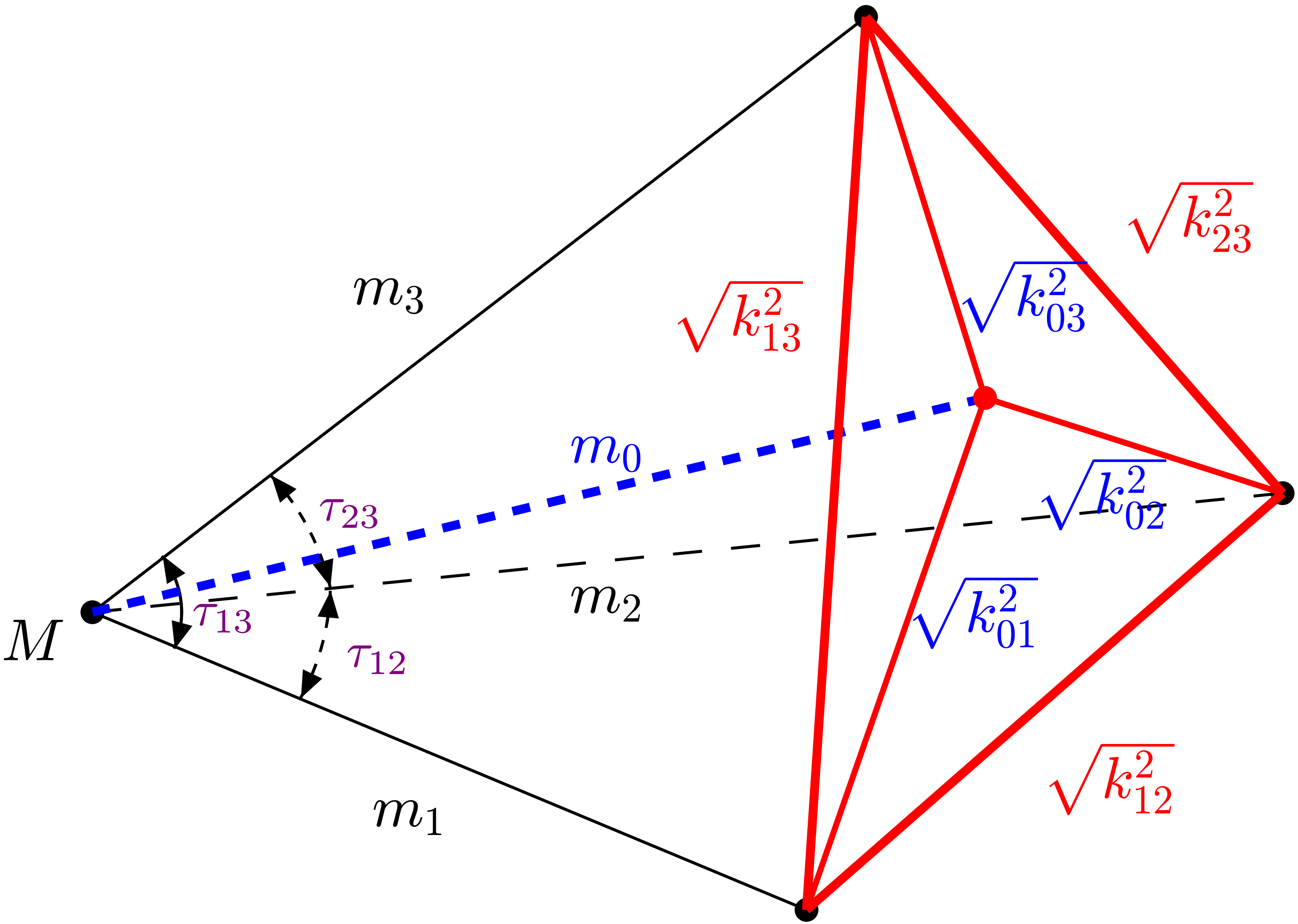
*assuming that $m_1 < m_3 < m_2$



in general,
not the right angle

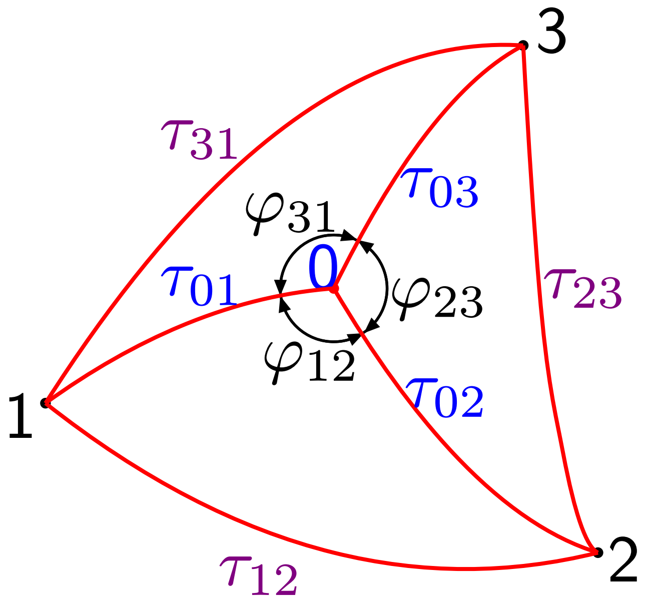
In this case, further splitting and
reduction is not obvious...

Three-point function: symmetric splitting using the height m_0



Three-point function: splitting the solid angle

Relation to the angles accociated with a spherical (or hyperbolic) triangle:



$$\varphi_{12}+\varphi_{23}+\varphi_{31}=2\pi$$

$$\cos\tau_{12}=\frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \text{ etc.}$$

$$\cos\tau_{0i}=\frac{m_0}{m_i} \qquad (i=1,2,3)$$

$$m_0=m_1m_2m_3\sqrt{\frac{D^{(3)}}{\Lambda^{(3)}}}$$

$$D^{(3)}=\left|\begin{array}{ccc}1 & c_{12} & c_{13}\\ c_{12} & 1 & c_{23}\\ c_{13} & c_{23} & 1\end{array}\right|, \quad \Lambda^{(3)}=\tfrac{1}{4}\left[2k_{12}^2k_{13}^2+2k_{13}^2k_{23}^2+2k_{23}^2k_{12}^2-(k_{12}^2)^2-(k_{13}^2)^2-(k_{23}^2)^2\right]$$

Three-point function: splitting the basic tetrahedron

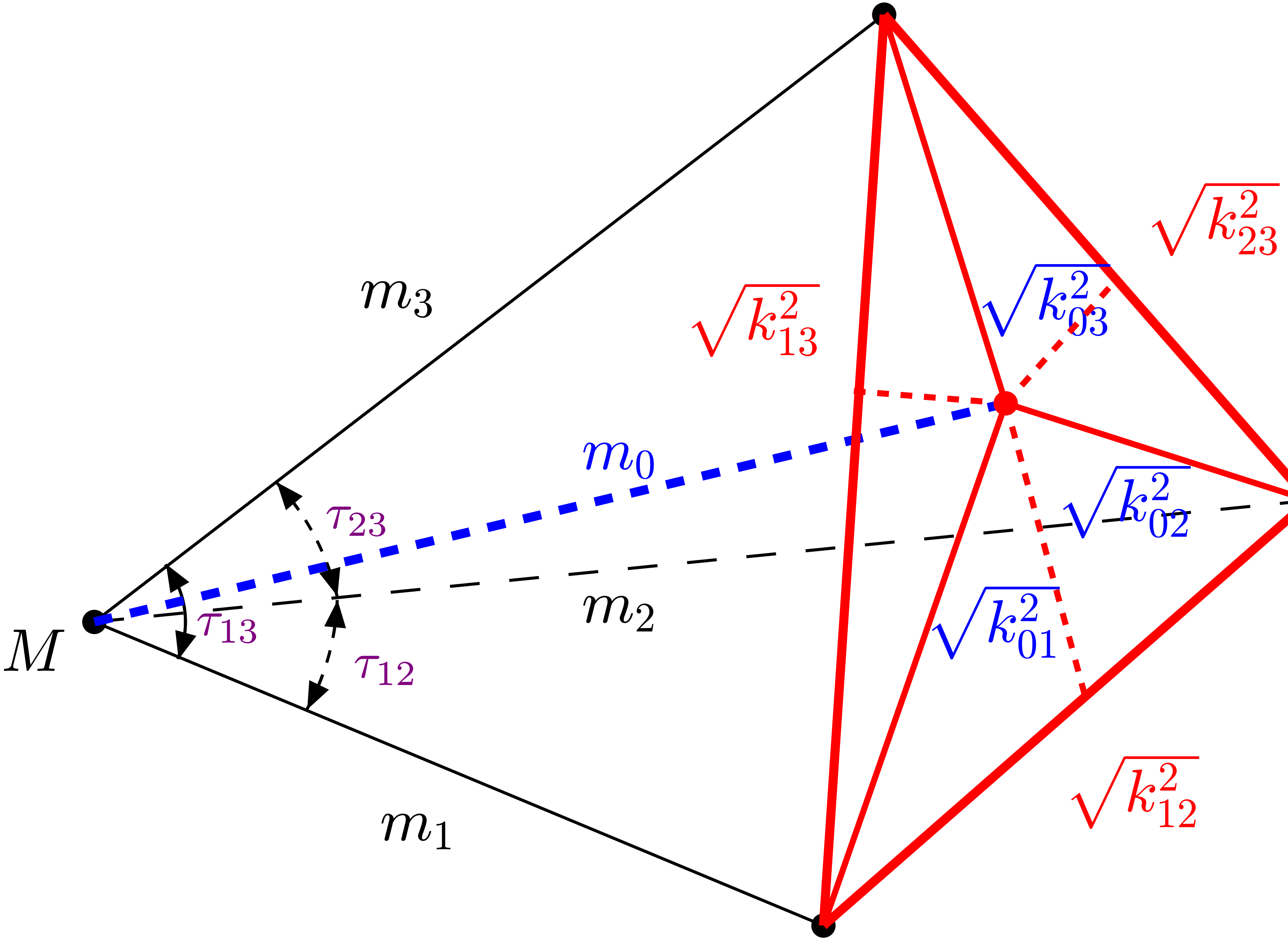
$$
\begin{aligned}
&J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)\\
&\quad=\frac{m_1^2m_2^2m_3^2}{\Lambda^{(3)}}\left\{\frac{F_1^{(3)}}{m_1^2}J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{03}^2,k_{02}^2;m_0,m_2,m_3\right)\right.\\
&\qquad+\frac{F_2^{(3)}}{m_2^2}J^{(3)}\left(n;1,1,1\middle|k_{03}^2,k_{13}^2,k_{01}^2;m_1,m_0,m_3\right)\\
&\qquad\left.+\frac{F_3^{(3)}}{m_3^2}J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)\right\}
\end{aligned}
$$

with $k_{01}^2=m_1^2-m_0^2,\qquad k_{02}^2=m_2^2-m_0^2,\qquad k_{03}^2=m_3^2-m_0^2,$

$$F_3^{(3)}=\frac{1}{4m_1^2m_2^2}\left[k_{12}^2\left(k_{13}^2+k_{23}^2-k_{12}^2+m_1^2+m_2^2-2m_3^2\right)-\left(m_1^2-m_2^2\right)\left(k_{13}^2-k_{23}^2\right)\right],$$ etc.

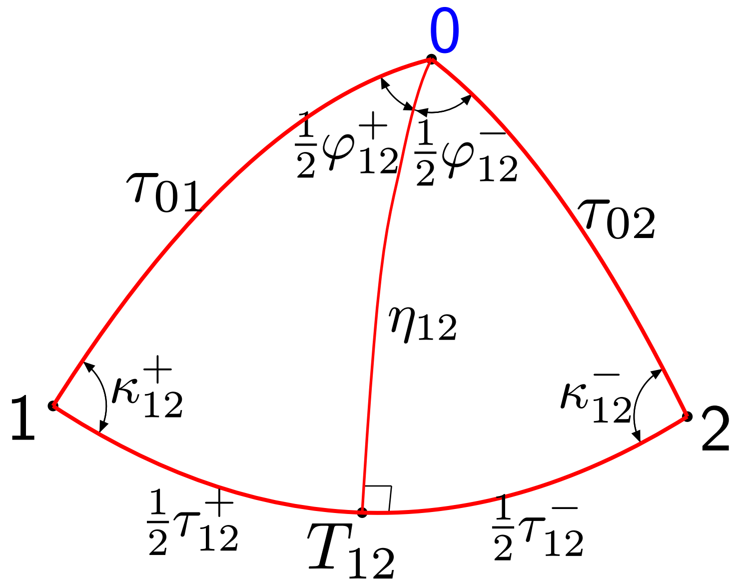
$$\frac{F_1^{(3)}}{m_1^2}+\frac{F_2^{(3)}}{m_2^2}+\frac{F_3^{(3)}}{m_3^2}=\frac{\Lambda^{(3)}}{m_1^2m_2^2m_3^2}$$

Three-point function: further splitting of the basic tetrahedron



Three-point function: further splitting

One of the three triangles ($\frac{1}{2}(\varphi_{12}^{+} + \varphi_{12}^{-}) = \varphi_{12}$):



Three-point function: reduction to integrals with two equal masses

$$J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$$

$$= \frac{1}{2k_{12}^2}\left\{(k_{12}^2+m_1^2-m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{01}^2,k_{01}^2,\frac{(k_{12}^2+m_1^2-m_2^2)^2}{k_{12}^2};m_1,m_1,m_0\right)\right.$$

$$\left.+(k_{12}^2-m_1^2+m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{02}^2,\frac{(k_{12}^2-m_1^2+m_2^2)^2}{k_{12}^2};m_2,m_2,m_0\right)\right\}$$

— similarly to the reduction of the two-point function

Three-point function: number of dimensionless variables

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$:

$$6-1(\text{dimension})=5$$

in $J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$:

$$6-2(\text{relations})-1(\text{dimension})=3$$

in $J^{(3)}\left(n;1,1,1\middle|k_{01}^2,k_{01}^2,\frac{(k_{12}^2+m_1^2-m_2^2)^2}{k_{12}^2};m_1,m_1,m_0\right)$

$$6-3(\text{relations})-1(\text{dimension})=2$$

Three-point function in arbitrary dimension

$$J^{(3)}(n;1,1,1) = -\frac{\mathrm{i}\pi^{n/2}}{\sqrt{\Lambda^{(3)}}}\,\Gamma\left(3-\frac{n}{2}\right)\; m_0^{n-4}\,\Omega^{(3;n)}\;,$$

$$\begin{aligned}\Omega^{(3;n)} = \int\!\!\int\limits_{\Omega^{(3)}} \frac{\sin^{n-2}\theta\;\mathrm{d}\theta\;\mathrm{d}\phi}{\cos^{n-3}\theta} \quad &= \quad \omega\left(\tfrac{1}{2}\varphi_{12}^{+},\eta_{12}\right) + \omega\left(\tfrac{1}{2}\varphi_{12}^{-},\eta_{12}\right) \\ &\quad\; +\omega\left(\tfrac{1}{2}\varphi_{23}^{+},\eta_{23}\right) + \omega\left(\tfrac{1}{2}\varphi_{23}^{-},\eta_{23}\right) \\ &\quad\; +\omega\left(\tfrac{1}{2}\varphi_{31}^{+},\eta_{31}\right) + \omega\left(\tfrac{1}{2}\varphi_{31}^{-},\eta_{31}\right)\;,\end{aligned}$$

with

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right) = -\frac{1}{n-4}\int\limits_0^{\varphi/2}\mathrm{d}\phi\left[1-\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{(n-4)/2}\right]$$

A.I.D., hep-th/9908032, Nucl.Instr.Meth. A559 (2006) 293

The result for arbitrary dimension $n = 4 - 2\varepsilon$ can be presented in terms of Appell's hypergeometric function F_1,

$$\omega\left(\tfrac{1}{2}\varphi, \eta\right) = \frac{1}{2\varepsilon}\left[\frac{\varphi}{2} - \sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\cos^{2\varepsilon}\tau_0\; F_1\left(1, 1, \varepsilon; \tfrac{3}{2}\middle|\; \sin^2\frac{\varphi}{2}, \sin^2\frac{\tau}{2}\right)\right],$$

with $\cos\tau_0 = \cos\eta\;\cos\frac{\tau}{2}$,

$$F_1(a, b, b', c|x, y) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a)_{j+l}\;(b)_j\;(b')_l}{(c)_{j+l}}\;\frac{x^j\;y^l}{j!\;l!}$$

Similar functions occurred in

O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Some special cases: L.G. Cabral-Rosetti, M.A. Sanchis-Lozano, hep-ph/0206081

Special value of n: $n = 4$ $(\varepsilon \to 0)$:

$$\int\limits_0^{\varphi/2} \mathrm{d}\phi \;\ln\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right) = \tfrac{1}{2}\tau\;\ln\left(\frac{1+\sin\eta}{1-\sin\eta}\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi+\tau\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi-\tau\right) - \mathrm{Cl}_2\left(\varphi\right)$$

Compare with: P. Wagner, Indag. Math. 7 (1996) 527

After analytical continuation, corresponds to

G. 'tHooft and M. Veltman, Nucl. Phys. B153 (1979) 365

Analytic Continuation: Arbitrary Dimension

Consider $$\int\limits_0^{\varphi_0} \mathrm{d}\phi \left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} .$$

Substitute $z \Rightarrow e^{2\mathrm{i}\phi}$, so that $\cos^2\phi \Rightarrow \dfrac{(1+z)^2}{4z}$,

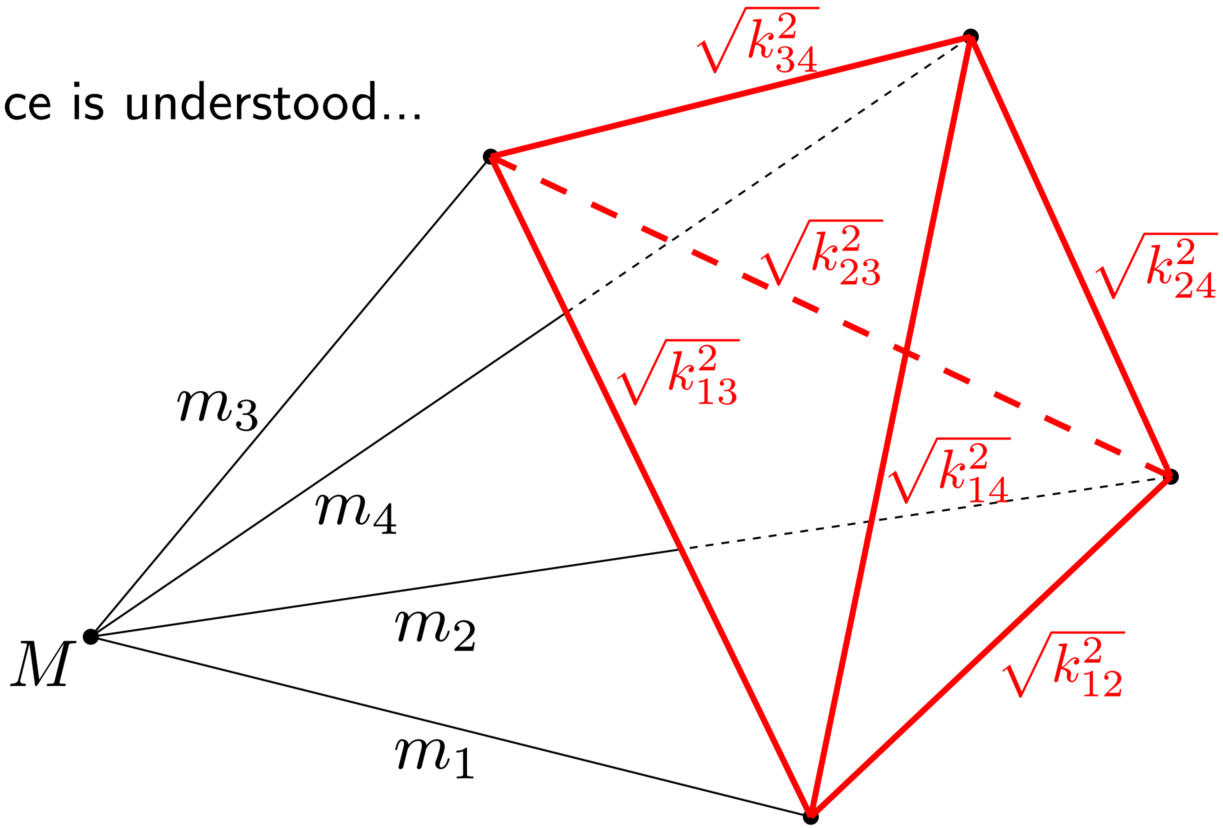
$$1 + \frac{\tan^2\eta}{\cos^2\phi} \Rightarrow \frac{(z+\rho)(z+1/\rho)}{(z+1)^2}, \quad \text{with} \quad \rho \equiv \frac{1-\sin\eta}{1+\sin\eta}$$

In this way, $$\int\limits_0^{\varphi_0} \mathrm{d}\phi \left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} \Rightarrow \frac{\mathrm{i}}{2} \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right]^{-\varepsilon},$$

with $z_0 \leftrightarrow e^{2\mathrm{i}\varphi_0}$.

Four-point function: the basic simplex for $N=4$

4-dim. space is understood...

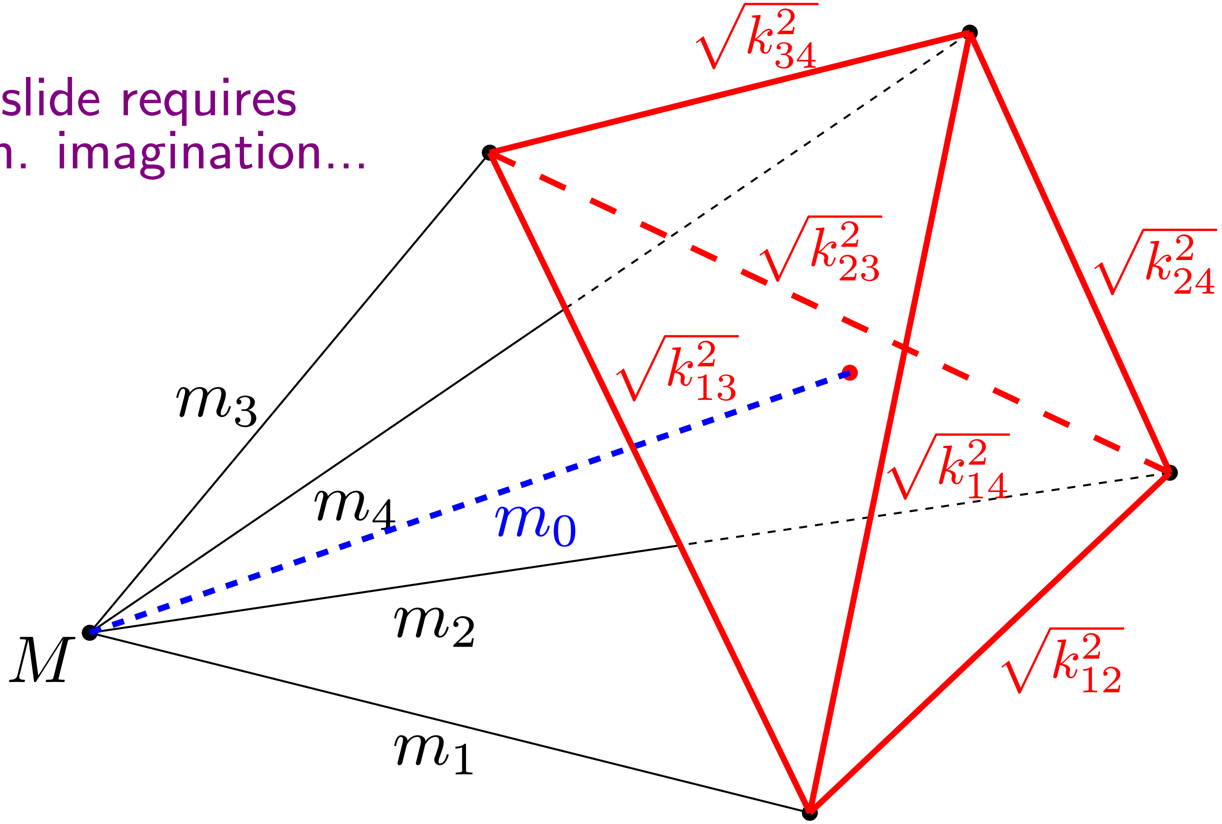


$$D^{(N)} = \det \|c_{jl}\| \,, \qquad \Lambda^{(N)} = \det \|(k_{jN} \cdot k_{lN})\| \,,$$

$$V^{(N)} = \frac{(\Pi m_i)}{N!}\sqrt{D^{(N)}}, \qquad \overline{V}_0^{(N-1)} = \frac{1}{(N-1)!}\sqrt{\Lambda^{(N)}}, \qquad m_0 = (\Pi m_i)\sqrt{\frac{D^{(N)}}{\Lambda^{(N)}}}$$

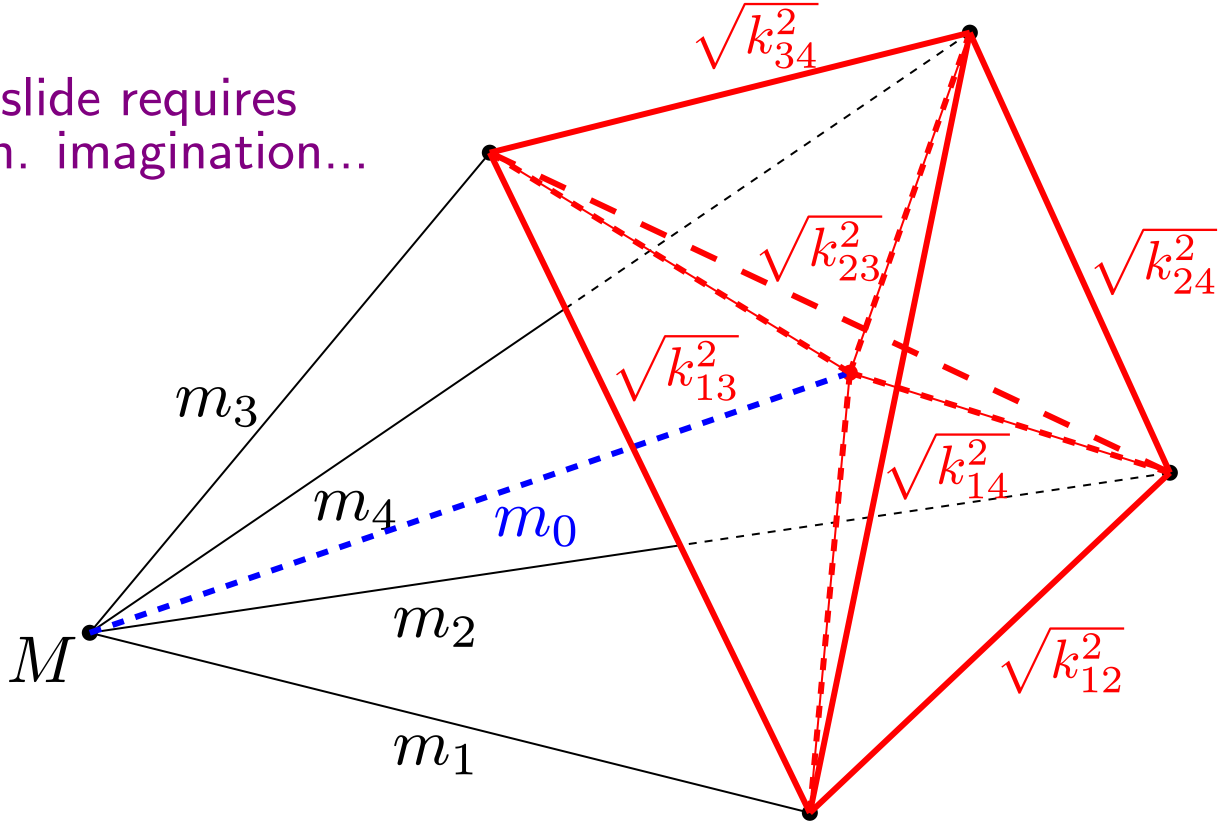
The basic simplex for $N = 4$

This slide requires
4-dim. imagination...



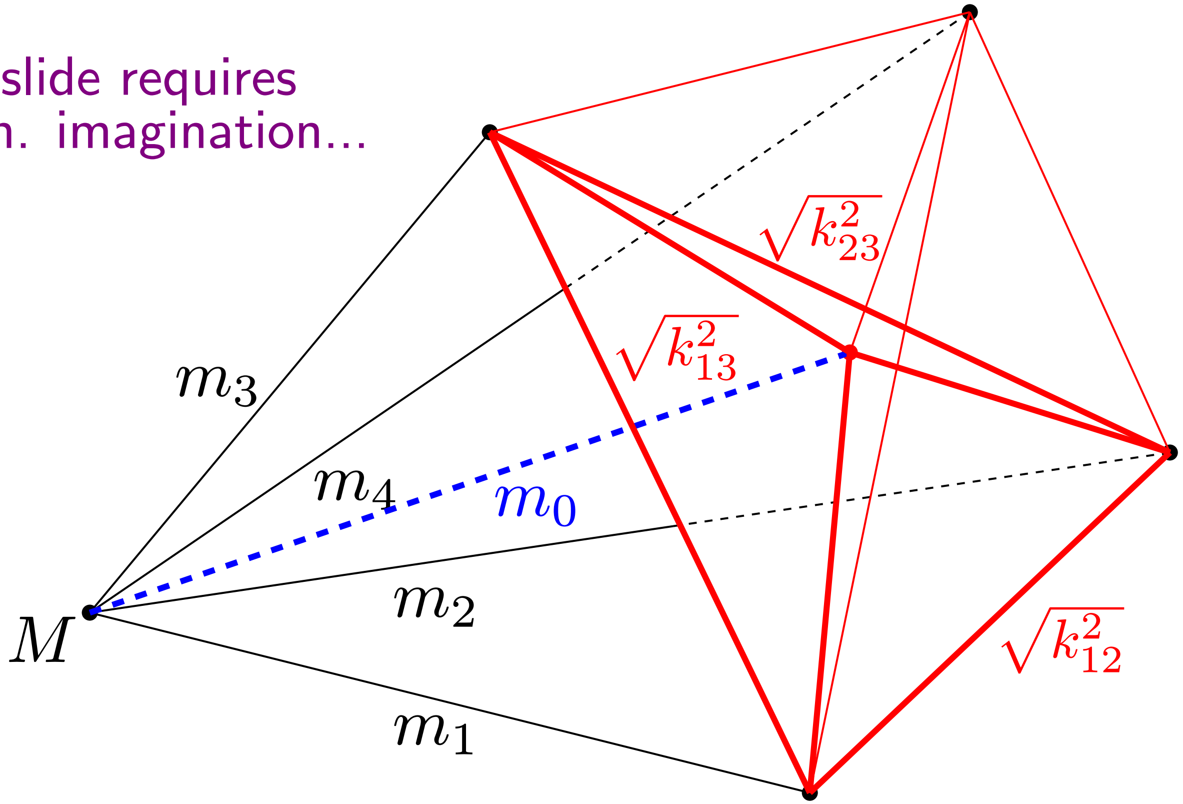
Four-point function: symmetric splitting using the height m_0

This slide requires
4-dim. imagination...



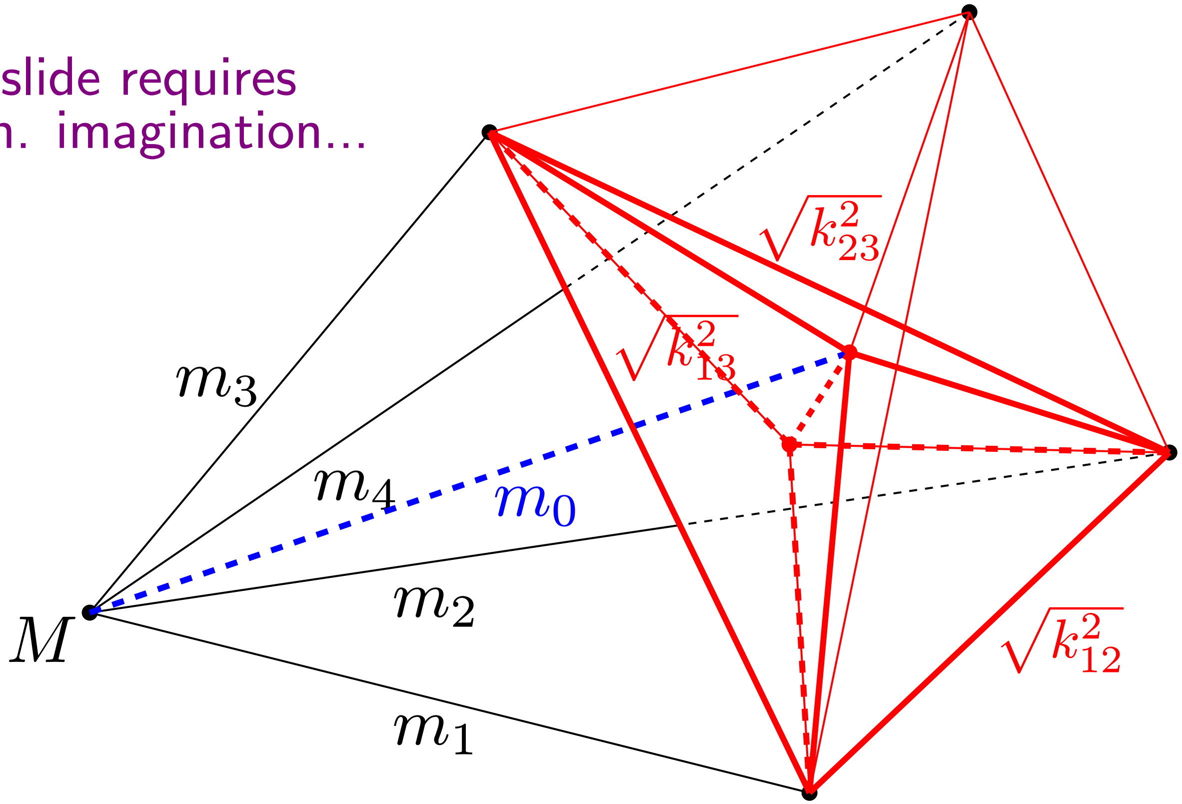
Four-point function: symmetric splitting using the height m_0

This slide requires
4-dim. imagination...



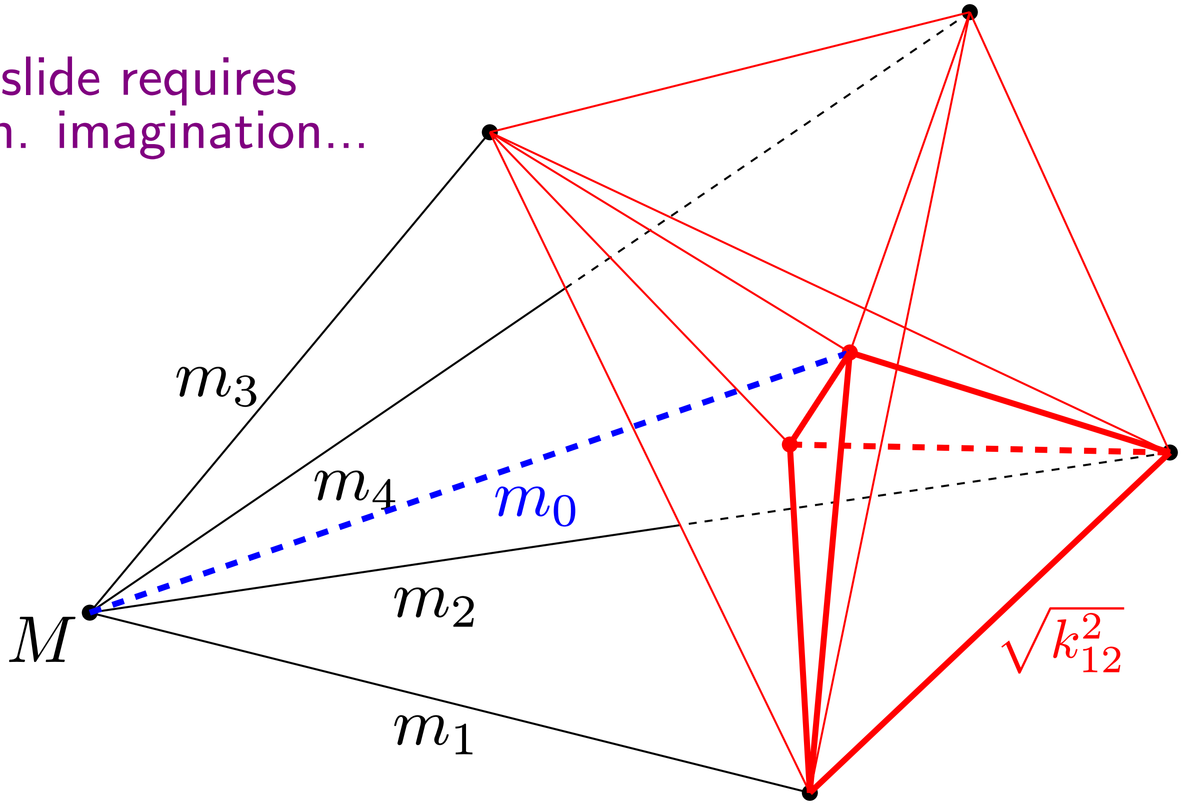
Four-point function: symmetric splitting using the height m_0

This slide requires
4-dim. imagination...



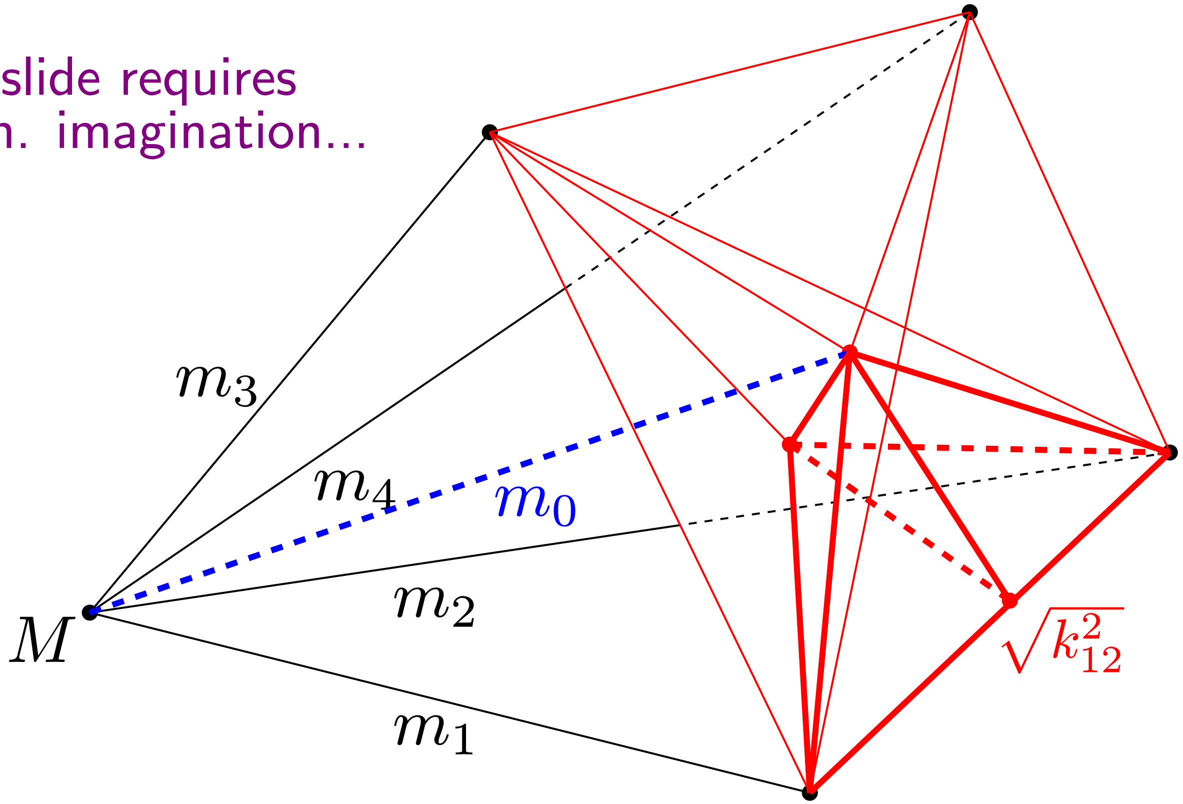
Four-point function: symmetric splitting using the height m_0

This slide requires
4-dim. imagination...



Four-point function: symmetric splitting using the height m_0

This slide requires
4-dim. imagination...



Four-point function: number of dimensionless variables

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{34}^2,k_{14}^2,k_{13}^2,k_{24}^2\};\{m_1,m_2,m_3,m_4\}\right)$:

$$10-1(\text{dimension})=9$$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{03}^2,k_{01}^2,k_{13}^2,k_{02}^2\};\{m_1,m_2,m_3,m_0\}\right)$:

$$10-3(\text{relations})-1(\text{dimension})=6$$

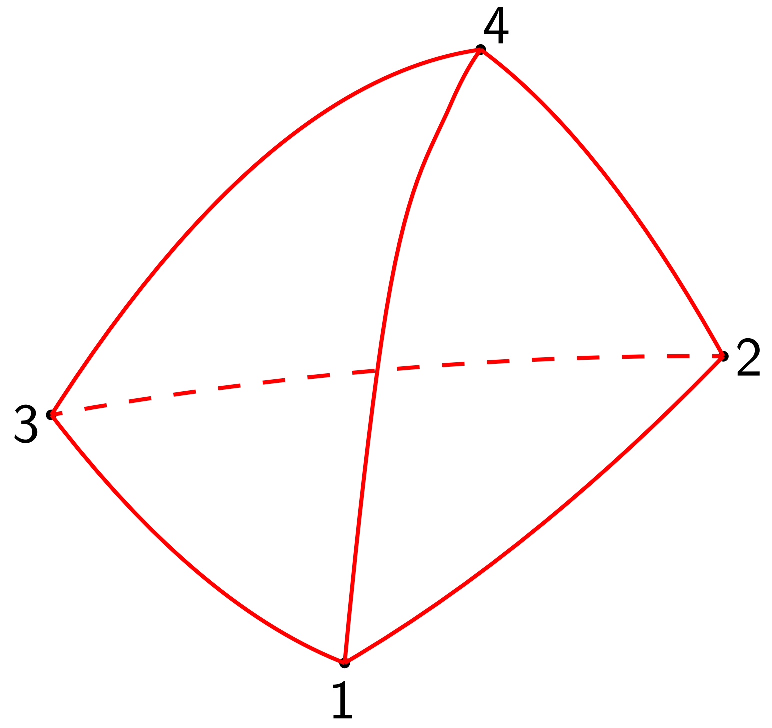
after splitting the tetrahedron 0123 into three tetrahedra:

$$10-5(\text{relations})-1(\text{dimension})=4$$

after splitting each of the resulting tetrahedra into two:

$$10-6(\text{relations})-1(\text{dimension})=3$$

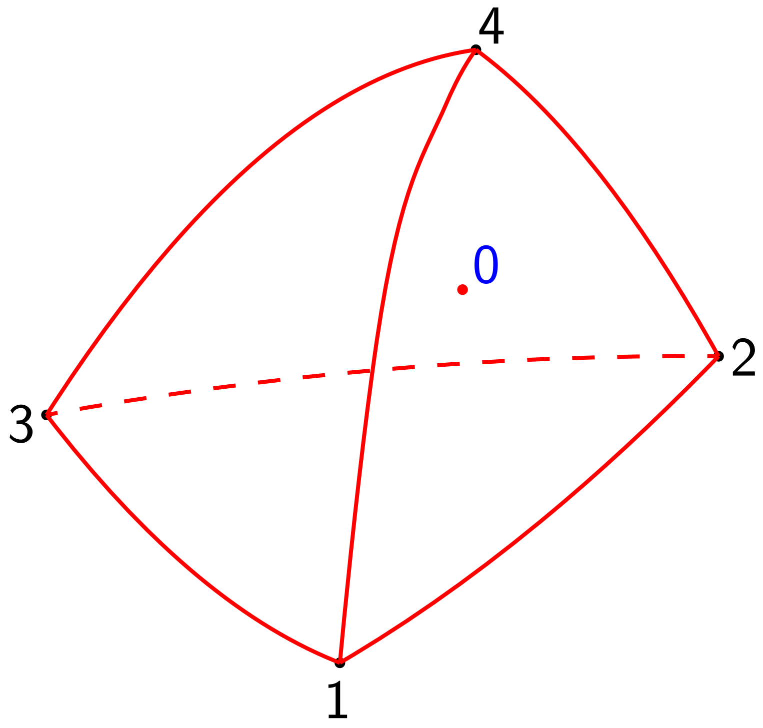
Geometrical approach: four-point function



The spherical tetrahedon

In four dimensions ($N = n = 4$), the result for the four-point function can be associated with the content of a spherical or hyperbolic tetrahedron in three-dimensional spherical or hyperbolic space (Lobachevsky, Schläfli, ...)

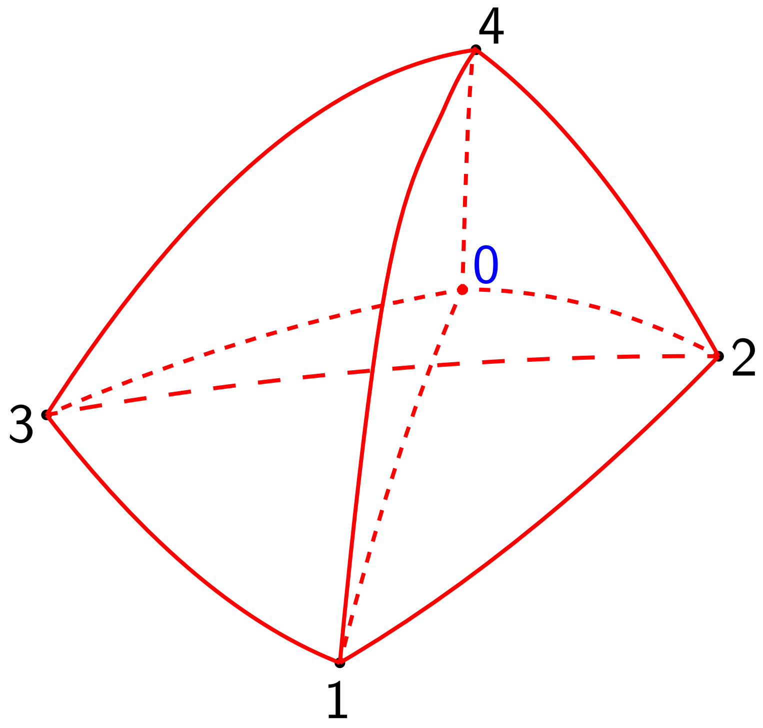
Geometrical approach: four-point function



The spherical tetrahedon

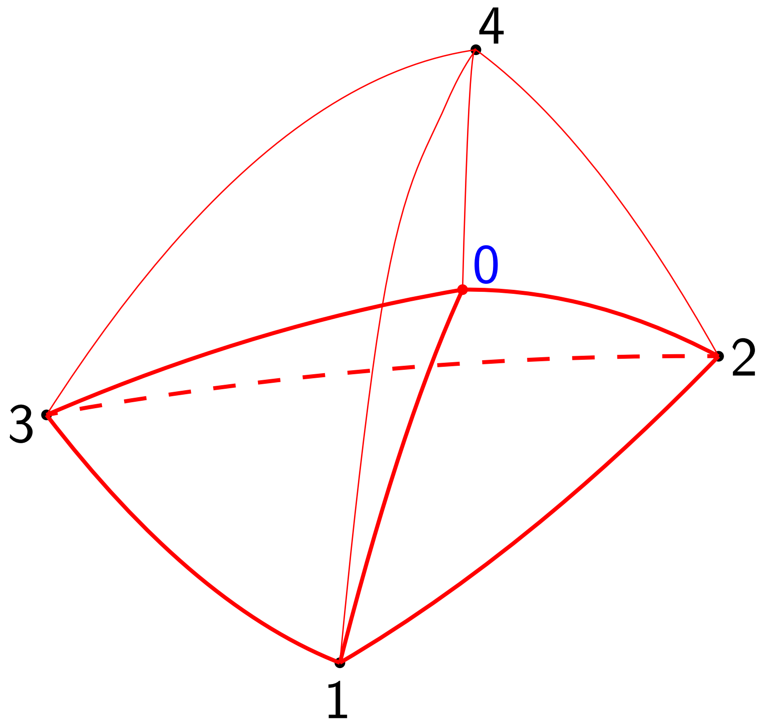
If $n \neq 4$, we need to do the splitting.

Geometrical approach: four-point function



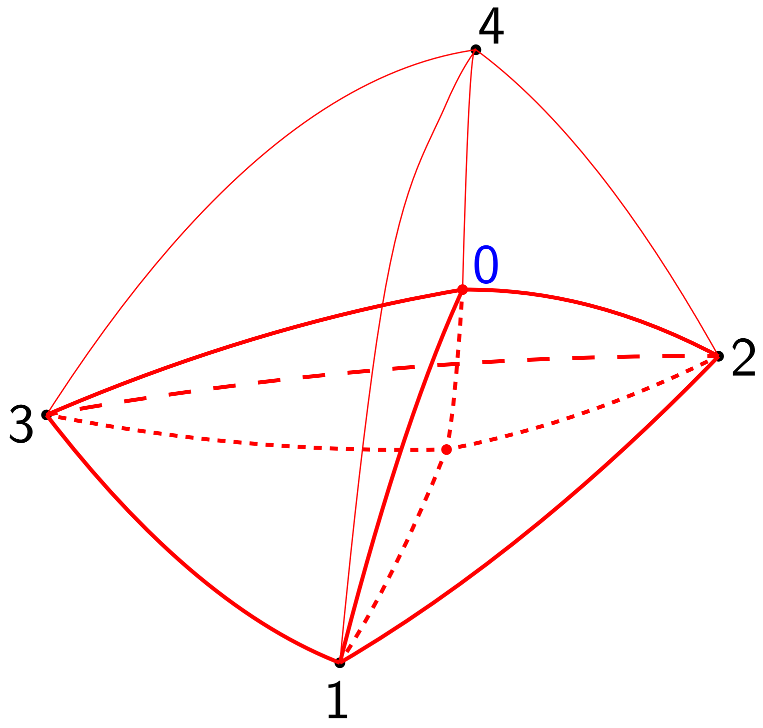
The spherical tetrahedon

Geometrical approach: four-point function



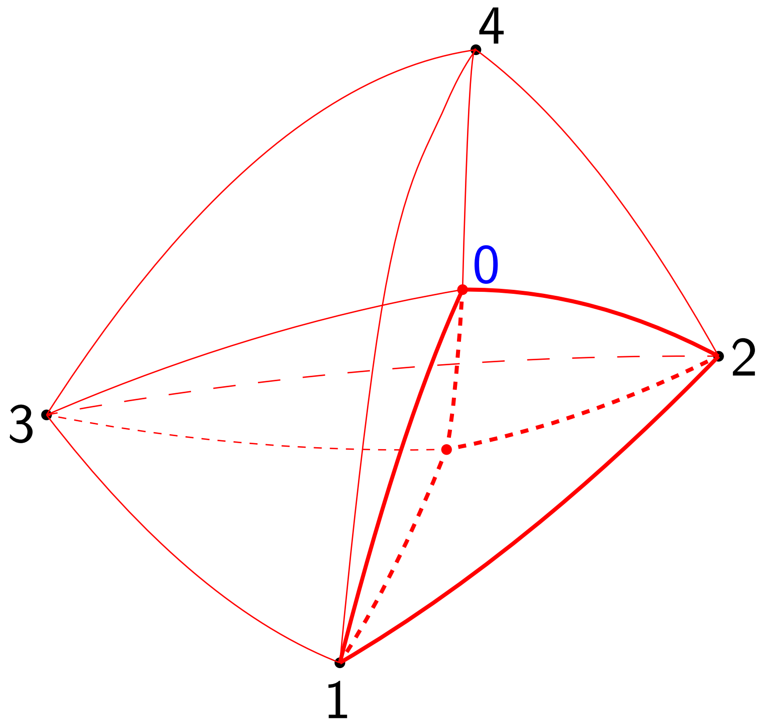
The spherical tetrahedon

Geometrical approach: four-point function



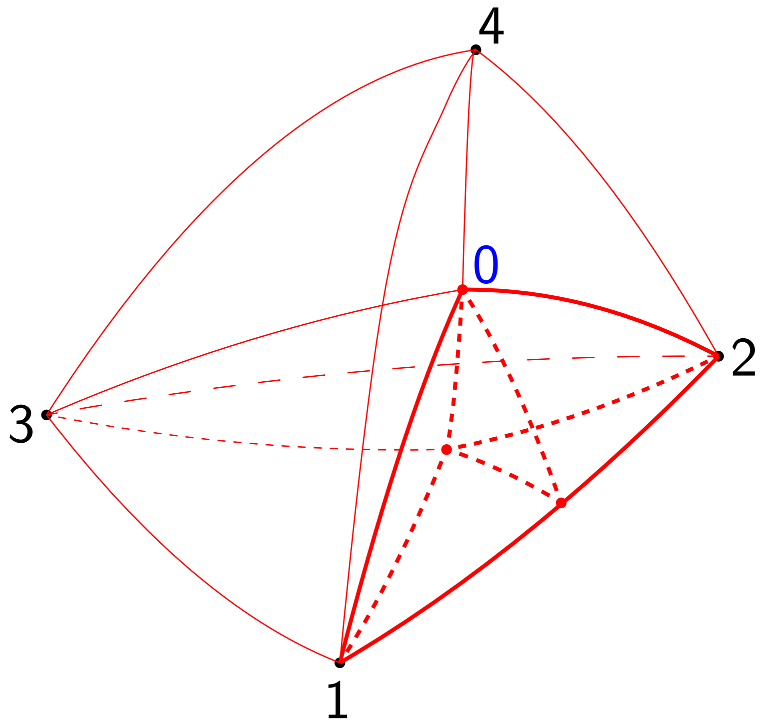
The spherical tetrahedon

Geometrical approach: four-point function



The spherical tetrahedon

Geometrical approach: four-point function



The spherical tetrahedon

Summary

- A geometrical way to calculate Feynman diagrams in arbitrary dimension is reviewed.
- In the one-loop N-point case, results can be related to certain volume integrals in non-Euclidean geometry.
- Analytical continuation of the results to other regions of kinematical variables (momenta and masses of the particles) is discussed.
- Geometrical splitting provides straightforward way of reducing general integrals to those with lesser number of independent variables and predict the set and the number of these variables in the resulting integrals, and allows to derive functional relations between integrals with different momenta and masses.

	total # of dimensionless variables	# of splitting pieces	reduced # of variables
$N = 2$	$3 - 1 = 2$	2	1
$N = 3$	$6 - 1 = 5$	6	2
$N = 4$	$10 - 1 = 9$	24	3
arbitrary N	$\frac{1}{2}(N-1)(N+2)$	$N!$	$N - 1 \quad (?)$