Calculating four-loop massless propagators with Forcer

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Collaboration with:
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Introduction
Introduction

- Massless propagator-type Feynman integrals
  - Example: gluon self-energy diagram

In pQCD, after factorizing a hard part and a soft part, the hard part contains only massless particles.

Evaluating massless propagator-type integrals (within dimensional regularization) has been one of basic components in pQCD since its early days.
Massless propagator-type integrals

- Without loss of generality, we can consider only “scalar” integrals

\[ A = A(q^2 g_{\mu\nu} - q_{\mu} q_{\nu}) \delta^{ab} \]

- Projection operator \( P^{\mu\nu,ab} \) to extract \( A \): all indices (Lorentz, color) contracted

\[ A = P^{\mu\nu,ab} \times \]

\[ P^{\mu\nu,ab} = \frac{q^{\mu} q^{\nu}}{q^2} \delta^{ab} \]

- At the 1-loop level, the following family of Feynman integrals

\[ F(n_1, n_2) = \int \frac{d^D p_1}{(2\pi)^D} \frac{1}{(p_1^2)^{n_1} (p_2^2)^{n_2}} \]

**indices: powers of the denominators**

- Dimensional regularization: \( D = 4 - 2\epsilon \)
- The external momenta \( Q \) is off-shell: \( Q^2 \neq 0 \)
Physics motivation

- Much effort has been made for partonic Higgs production cross section in $N^3$LO QCD for the LHC Run2
  
  Anastasiou, Duhr, Dulat, Herzog, Mistlberger ’15 (soft expansion, over 35 terms)
  Anzai, Hasselhuhn, Höschele, Hoff, Kilgore, Steinhauser, TU ’15 (only qq', exact in $m_h^2/s$)
  and many works

- In principle, $N^3$LO partonic cross section should be used with $N^3$LO parton distribution function $\rightarrow$ 4-loop splitting functions (unknown)

- In Mellin $N$-space, computed from

\[
\frac{Q\{\mu_1 \ldots \mu_N\}}{N!} \frac{\partial^N}{\partial P^{\mu_1} \ldots \partial P^{\mu_N}} \bigg|_{P=0}
\]

Massless propagator-type integrals with $N$-dependence
Physics motivation

- Fixing $N = 2, 4, 6, \ldots$, one can compute them as massless propagator-type integrals. Their information gives an approximation to / upper bounds of uncertainty of the result (especially at large $x$).

- If we have software that can compute such integrals at the 4-loop level more efficiently, we can go for more higher $N$, get a better estimation known up to $N=4$ (NS) at 4-loop [Velizhanin ’14]

“Mincer” approach
Feynman integral calculus

- Integration-by-parts identities (IBPs): the Gauss theorem in D-dimension gives linear identities among Feynman integrals
  
  [Chetyrkin, Tkachov '81]

- The “standard” way of Feynman integral calculus

  various integrals $\Rightarrow$ (irreducible) master integrals (MIs)

  (1) reduction via IBPs $\quad$ (2) evaluation of MIs

- Generic reduction algorithms (for any processes)

  - Laporta algorithm, Baikov's method, Lee's “LiteRed”
    
    [Laporta '01] $\quad$ [Baikov '96; '05] $\quad$ [Lee ’12; ’13]

  - “Mincer” algorithm $\quad$ [Chetyrkin, Tkachov '81]

  - Specialized reduction for massless propagator-type integrals (up to 3-loops), very efficient
General 1-loop formula

- General formula for **arbitrary indices** $\alpha$ and $\beta$

$$
\int \frac{d^D p}{(2\pi)^D} \frac{p^{\mu_1} \ldots p^{\mu_n}}{(p^2)^\alpha [(Q - p)^2]^\beta} = \frac{1}{(4\pi)^2} \frac{1}{(Q^2)^{\alpha+\beta-2+\epsilon}} \sum_{\sigma=0}^{[n/2]} G(\alpha, \beta, n, \sigma)(Q^2)^\sigma \left[ \frac{1}{\sigma!} \left( \frac{\Box p}{4} \right)^\sigma p^{\mu_1} \ldots p^{\mu_n} \right]_{p=Q}
$$

$$
G(\alpha, \beta, n, \sigma) = (4\pi)^\epsilon \frac{\Gamma(\alpha + \beta - \sigma - 2 + \epsilon)}{\Gamma(\alpha)\Gamma(\beta)} B(2 - \epsilon - \alpha + n - \sigma, 2 - \epsilon - \beta + \sigma)
$$

- The result gets a non-integer power $1/(Q^2)^\epsilon$

- Can be used as convolutions for higher loops

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**“Carpet” 1-loop integral**

- We can perform another type of 1-loop integral (outer loop of $p_1$) for **arbitrary indices $\alpha$ and $\beta$** possibly with **numerators** $p_i \cdot p_j$, $p_i \cdot Q$

  ![Diagram](image)

- Used for the following topology at the 3-loop level

  ![Diagram](image)
Triangle rule

- IBP \( \left( \frac{\partial}{\partial k} \cdot k \right) \) in one-loop triangle-shaped (sub-)diagrams

\[
\begin{align*}
\begin{array}{c}
\text{Triangle rule} \\
\text{any # of lines} \\
\text{numerator } k^{\mu_1} \ldots k^{\mu_N} \text{ OK}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Decrease } b \text{ or } c_1 \text{ or } c_2 \text{ by 1} \\
\text{at the cost of increasing } a_1 \text{ or } a_2 \text{ in the right-hand side}
\end{array}
\end{align*}
\]

- Recursive use of the triangle rule makes

\[
\begin{align*}
\begin{array}{c}
b = 0 \text{ or } c_1 = 0 \text{ or } c_2 = 0 \text{ (removal of a line)}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\rightarrow \text{ sums of integrals in simpler topologies}
\end{array}
\end{align*}
\]
Diamond rule

- Extension of the triangle rule to multi-loop diamond-shaped (sub-)diagrams

Increases: \( a_1, a_2, a_3 \)

Decreases: \( b_1, b_2, b_3 \)

\( c_1, c_2, c_3 \)

Increases: \( a_1, a_2, a_3, a_4 \)

Decreases: \( b_1, b_2, b_3, b_4 \)

\( c_1, c_2, c_3, c_4 \)

[References: Ruijl, TU, Vermaseren '15]
2-loop topologies

- All integrals can be evaluated by using the triangle rule and performing one-loop integrals.

Expressed in terms of $\Gamma$-functions.

Topologies in which
- One loop-insertion
- Triangle rule can be applied

Tadpoles are dropped.
3-loop topologies

- All integrals are expressed by the 2 MIs and $\Gamma$-functions

- Special rules for 2 topologies
  - no, t1star5  ➔ 2 MIs

One loop-insertion
Carpet integral
Triangle rule

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Mincer approach

- Many topologies for massless propagator-type integrals can be reduced to simpler ones by
  - performing one-loop integrals
  - use of triangle rules to remove one of lines
- Special cases (where we need to solve IBPs) are not too many

Algorithm: [Chetyrkin, Tkachov '81]
Schoonschip implementation: [Gorishny, Larin, Surguladze, Tkachov '89]
Form implementation: [Larin, Tkachov, Vermaseren '91]

- In principle, this approach can be extended to the 4-loop level...
  - Laurent series of the MIs in $\epsilon$ are known

Baikov, Chetyrkin '10 (via glue-and-cut symmetry)
Lee, Smirnov$^2$ '11 (up to weight 12, via DRA, Mellin-Barnes, PSLQ)
Extension to the 4-loop level: “Forcer”
4-loop topologies

11 top-level topologies

- One loop-insertion (335)
- Carpet integral (24)
- Triangle rule (53)
- Diamond rule (4)
- Special rules (21) (pure 4-loop: 9)

- Enormous number of cases!!
- Coding such a reduction by hand is impractical

→ Automatization
How to handle

- Each topology as an “undirected graph” in graph theory

![Graph example with labeled vertices and edges](http://igraph.org)

- Implementation: Python3 with a graph library “igraph”

  [Implementation link](http://igraph.org)

- Easy to detect one-loop insertion, carpet, triangles, diamonds and tadpoles

  [Detection example](http://igraph.org)

  could be difficult by human eyes (diamond example)
How to handle

- **Graph isomorphism**
  - Detects equivalent graphs and finds mappings among them

- **Graph automorphism**
  - Finds symmetry / mappings in each graph
How to handle

- Input: top-level topologies (11 for 4-loops)

- From each topology, remove a line in all possible ways while taking graph isomorphism and dropping tadpoles into account

- For each topology, the next action is decided (one-loop, carpet, triangle, diamond, otherwise special rule needed)
  - Irreducible numerators (dot products) are chosen such as they do not interfere with the next action
Code generation

- In the end, we get
- Code generation from
  - Adequate subroutines are called at each topology
  - Symmetries from graph automorphism
  - Rewriting propagators and irreducible numerators at all transitions from a topology to another
  - Generates 41240 lines of FORM code for 4-loops
- Works even for 5-loops
  (64 top-topologies, 6570 in total, but 284 special rules)

visualized via Graphviz
Manual reduction rules

- 21 topologies require special rules, manually constructed
Finding manual reduction rules

- Shift an index by 1 in IBPs in all possible ways
  \[ n_1 \rightarrow n_1 + 1, \; n_2 \rightarrow n_2 + 1, \; n_3 \rightarrow n_3 + 1, \ldots \]

- At the 4-loop, 20 IBPs and 14 indices
  \[ 20 \times 14 = 280 \text{ equations} \]

- Eliminate “complicated” integrals that increase indices from the system of equations as possible
  \[ F(n_1 + 1, \ldots) \]

- This is not complete, but helps a lot. Human eyes for finding good rules for reducing integrals

- Everything should be done by using computer algebra systems (even for just rational arithmetic or simple substitutions)
  A rule can easily be over 1000 lines
Result looks like...

\[
\text{[no1}(2,2,2,2,2,2,2,2,-1,-1,-1)]=
\]

\[-10/9\text{num}(1+2\text{ep})^2\text{num}(2+5\text{ep})\text{num}(3+2\text{ep})^2\text{num}(3+5\text{ep})\text{num}(4+5\text{ep})\text{num}(6+5\text{ep})\text{num}(7+5\text{ep})\text{num}(8+5\text{ep})\text{num}(9+5\text{ep})\text{num}(36141384+167650024\text{ep}+369157793\text{ep}^2+504389598\text{ep}^3+470560515\text{ep}^4+312347786\text{ep}^5+149770838\text{ep}^6+51734214\text{ep}^7+12609912\text{ep}^8+2060632\text{ep}^9+203648\text{ep}^{10}+9216\text{ep}^{11})\text{den}(1+\text{ep})^2\text{den}(2+\text{ep})^2\text{den}(2+3\text{ep})\text{den}(3+\text{ep})^2\text{den}(4+3\text{ep})\text{den}(5+3\text{ep})\text{den}(7+3\text{ep})\text{den}(8+3\text{ep})\text{Master}(\text{no1})
\]

\[+2/9\text{num}(1+2\text{ep})\text{num}(1+4\text{ep})^2\text{num}(3+2\text{ep})\text{num}(39337381531008+422506983834144\text{ep}+2126828256064272\text{ep}^2+6682923999248124\text{ep}^3+14715479582570820\text{ep}^4+24142592323497543\text{ep}^5+30660697414180459\text{ep}^6+306638686432898\text{ep}^7+724613240685251374\text{ep}^8+15944111435166437\text{ep}^9+8353498138352567\text{ep}^{10}+3530865125135640\text{ep}^{11}+119505256887120\text{ep}^{12}+319518770241334\text{ep}^{13}+66015276933500\text{ep}^{14}+1017324432808\text{ep}^{15}+1101566031376\text{ep}^{16}+74820000000\text{ep}^{17}+24000000000\text{ep}^{18})\text{den}(1+\text{ep})^2\text{den}(2+\text{ep})^2\text{den}(2+3\text{ep})\text{den}(3+\text{ep})^2\text{den}(4+3\text{ep})\text{den}(5+3\text{ep})\text{den}(7+3\text{ep})\text{den}(8+3\text{ep})\text{Master}(\text{no6})
\]

\[-4/9\text{num}(\text{ep})^2\text{num}(1+2\text{ep})\text{num}(3+2\text{ep})\text{num}(4970268890523930816+65981799142896214872\text{ep}+416366999743872130470\text{ep}^2+16638417114114631648\text{ep}^3+4730696245772216998455\text{ep}^4+10190116418687033776180\text{ep}^5+1728315225628577356718\text{ep}^6+23674869565869007631757\text{ep}^7+26649665048245779930527\text{ep}^8+24945440138420991798098\text{ep}^9+19571616320498135318722\text{ep}^{10}+12933081950816489701881\text{ep}^{11}+7214358008133792648788\text{ep}^{12}+3396810005568803286172\text{ep}^{13}+13146680080420400500352\text{ep}^{14}+447318716827968227680\text{ep}^{15}+123500240869574621248\text{ep}^{16}+28008584917867939712\text{ep}^{17}+5129371482425778688\text{ep}^{18}+739826312414941184\text{ep}^{19}+80910145880457216\text{ep}^{20}+630638611958816\text{ep}^{21}+312125440000000\text{ep}^{22}+7372800000000\text{ep}^{23})\text{den}(1+\text{ep})^2\text{den}(2+\text{ep})^2\text{den}(2+3\text{ep})\text{den}(3+\text{ep})^2\text{den}(4+3\text{ep})\text{den}(4+3\text{ep})\text{den}(5+3\text{ep})\text{den}(5+4\text{ep})\text{den}(7+3\text{ep})\text{den}(7+4\text{ep})\text{den}(8+3\text{ep})\text{den}(9+4\text{ep})\text{den}(11+4\text{ep})\text{den}(13+4\text{ep})\text{Master}(\text{lala})
\]

(cont'd on next page)


+ ( other 17 terms )

+1/69984*num(-1+2*ep)^3*num(7241916201944976216509644800000+319933970273126101430280830976000*ep+6756458046923224428114990694400*ep^2+91350459184391655670944774398730240*ep^3+8912898867758173036005961791447318336*ep^4+66930375969349545757286105298423578624*ep^5+40235415927130336283161325668026920448*ep^6+198716317145927597305008924953882368*ep^7+820916970447314630727085836746702354688*ep^8+28738816351713697280438878686838131008*ep^9+8607952779410951522413494094313728793280*ep^10+22214777336844473013242466700149857424432*ep^11+49635643135503998398733139623853337720392*ep^12+96271970320109519554706474095019058884972*ep^13+16209202004423336736948422894086915254188710*ep^14+23601102551449738395477852326524380246208*ep^15+294116625122127952136996687381588045466897*ep^16+306225945611093340989894115787780494656622*ep^17+2509535713886278522318155623245534964532*ep^18+18728490682309307662131972394762735418774*ep^19-3231169865841642165998037752400422510997*ep^20-183932395284770686557606510999638263724434*ep^21-284795600836347547186578491353435307870326*ep^22-315022919192483812286675434478208160276910*ep^23-2382035508037611102151358311949984849163164*ep^24-212936199022871691907574075383055371432706*ep^25-13644599351865786696347804015820540166832*ep^26-73490078278873945786210757806097518867018*ep^27-31898465850321924924213285598334089035466304*ep^28-2965138248495148928414372757109220693984414*ep^29-421236561806300181608330799113484613884*ep^30+20170024686444720352265271521884303326282*ep^31+18328747615626866866983059056844702899*ep^32+1096707581456919232171368580075436795092*ep^33+52452370266277336661140017451441153706*ep^34+21214975274167170043985032559095034796*ep^35+74250901029010212070735988610887333749*ep^36+227205125224318403811895754585205484412*ep^37+6103292399690879903999959233123243684*ep^38+1439736393111277351174910769581576800*ep^39+29752853032297579806839220046747248*ep^40+5360618209191663270872765495066048*ep^41+835965908927948935684267151014592*ep^42+1117001205642226519475032390006784*ep^43+126129308732809465322496308824064*ep^44+1181003341832521121454334027776*ep^45+892823848935298884179601309696*ep^46+5238433612681661107473928704*ep^47+2238933537865339140484104192*ep^48+6202902113698185216000000*ep^49+83626417685790720000000*ep^50*den(ep)^6*den(-1+ep)*den(1+ep)^6*den(1+2*ep)^2*den(1+3*ep)^2*den(-2*ep)*den(2+ep)^6*den(2+3*ep)^2*den(3+ep)^3*den(3+2*ep)^2*den(3+4*ep)*den(4+3*ep)^2*den(5+2*ep)^2*den(5+3*ep)^2*den(5+4*ep)*den(7+3*ep)^2*den(7+4*ep)^2*den(8+3*ep)^2*den(9+4*ep)^2*den(11+4*ep)^2*den(13+4*ep)^2*G10*G20*G30

Optionally: $\epsilon$-expansions in intermediate steps
Checks and status

- Recomputing known results – strong non-trivial checks
  - Reproduced the 4-loop QCD $\beta$-function via massless propagators
  - Checked the gauge invariance
  - Also with using background field method
  - Reproduced N=2 and N=4 non-singlet splitting functions

- Statistics of these computations tell us bottlenecks

- Trying to optimize the program before going for N=6

Collaboration with Andreas Vogt
Summary
Summary

- Evaluating Massless propagator-type Feynman integrals is one of key components of pQCD
- “Mincer” up to 3-loops
- We have been developing “Forcer” for 4-loops
  - Highly complicated structure of the program / equations
    - Automatization: write a program for generating a program
      Manual rules are derived with the aid of computers
- Status:
  - Checked for known results
  - More optimizations
Backup
Integration-by-parts identities (IBPs) [Chetyrkin, Tkachov ’81]

- The Gauss theorem in $D$-dimension (the surface term vanishes)
  \[ \int d^D p \frac{\partial}{\partial p^\mu} X^\mu = 0 \]

- Example:

\[
F(n_1, n_2, n_3, n_4, n_5) = \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{1}{(p_1^2)^{n_1} (p_2^2)^{n_2} (p_3^2)^{n_3} (p_4^2)^{n_4} (p_5^2)^{n_5}} \]

\[
\int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{\partial}{\partial p_1^\mu} \frac{p_2^\mu}{(p_1^2)^{n_1} (p_2^2)^{n_2} (p_3^2)^{n_3} (p_4^2)^{n_4} (p_5^2)^{n_5}} = 0
\]

- Performing $\frac{\partial}{\partial p_1^\mu}$ and multiplying $p_2^\mu$ give various dot products (e.g., $p_1 \cdot p_2$), which are all decomposed to sums of the propagators $\rightarrow$ linear identity among $F$
Integration-by-parts identities (IBPs)

- $\{p_1, p_2; Q\} \rightarrow 2 \times 3 = 6$ identities (and their linear combination)

\[ F(n_1 - 1, n_2, n_3, n_4 + 1, n_5) \times (-n_4) + F(n_1 - 1, n_2, n_3, n_4, n_5 + 1) \times (-n_5) + F(n_1, n_2 - 1, n_3, n_4, n_5 + 1) \times (n_5) + F(n_1, n_2, n_3, n_4 + 1, n_5) \times (Q^2 n_4) + F(n_1, n_2, n_3, n_4, n_5) \times (-2n_1 - n_4 - n_5 + 4 - 2\epsilon) = 0 \]

\[ \frac{\partial}{\partial p_1} \cdot p_1 - \frac{\partial}{\partial p_1} \cdot p_2 = ? \]

and other 4 identities

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Integration-by-parts identities (IBPs)

\[
\frac{\partial}{\partial p_1} \cdot p_1 - \frac{\partial}{\partial p_1} \cdot p_2 \Rightarrow \\
F(n_1, n_2, n_3, n_4, n_5) \times (-n_1 - n_4 - 2n_5 + 4 - 2\epsilon) \\
+ F(n_1 + 1, n_2 - 1, n_3, n_4, n_5) \times (n_1) \\
+ F(n_1 + 1, n_2, n_3, n_4, n_5 - 1) \times (-n_1) \\
+ F(n_1, n_2, n_3 - 1, n_4 + 1, n_5) \times (n_4) \\
+ F(n_1, n_2, n_3, n_4 + 1, n_5 - 1) \times (-n_4) = 0
\]

- Decreases \((n_2 + n_3 + n_5)\) by 1 at the cost of increasing \((n_1 + n_4)\)
- Repeated use of this rule gives integrals with \(n_2 = 0, \ n_3 = 0\) or \(n_5 = 0\) → simpler topologies with removing one of the lines
Diamond rule

\[ F(\{a_i\}, \{b_i\}) = \left[ \prod_{i=1}^{L} \int d^{D}k_{i} \right] \left[ \prod_{i=1}^{S} \int d^{D}l_{i} \right] \times \left[ \prod_{i=1}^{L+1} \frac{k_{i}^{\mu_{1}^{(i)}} \ldots k_{N_{i}}^{\mu_{N_{i}}^{(i)}}}{(k_{i} + p_{i})^2 + m_{i}^2} \right] \left[ \sum_{i=1}^{S} \frac{l_{i}^{\nu_{1}^{(i)}} \ldots l_{i}^{\nu_{N_{i}}^{(i)}}}{(l_{i}^2)^{s_{i}}} \right] \]

\[ (L+S)D + \sum_{i=1}^{L+1} (N_{i} - a_{i} - 2b_{i}) + \sum_{i=1}^{S} (R_{i} - 2s_{i}) = \sum_{i=1}^{L+1} a_{i} A_{i}^{+} \left[ B_{i}^{\mp} - (p_{i}^2 + m_{i}^2) \right] \]
Diamond rule

- Explicit summation formula

\[ F(\{a_i\}, \{b_i\}, \{c_i\}) = \sum_{r=1}^{L+1} \left[ \prod_{i=1}^{L+1} \sum_{k_i^+=0}^{b_i-1} \prod_{i=1}^{L+1} \sum_{k_i^-=-0}^{c_i-1} (-1)^{k_i^-} \frac{k_r^+ (k_r^+ + k_r^-) - 1!}{\prod_{i=1}^{L+1} k_i^+! k_i^-!} (E + k_r^+) - k_r^+ - k_r^- \right] \]

\[ \times \left( \prod_{i=1}^{L+1} (a_i)_{k_i^+ + k_i^-} \right)^2 \]

\[ F(\{a_i + k_i^+ + k_i^-\}, \{b_i - k_i^+\}, \{c_i - k_i^-\}) \]

\[ \left. \begin{array}{l} k_r^+ = b_r \\ \end{array} \right| \]

\[ + \sum_{r=1}^{L+1} \left[ \prod_{i=1}^{L+1} \sum_{k_i^+=0}^{b_i-1} \prod_{i=1}^{L+1} \sum_{k_i^-=-0}^{c_i-1} (-1)^{k_i^-} \frac{k_r^- (k_r^+ + k_r^- - 1)!}{\prod_{i=1}^{L+1} k_i^+! k_i^-!} (E + k_r^+ + 1) - k_r^+ - k_r^- \right] \]

\[ \times \left( \prod_{i=1}^{L+1} (a_i)_{k_i^+ + k_i^-} \right)^2 \]

\[ F(\{a_i + k_i^+ + k_i^-\}, \{b_i - k_i^+\}, \{c_i - k_i^-\}) \]

\[ \left. \begin{array}{l} k_r^- = c_r \\ \end{array} \right| \]

\[ E = (L + S)D + \sum_{i=1}^{L+1} (N_i - a_i - 2b_i) + \sum_{i=1}^{S} (R_i - 2s_i) \]

\[ k^+ = \sum_{i=1}^{L+1} k_i^+ \quad k^- = \sum_{i=1}^{L+1} k_i^- \]

Avoids spurious poles
2-loop: easy integrals

- Products / convolutions of 1-loop integrals

\[
\begin{align*}
\begin{array}{c}
\text{n}_1 & \text{n}_3 \\
\text{n}_2 & \text{n}_4
\end{array} &=
\begin{array}{c}
\text{n}_1 \\
\text{n}_2
\end{array} \times
\begin{array}{c}
\text{n}_3 \\
\text{n}_4
\end{array}
\end{align*}
\]

\[
\begin{align*}
n_1 + n_2 - 2 + \epsilon & \quad n_3 + n_4 - 2 + \epsilon & \quad n_1 + n_2 + n_3 + n_4 - 4 + 2\epsilon \\
\sim & \quad \ast \times \ast & \quad \ast \ast
\end{align*}
\]

one-loop insertion

\[
\begin{align*}
\begin{array}{c}
\text{n}_1 & \text{n}_2 \\
\text{n}_3 & \text{n}_4
\end{array} &=
\begin{array}{c}
\text{n}_1
\end{array} \sim
\begin{array}{c}
\text{n}_1 \\
\text{n}_4
\end{array}
\end{align*}
\]

\[
\begin{align*}
n_1 + n_2 + n_3 - 2 + \epsilon & \quad n_1 + n_2 + n_3 + n_4 - 4 + 2\epsilon \\
\sim & \quad \ast \ast
\end{align*}
\]
2-loop: generic topology

- “Generic” topology with the maximal number of lines at the 2-loop:

\[
F(n_1, n_2, n_3, n_4, n_5) = \int \frac{d^Dp_1}{(2\pi)^D} \frac{d^Dp_2}{(2\pi)^D} \frac{1}{(p_1^2)^{n_1}(p_2^2)^{n_2}(p_3^2)^{n_3}(p_4^2)^{n_4}(p_5^2)^{n_5}}
\]

- The propagators are “complete”: e.g., \(2p_1 \cdot p_2 = p_1^2 + p_2^2 - p_5^2\)

- Easy when one of the indices is \(\leq 0\)

- In the case that all the indices are \(\geq 1\):
  - In early days: special techniques (e.g., GPXT)
  - Nowadays: IBPs
3-loop level

- Generic topologies have 8 lines, but there are 9 dot products. Need to introduce another index for an irreducible numerator.

\[
F(n_1, \ldots, n_9) = \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \times \frac{(p_2 \cdot Q)^{-n_9}}{(p_1^2)^{n_1} \cdots (p_8^2)^{n_8}}
\]

- Our convention:
  - positive indices: denominators
  - negative indices: numerators

- The irreducible numerator \((p_2 \cdot Q)\) is chosen such that it does not interfere with use of the triangle rule (next page).
Reduction of the Benz topology

1,3 → 1,3

1-loop

carpet

1-loop

same as 2-loop diagrams

“carpet” integral
Perform the outer one-loop

easy integrals
Special topologies at 3-loop

- At the 3-loop level, there are 2 topologies that require special treatment.

Repeated use of the triangle rule cannot remove the central line because of the non-integer $n_5 + \epsilon$.

2-loop topology with $\epsilon$ at the central line.

No triangle.

Generic “non-planer” topology.

- Need to “solve” IBPs: solve a system of recurrence relations to reduce these integrals as much as possible.
Special topologies at 3-loop

• Fortunately, this is possible:

\[
F(n_1, n_2, n_3, n_4, n_5 + \epsilon) \times (n_1 - 1) \quad \text{valid for } n_1 \neq 1 \\
+ F(n_1 - 1, n_2, n_3, n_4, n_5 + \epsilon) \times (-1) \times (n_1 + 2n_4 + n_5 - 5 + 3\epsilon) \\
+ F(n_1, n_2, n_3, n_4 - 1, n_5 + \epsilon) \times (-1) \times (n_1 - 1) \\
+ F(n_1 - 1, n_2, n_3 - 1, n_4, n_5 + 1 + \epsilon) \times (n_5 + \epsilon) \\
+ F(n_1 - 1, n_2, n_3, n_4 - 1, n_5 + 1 + \epsilon) \times (-1) \times (n_5 + \epsilon) = 0
\]

Decreases $n_1$ or $n_4$ (or both) until $n_1 = 1$. Similar rules for $n_2, n_3, n_4$

* They can be derived from the triangle rule identity [Ruijl, TU, Vermaseren '15]

Then $n_5$ can be increased/decreased until $n_5 = 1$ by

\[
F(1, 1, 1, 1, n_5 + \epsilon) \times (-1) \times (n_5 - 1 + 2\epsilon) \\
+ F(1, 1, 1, 1, n_5 - 1 + \epsilon) \times (-1) \times (n_5 - 2 + 3\epsilon) \\
+ F(0, 1, 1, 1, n_5 + \epsilon) \times (-3 + 2n_5 + 5\epsilon) \\
+ F(1, 1, 1, 0, n_5 + \epsilon) \times (-3 + 2n_5 + 5\epsilon) = 0
\]

• “Master” integral $F(1, 1, 1, 1, 1 + \epsilon)$ cannot be reduced any more
Special topologies at 3-loop

- One can make a reduction scheme also for the non-planer topology.
  * Rules for reducing $n_1, n_3, n_4, n_6$ to 1 can be derived from the diamond rule identity [Ruijl, TU, Vermaseren '15]

- Master integral $F(1, 1, 1, 1, 1, 1, 1, 1, 0)$

- The 2 master integrals (MIs) at the 3-loop level

- They cannot be expressed in terms of $\Gamma$-functions. But their expansions with respect to $\epsilon$ are known (at least, up to enough orders)