

Fun with higher-loop Feynman diagrams

York Schröder

(Univ del Bío-Bío, Chillán, Chile)

recent work with Thomas Luthe

and earlier work with:

J. Möller, C. Studerus

ACAT 2016, Valparaíso, 19 Jan 2016

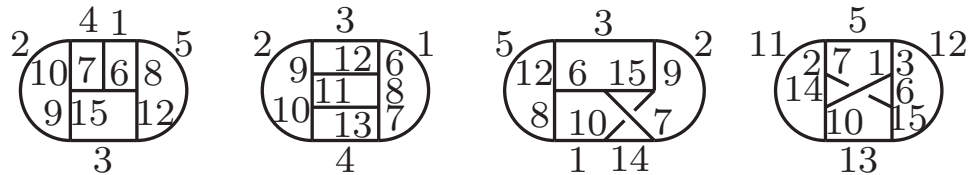
Motivation

- pressure of hot QCD
 - ▷ phenomenology needs physical NLO [Braaten/Nieto 96; KLRS 04]
 - ▷ hardest building block: 5-loop tadpoles(m,0) at $d = 3 - 2\epsilon$
- QCD beta function and anomalous dimensions
 - ▷ known since ~ 18 years at 4-loop accuracy [vanRitbergen/Vermaseren/Larin 97]
 - ▷ 5-loop needed e.g. for improved ρ -parameter, decoupling equation, ... [4loop: 06]
 - ▷ partial results appear since ~ 5 years
mainly from Karlsruhe (β^{QED} , γ_m , ...) [Baikov/Chetyrkin/Kühn/Rittinger 08-13]
- moments
 - ▷ many problems allow for asymptotic expansions
 - ▷ mapping on tadpoles, often for price of many dots
- in this talk: focus on master integrals
 - ▷ basic building block for 5-loop problems
 - ▷ interested in methods that allow to choose d in the end
 - ▷ main progress via refinement of Laporta approach
 - ▷ (problem-specific) reduction not covered here

Classification

- consider fully massive 5-loop tadpoles
 - ▷ Euclidean space-time
 - ▷ same mass in all propagators $\Rightarrow 1/(q_i^2 + 1)$
- the 5-loop integral family needs 15 propagators / lines
 - ▷ $q_i \in \{k_1, k_2, k_3, k_4, k_5, k_{13}, k_{14}, k_{15}, k_{23}, k_{24}, k_{25}, k_{35}, k_{45}, k_{124}, k_{34}\}$
 where $k_{a\dots bc} = k_a + \dots + k_b - k_c$

- ▷ trivalent graphs have 12 lines:



- classification: label sectors by binary rep
 - ▷ identify unique graphs
 - ▷ find all isometries and corresponding momentum shifts
 - ▷ choose largest representative from each class
- normalization: divide out $[1\text{-loop tadpole}]^{\#loops}$
 - ▷ recall that in 4d, $[1\text{-loop tadpole}] \sim 1/\epsilon$

Classification: non-trivial shift relations

- suppose you were given a 5-loop massive tadpole

[ID 2778 of our list above]

$$\int_{k_{1..5}}^{(d)} \frac{1}{k_4^2+1} \frac{1}{(k_1-k_3)^2+1} \frac{1}{(k_1-k_5)^2+1} \frac{1}{(k_2-k_3)^2+1} \frac{1}{(k_2-k_5)^2+1} \frac{1}{(k_3-k_5)^2+1} \frac{1}{(k_1+k_2-k_4)^2+1}$$

$$\text{shift } k_i^\mu \rightarrow k'^\mu_i = M_{ij} k_j^\mu \quad \text{with } M = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ -2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and $|\det M| = \frac{1}{2}$ then gives

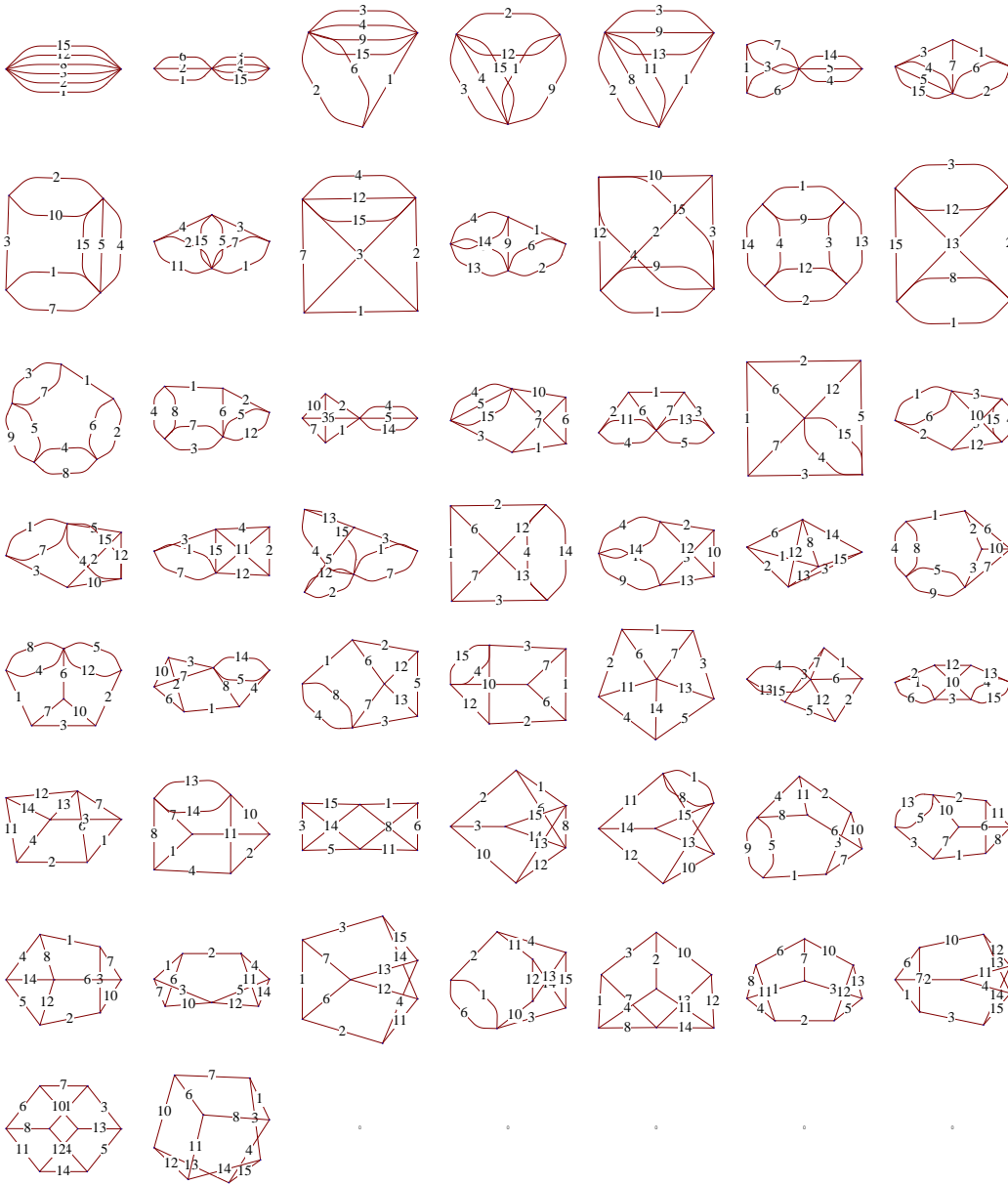
$$= 2^{-d} \int_{k_{1..5}}^{(d)} \frac{1}{k_2^2+1} \frac{1}{k_4^2+1} \frac{1}{(k_1-k_4)^2+1} \frac{1}{k_3^2+1} \frac{1}{(k_1-k_3)^2+1} \frac{1}{k_1^2+1} \frac{1}{k_5^2+1}$$

$$= 2^{-d} \times \text{diagram}$$

[ID 32512 of our list above]

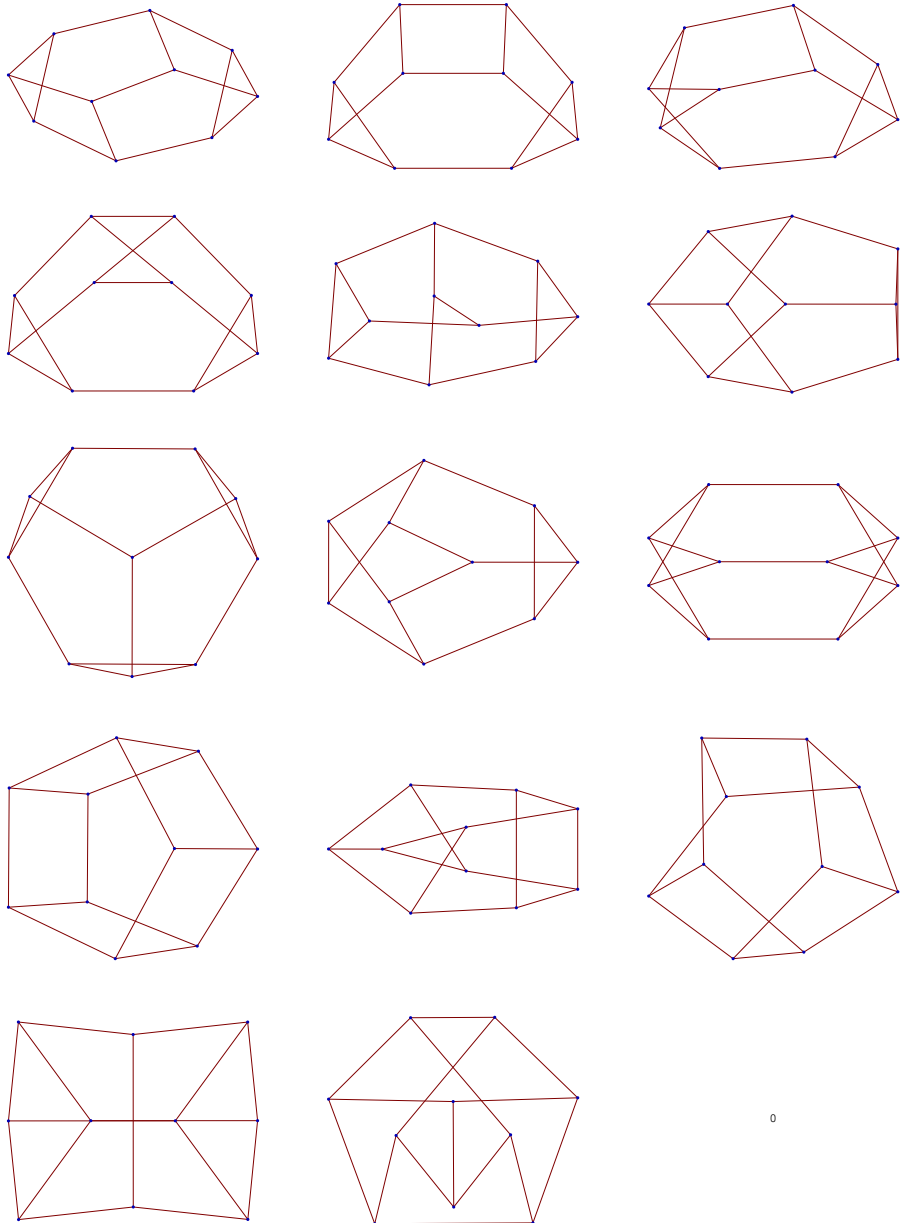
- does this (evil) momentum-labelling occur in practice?

Classification: 5-loop



- some numerology
 - ▷ for specific momentum list
 - ▷ combinatorics wins!
- $L(L+1)/2 + LE = 15$ scalar products
- $2^{15} = 32768$ possible sectors
- 5151 do not correspond to a Feynman graph
- $1941 + 3625$ zero-sectors
- $22 + 21962$ shifts
- $19 + 48$ unique sectors
(16 with 1-loop factors not shown)

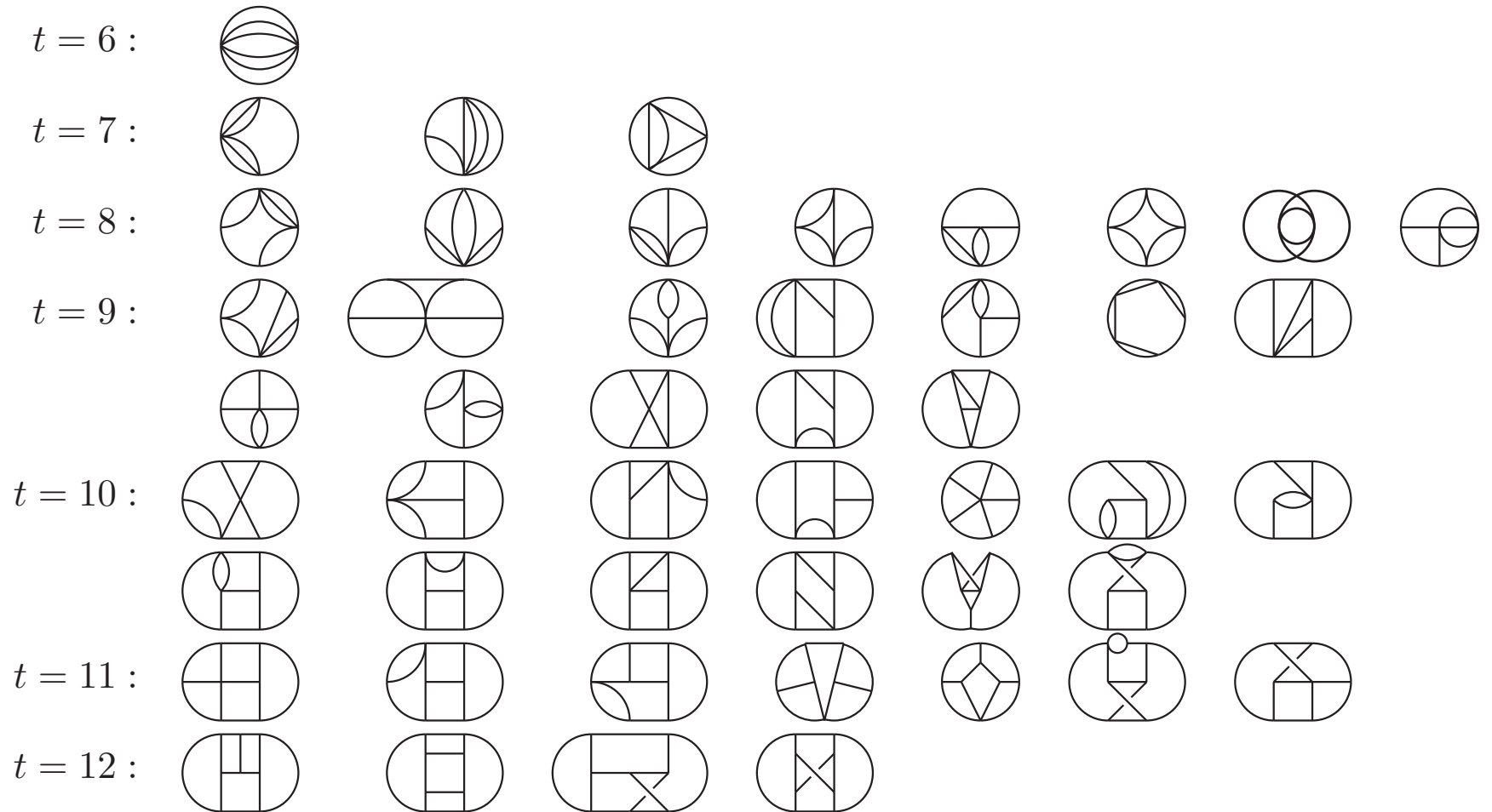
Classification: 6-loop



- 21 sc prod
- $2^{21} = 2.1\text{M}$ sectors
- 1.3M no graph
- 178K zeros
- 569K shifts
- 487 unique
(show 3-conn. cubic graphs)

5-loop Sectors

- arrive at 48 unique 5-loop sectors (+19 factorized ones not shown)



▷ recall that at 1/2/3/4-loop there were 1/1/3/10 unique sectors (plus 0/1/2/6 fact)

5-loop Masters

- a (small) IBP reduction reveals that some sectors contain multiple master integrals
 - ▷ need in addition 62 (+3 factorized ones) masters with 'dots'. some examples:



- ▷ recall that at 1/2/3/4-loop there were 0/0/0/3 masters with 'dots'

- how to evaluate these 48+62 (+19+3) zero-scale master integrals?
various methods, e.g.

- ▷ explicit integration in x-space
- ▷ differential eqs (in mass ratio); solve iteratively with HPLs
- ▷ explicit solution of low-order difference equations: ${}_P F_{P-1}$ etc.
- ▷ numerical solution of difference equations via factorial series

[Laporta 00]

- Mathematical structure

- ▷ interested in the coefficients of an ϵ expansion
- ▷ in many cases, these are from a generic class of functions/numbers
- ▷ e.g. harmonic polylogarithms HPL(x)
- ▷ e.g. harmonic sums $S(N)$
- ▷ relation: $H_{\bar{m}}(1) \rightarrow S_{\bar{m}}(\infty)$
- ▷ if solution numerical: use some PSLQ

[Remiddi/Vermaseren 00]

[Vermaseren 98]

Evaluation: differential equations

- perform IBP reduction with two masses: M, m
 - ▷ get differential eqn in $z = M/m$
 - ▷ use boundary values at $z = 0$ ($z = 1$)
 - ▷ use symmetry relations like $z \leftrightarrow 1/z$
 - ▷ typically, want the integral at $z = 1$ ($z = 0$)
- simple 3-loop example (basketball type)

$$B_{24}(z) \equiv \frac{1}{J^3} \int_{p_{1..3}}^{(d)} \frac{1}{p_1^2 + z^2} \frac{1}{p_2^2 + z^2} \frac{1}{p_3^2 + 1} \frac{1}{(p_1 + p_2 + p_3)^2 + 1}$$

$$B_{24}(z) = x^{3d-8} B_{24}(1/z) \quad , \quad B_{24}(0) = 2^{d-3} \frac{\Gamma(\frac{8-3d}{2})\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{7-2d}{2})\Gamma(\frac{2-d}{2})}$$

satisfies $\left\{ z(1-z^2)\partial_z^2 - 2(1-2z^2)(d-3)\partial_z - z(d-3)(3d-8) \right\} B_{24}(z) = (d-2)^2 z^{d-3} (z^{d-2} - 1)$

- solution standard, via variation of constants, in terms of HPL(z); set $z = 1$ and use algebra of HPL(1) resp. $S(\infty)$

$$B_4 = -2 + \dots$$

$$+ \epsilon^4 * (1141/24 - 112/3 * z^3) + \dots$$

$$+ \epsilon^7 * (418903/192 + 2278/45 * \pi^4 + 32/5 * \pi^6 - 3840 * s_6 - \dots) + \dots$$

Evaluation: difference equations

- perform IBP reduction with symbolic power \boldsymbol{x} on one line
- derive **difference equation** for generalized master $I(\boldsymbol{x}) \equiv \int \frac{1}{D_1^{\boldsymbol{x}} D_2 \dots D_N}$

$$\sum_{j=0}^R p_j(\boldsymbol{x}) I(\boldsymbol{x} + j) = F(\boldsymbol{x})$$

- typically, want $I(\mathbf{1})$; solve the difference equation
 - ▷ explicitly (if 1st order)
 - ▷ numerically (very general setup) [Laporta 00]
- solve via **factorial series** $I(\boldsymbol{x}) = I_0(\boldsymbol{x}) + \sum_{j=1}^R I_j(\boldsymbol{x})$, where

$$I_j(\boldsymbol{x}) = \mu_j^{\boldsymbol{x}} \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(\boldsymbol{x} + 1)}{\Gamma(\boldsymbol{x} + 1 + s - K_j)}$$

- need boundary condition for fixing, say, $a_j(0)$: use decoupling at large \boldsymbol{x}
 - ▷ $I(\boldsymbol{x}) = \int_{k_1} g(k_1)/(k_1^2 + 1)^{\boldsymbol{x}} \Rightarrow I(\boldsymbol{x}) \sim (1)^{\boldsymbol{x}} \boldsymbol{x}^{-d/2} g(0)$

Evaluation: difference equations

- Advantages
 - ▷ high level of automation
 - ▷ works well with divergent integrals
 - ▷ does not rely on specific function classes
 - ▷ high-precision results for arbitrary ε orders
 - ▷ can expand around any dimension
 - ▷ cross-checks by putting x on different lines
- Problems and limitations
 - ▷ limited use for multi-scale integrals
 - ▷ complexity of coefficients in high-order equations
 - ▷ high orders of recurrence relations
 - ▷ instability of factorial series in numerical evaluation
- Progress and fixes
 - ▷ use coupled equations
 - ▷ reduce recurrence relations
 - ▷ predict instability factors

Choice of basis of transcendentals

- to absorb single powers of π as well as powers of $\ln 3$, def

$$h_n \equiv \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)}{\Gamma(k + 1)\Gamma(1/2)} \frac{(3/4)^k}{(2k + 1)^n}$$

$$H_n \equiv h_n + h_1 \text{ Coefficient} \left[1 - \frac{3^{\epsilon/2}\Gamma(1 - \epsilon)}{\Gamma^2(1 - \epsilon/2)} + \mathcal{O}(\epsilon^n), \epsilon, n - 1 \right]$$

$$H_1 = h_1 = \frac{2\pi}{3\sqrt{3}}, \quad H_2 = h_2 - \frac{1}{2}h_1 \ln 3, \quad \text{etc}$$

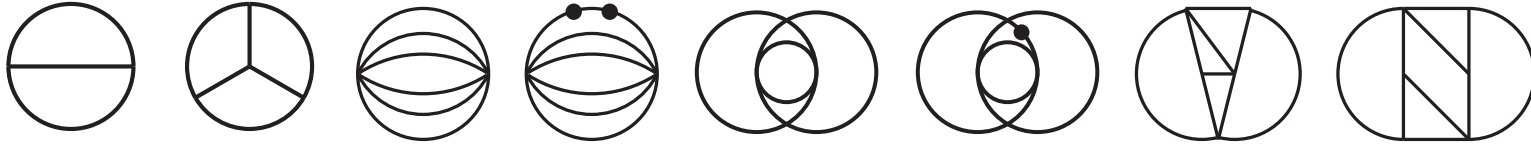
$$\triangleright I_{7.1.1} = +(-\frac{3}{2})\epsilon^0 + (-\frac{3}{2})\epsilon^1 + (9H_2 - 3)\epsilon^2 + (9H_2 - 18H_3 - 6)\epsilon^3 + (18H_2 - 18H_3 + 36H_4 - 12)\epsilon^4 + \dots$$

- to absorb powers of $\ln 2$, def as elements of the MZV basis

$$A_n \equiv Li_n\left(\frac{1}{2}\right) + (-1)^n \frac{\ln^n 2}{n!} \left(1 - \frac{n(n-1)}{2} \frac{\zeta_2}{\ln^2 2} \right)$$

$$\triangleright I_{63.1.1} = +(0)\epsilon^0 + (0)\epsilon^1 + (-2\zeta_3)\epsilon^2 + (-16A_4 + 27H_2^2 + \frac{34\zeta_2^2}{5})\epsilon^3 + \dots$$

Sample results (4d)



$$I_{28686.1.1} = +(-3)\epsilon^0 + \left(-\frac{3}{2}\right)\epsilon^1 + \left(\frac{13}{24}\right)\epsilon^2 + \left(-\frac{1267}{1440}\right)\epsilon^3 + \left(-\frac{4193}{3456}\right)\epsilon^4 + \\ +135.95072868792871461956492733702218574897992953584\epsilon^5 + \dots$$

$$I_{28686.1.3} = +(0)\epsilon^0 + \left(\frac{3}{2}\right)\epsilon^1 + \left(-\frac{1}{2}\right)\epsilon^2 + \left(-\frac{443}{360}\right)\epsilon^3 + \left(\frac{95}{216}\right)\epsilon^4 + \\ -38.292059175062436961881799538284449799148385376441\epsilon^5 + \dots$$

$$I_{30862.1.1} = +\left(-\frac{3}{5}\right)\epsilon^0 + \left(-\frac{27}{10}\right)\epsilon^1 + \left(-\frac{4\zeta_3}{5} - \frac{421}{60}\right)\epsilon^2 + \left(-\frac{12\zeta_2^2}{25} + \frac{24\zeta_3}{5} + \frac{211}{24}\right)\epsilon^3 + \left(\frac{72\zeta_2^2}{25} - 98\zeta_3 + \frac{32\zeta_5}{5} + \frac{12959}{48}\right)\epsilon^4 + \\ +1143.1838307558764599466030303839590323268318605888\epsilon^5 + \dots$$

$$I_{30862.1.2} = +\left(\frac{1}{5}\right)\epsilon^0 + \left(\frac{11}{30}\right)\epsilon^1 + \left(-\frac{1}{30}\right)\epsilon^2 + \left(-\frac{12\zeta_3}{5} - 9\right)\epsilon^3 + \left(-\frac{36\zeta_2^2}{25} + \frac{548\zeta_3}{15} - \frac{1229}{15}\right)\epsilon^4 + \\ -102.42854342605587086319606311891941160276036031953\epsilon^5 + \dots$$

$$I_{30231.1.1} = +(0)\epsilon^0 + (0)\epsilon^1 + \left(\frac{3\zeta_3}{5}\right)\epsilon^2 + \left(\frac{9\zeta_2^2}{25} + \frac{21\zeta_3}{5} + 3\zeta_5\right)\epsilon^3 + \left(-36H_2\zeta_3 + \frac{12\zeta_2^3}{7} + \frac{63\zeta_2^2}{25} - \frac{21\zeta_3^2}{5} + 27\zeta_3 - \frac{24\zeta_5}{5}\right)\epsilon^4 + \\ -531.32391547725635267943444561495368318398901378435\epsilon^5 + \dots$$

$$I_{31420.1.1} = +(0)\epsilon^0 + (0)\epsilon^1 + (0)\epsilon^2 + (0)\epsilon^3 + \left(-\frac{36\zeta_3^2}{5}\right)\epsilon^4 + \\ +167.81535305918474061962120601112466233675898298296\epsilon^5 + \dots$$

Summary

- studied class of fully massive vacuum diagrams
 - ▷ useful for e.g. hot QCD / anomalous dimensions / moments
- classification done at 5-loop level
 - ▷ complete set of (48+62) master integrals identified
- automated numerical evaluation via IBP / difference eqs / factorial series
 - ▷ **C++** implementation and parallelization, using **Fermat** for polynomial algebra
 - ▷ substantial fine-tuning of Laporta approach
 - ▷ at 5 loops: difference eqs of order 20, recurrence relations of order 28
- solved 37 of the 48 5-loop master-sectors numerically
 - ▷ ca 300 digits precision, ≥ 10 ε -orders around $d = 4 - 2\varepsilon$ and $d = 3 - 2\varepsilon$