Calculating four-loop massless propagators with Forcer

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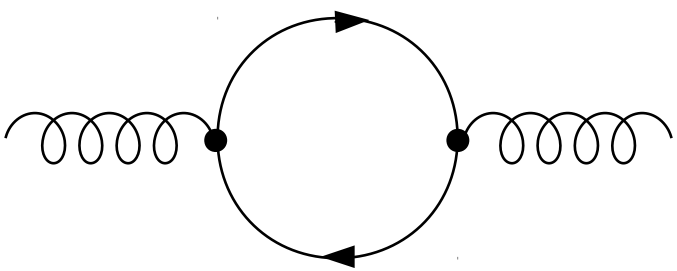
Contents

- Introduction
  - Massless propagator-type integrals
  - Physics motivation (4-loop splitting funcs.)
- “Mincer” approach
  - How did people compute them up to 3-loops?
- Extension to the 4-loop level: our program “Forcer”
  - Automatization
- Summary

Introduction

Introduction

- Massless propagator-type Feynman integrals
    - Example: gluon self-energy diagram

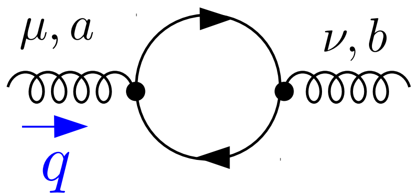


gluon: massless
(light) quark: massless

- In pQCD, after factorizing a hard part and a soft part, the hard part contains only massless particles
- Evaluating massless propagator-type integrals (within dimensional regularization) has been one of basic components in pQCD since its early days

Massless propagator-type integrals

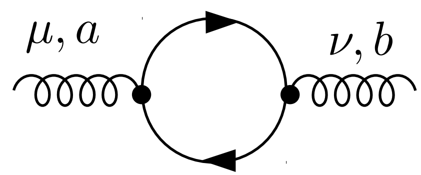
- Without loss of generality, we can consider only “scalar” integrals



$$= A(q^2 g_{\mu\nu} - q_\mu q_\nu)\delta^{ab}$$

- Projection operator $P^{\mu\nu,ab}$ to extract A: all indices (Lorentz, color) contracted

$$A = P^{\mu\nu,ab} \times$$

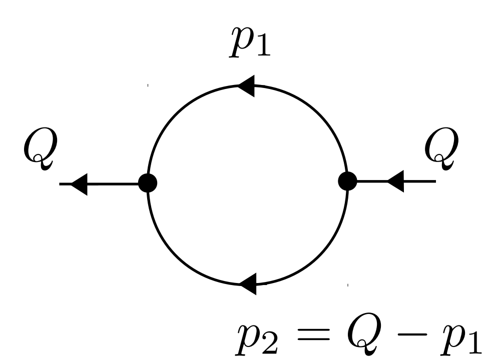


$$P^{\mu\nu,ab} = \frac{q^\mu q^\nu}{q^2}\delta^{ab}$$

- At the 1-loop level, the following family of Feynman integrals

$$\underbrace{F(n_1, n_2)}_{\text{indices: powers of the denominators}} = \int \frac{d^D p_1}{(2\pi)^D} \frac{1}{(p_1^2)^{n_1} (p_2^2)^{n_2}}$$

indices: powers of the denominators



  - Dimensional regularization: $D = 4 - 2\epsilon$
  - The external momenta Q is off-shell: $Q^2 \neq 0$

Physics motivation

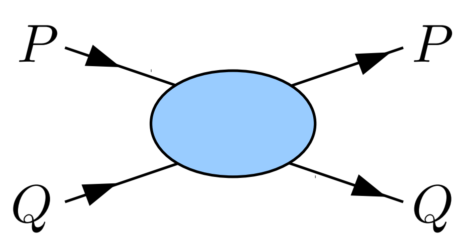
- Much effort has been made for partonic Higgs production cross section in N^3LO QCD for the LHC Run2

  Anastasiou, Duhr, Dulat, Herzog, Mistlberger '15 (soft expansion, over 35 terms)
  Anzai, Hasselhuhn, Höschele, Hoff, Kilgore, Steinhauser, TU '15 (only qq', exact in m_h^2/s)
  and many works

- In principle, N^3LO partonic cross section should be used with N^3LO parton distribution function ⇒ 4-loop splitting functions (unknown)

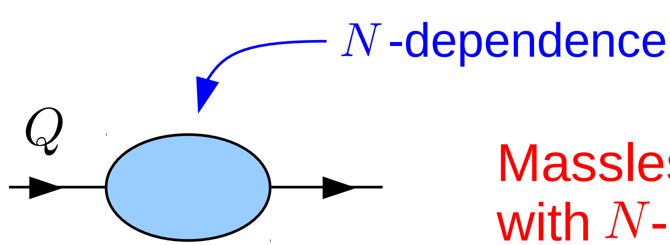
- In Mellin N-space, computed from

$$\frac{Q^{\{\mu_1}\ldots Q^{\mu_N\}}}{N!}\frac{\partial^N}{\partial P^{\mu_1}\ldots\partial P^{\mu_N}}$$



$$\Bigg|_{P=0}$$

⇒

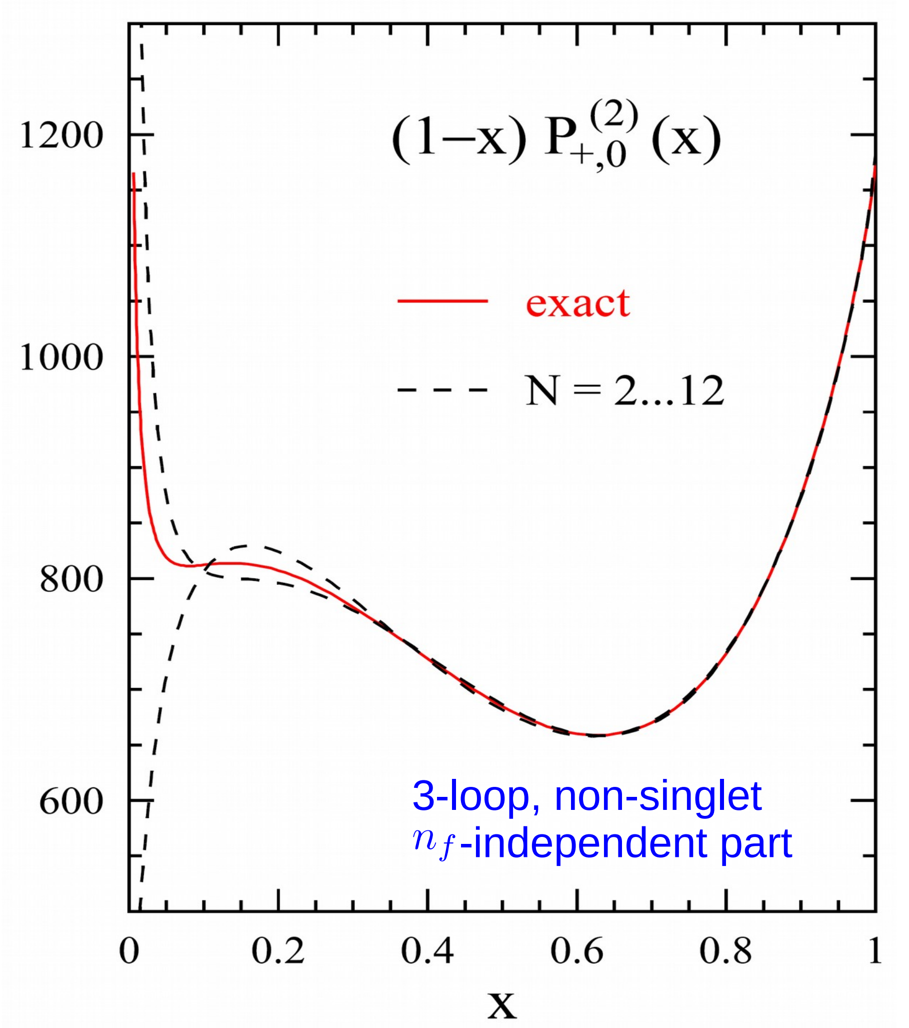


Massless propagator-type integrals with N-dependence

Physics motivation

- Fixing $N = 2, 4, 6, \ldots$, one can compute them as massless propagator-type integrals. Their information gives an approximation to / upper bounds of uncertainty of the result (especially at large x)
- If we have software that can compute such integrals at the 4-loop level more efficiently, we can go for more higher N, get a better estimation

known up to N=4 (NS) at 4-loop
[Velizhanin '14]



From Moch, Vermaseren, Vogt NPB688(2004)101 [arXiv:hep-ph/0403192]

“Mincer” approach

Feynman integral calculus

- Integration-by-parts identities (IBPs): the Gauss theorem in D-dimension gives linear identities among Feynman integrals [Chetyrkin, Tkachov '81]
- The "standard" way of Feynman integral calculus

  various integrals → (irreducible) master integrals (MIs)

  (1) reduction via IBPs (2) evaluation of MIs

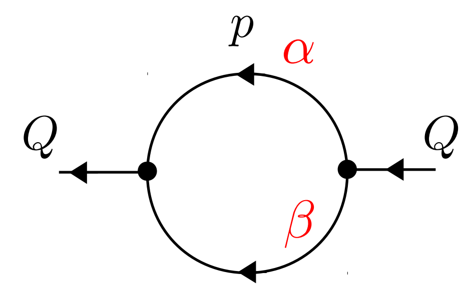
- Generic reduction algorithms (for any processes)
  - Laporta algorithm, Baikov's method, Lee's "LiteRed"
    [Laporta '01] [Baikov '96; '05] [Lee '12; '13]
- "Mincer" algorithm [Chetyrkin, Tkachov '81]
  - Specialized reduction for massless propagator-type integrals (up to 3-loops), very efficient

General 1-loop formula

- General formula for arbitrary indices α and β

$$\int \frac{d^Dp}{(2\pi)^D} \frac{p^{\mu_1} \dots p^{\mu_n}}{(p^2)^{\alpha}[(Q-p)^2]^{\beta}}$$

allowing numerators $(n \geq 0)$

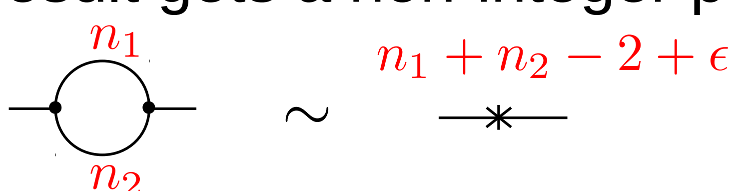


$$= \frac{1}{(4\pi)^2} \frac{1}{(Q^2)^{\alpha+\beta-2+\epsilon}} \sum_{\sigma=0}^{\lfloor n/2 \rfloor} G(\alpha,\beta,n,\sigma)(Q^2)^{\sigma} \left[\frac{1}{\sigma!} \left(\frac{\Box_p}{4} \right)^{\sigma} p^{\mu_1} \dots p^{\mu_n} \right]_{p=Q}$$

$$G(\alpha,\beta,n,\sigma) = (4\pi)^{\epsilon} \frac{\Gamma(\alpha+\beta-\sigma-2+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)} B(2-\epsilon-\alpha+n-\sigma, 2-\epsilon-\beta+\sigma)$$

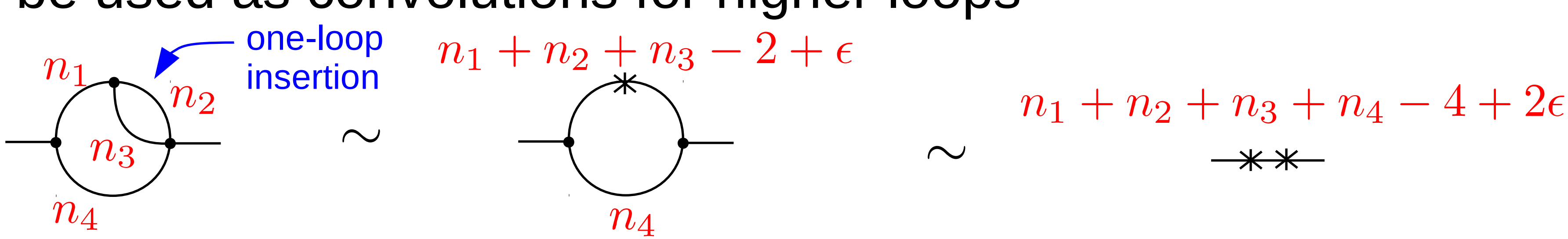
[Chetyrkin, Tkachov '81]

- The result gets a non-integer power $1/(Q^2)^{\epsilon}$



$\ast$: non-integer part ϵ

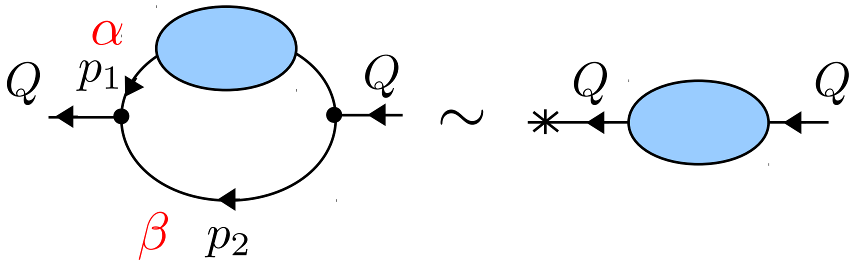
- Can be used as convolutions for higher loops



"Carpet" 1-loop integral

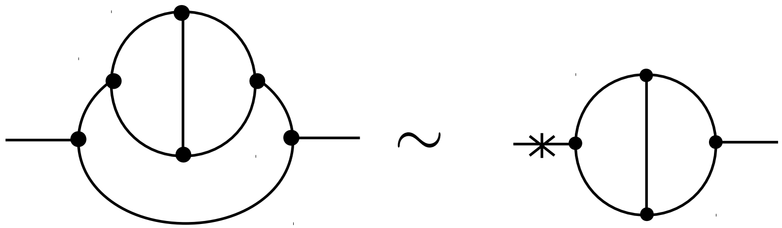
- We can perform another type of 1-loop integral (outer loop of p_1) for arbitrary indices α and β possibly with numerators

$$p_i \cdot p_j\ ,\ \ p_i \cdot Q$$



- Used for the following topology at the 3-loop level

[Chetyrkin, Tkachov '81]



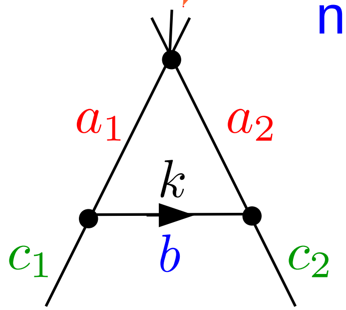
Triangle rule

[Chetyrkin, Tkachov '81]

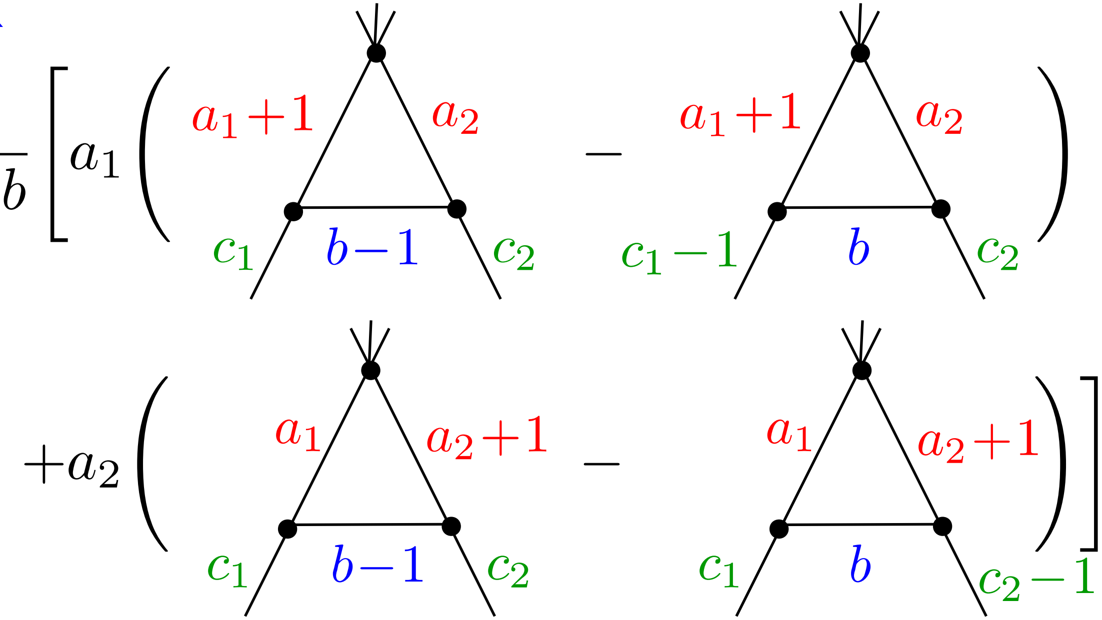
- IBP ($\frac{\partial}{\partial k} \cdot k$) in one-loop triangle-shaped (sub-)diagrams

any # of lines

numerator $k^{\mu_1} \ldots k^{\mu_N}$ OK



$$= \frac{1}{D + N - a_1 - a_2 - 2b}$$

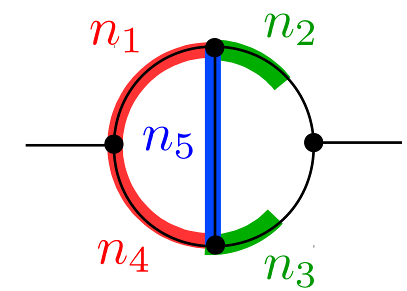


Decreases b or c_1 or c_2 by 1
at the cost of increasing a_1 or a_2
in the right-hand side

- Recursive use of the triangle rule makes

$b=0$ or $c_1=0$ or $c_2=0$ (removal of a line)

sums of integrals in simpler topologies



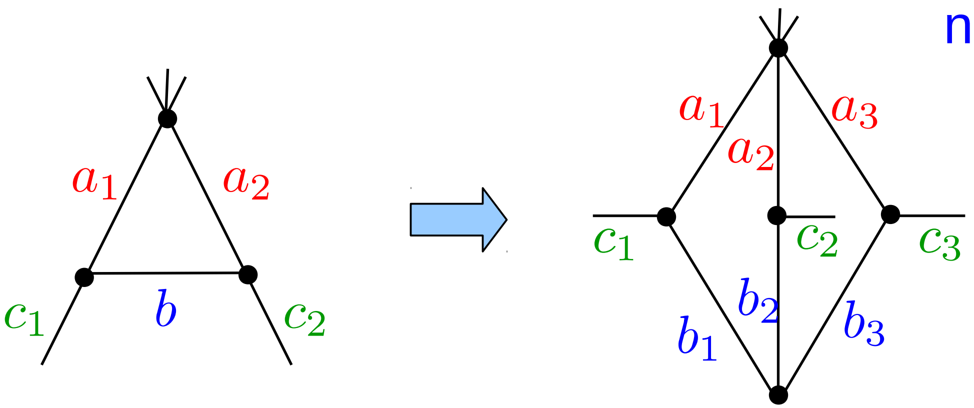
Diamond rule

[Ruijl, TU, Vermaseren ’15]

- Extension of the triangle rule to multi-loop diamond-shaped (sub-)diagrams

any # of lines

numerator OK

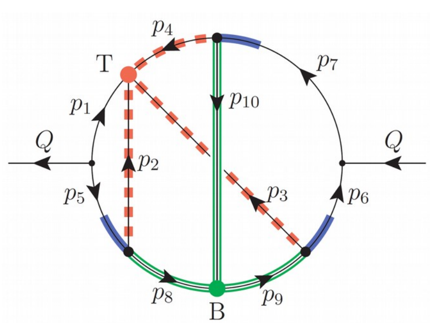


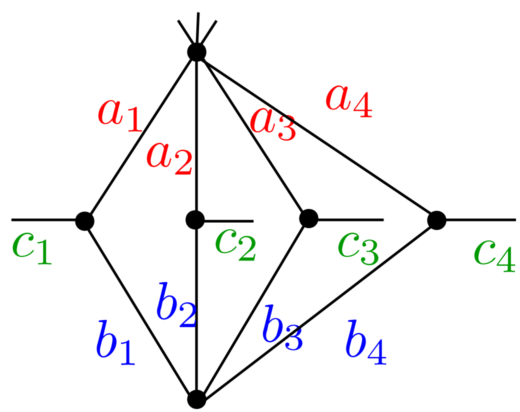
Increases a_1, a_2, a_3

Decreases b_1, b_2, b_3

c_1, c_2, c_3

Appear from 4-loops



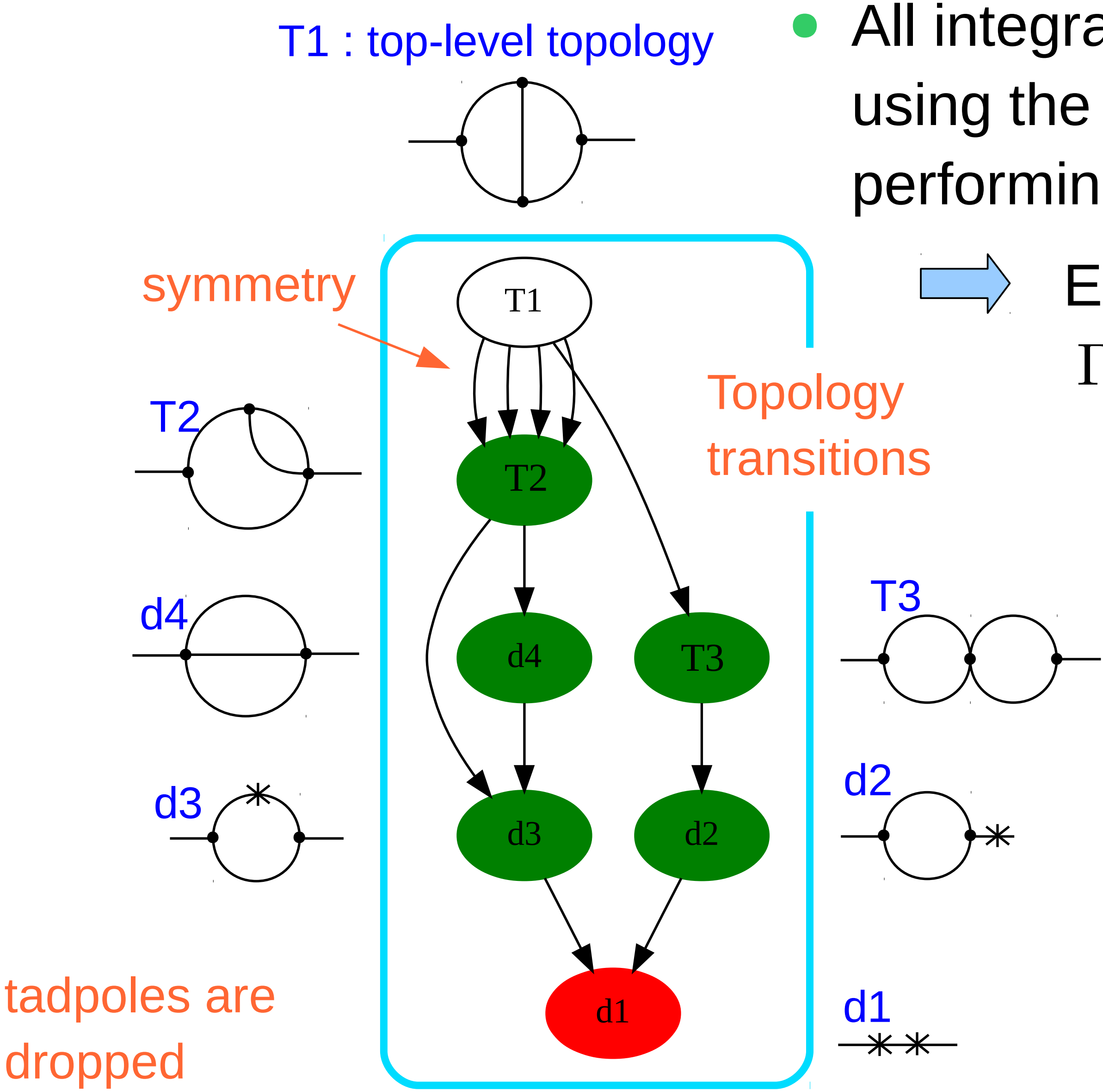


Increases a_1, a_2, a_3, a_4

Decreases b_1, b_2, b_3, b_4

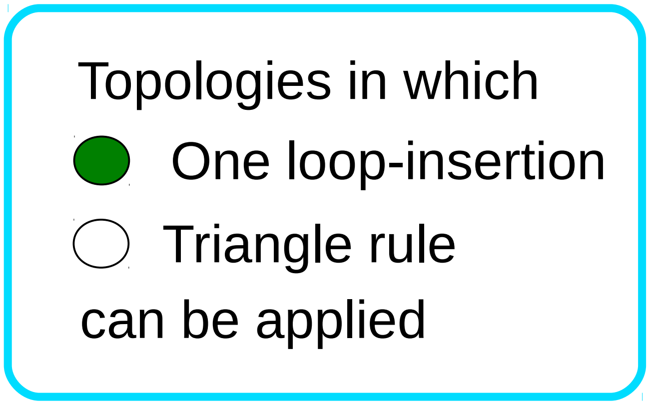
c_1, c_2, c_3, c_4

2-loop topologies



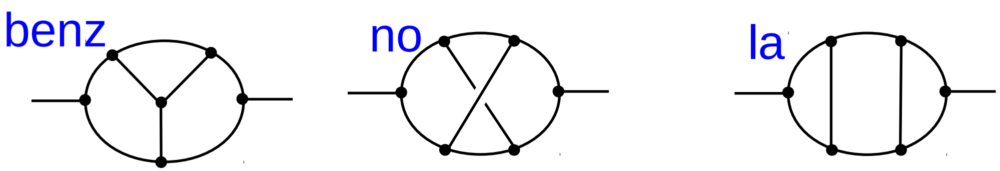
- All integrals can be evaluated by using the triangle rule and performing one-loop integrals

→ Expressed in terms of Γ -functions

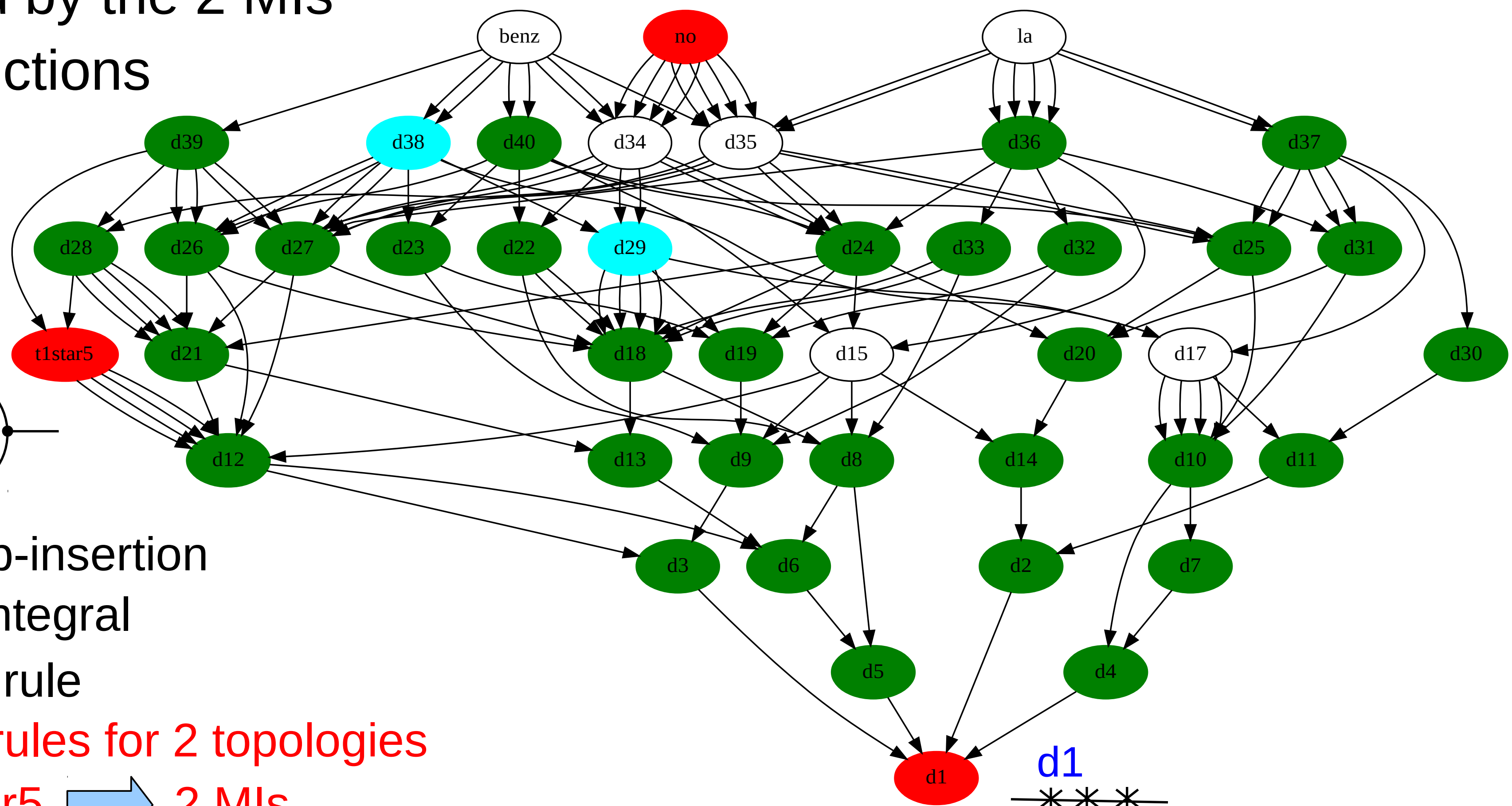


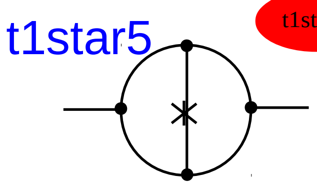
3-loop topologies

3 top-level topologies



- All integrals are expressed by the 2 MIs and Γ-functions





One loop-insertion

Carpet integral

Triangle rule

Special rules for 2 topologies

no, t1star5 ⇨ 2 MIs

Mincer approach

- Many topologies for massless propagator-type integrals can be reduced to simpler ones by
  - performing one-loop integrals
  - use of triangle rules to remove one of lines
- Special cases (where we need to solve IBPs) are not too many

Algorithm: [Chetyrkin, Tkachov ’81]
Schoonschip implementation: [Gorishny, Larin, Surguladze, Tkachov ’89]
Form implementation: [Larin, Tkachov, Vermaseren ’91]

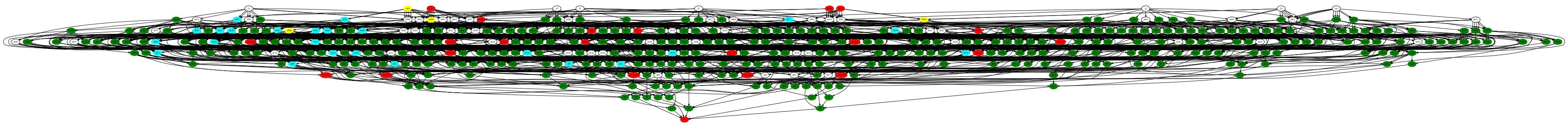
- In principle, this approach can be extended to the 4-loop level...
  - Laurent series of the MIs in ϵ are known

Baikov, Chetyrkin ’10 (via glue-and-cut symmetry)
Lee, Smirnov^2 ’11 (up to weight 12, via DRA, Mellin-Barnes, PSLQ)

Extension to the 4-loop level: “Forcer”

4-loop topologies

11 top-level topologies

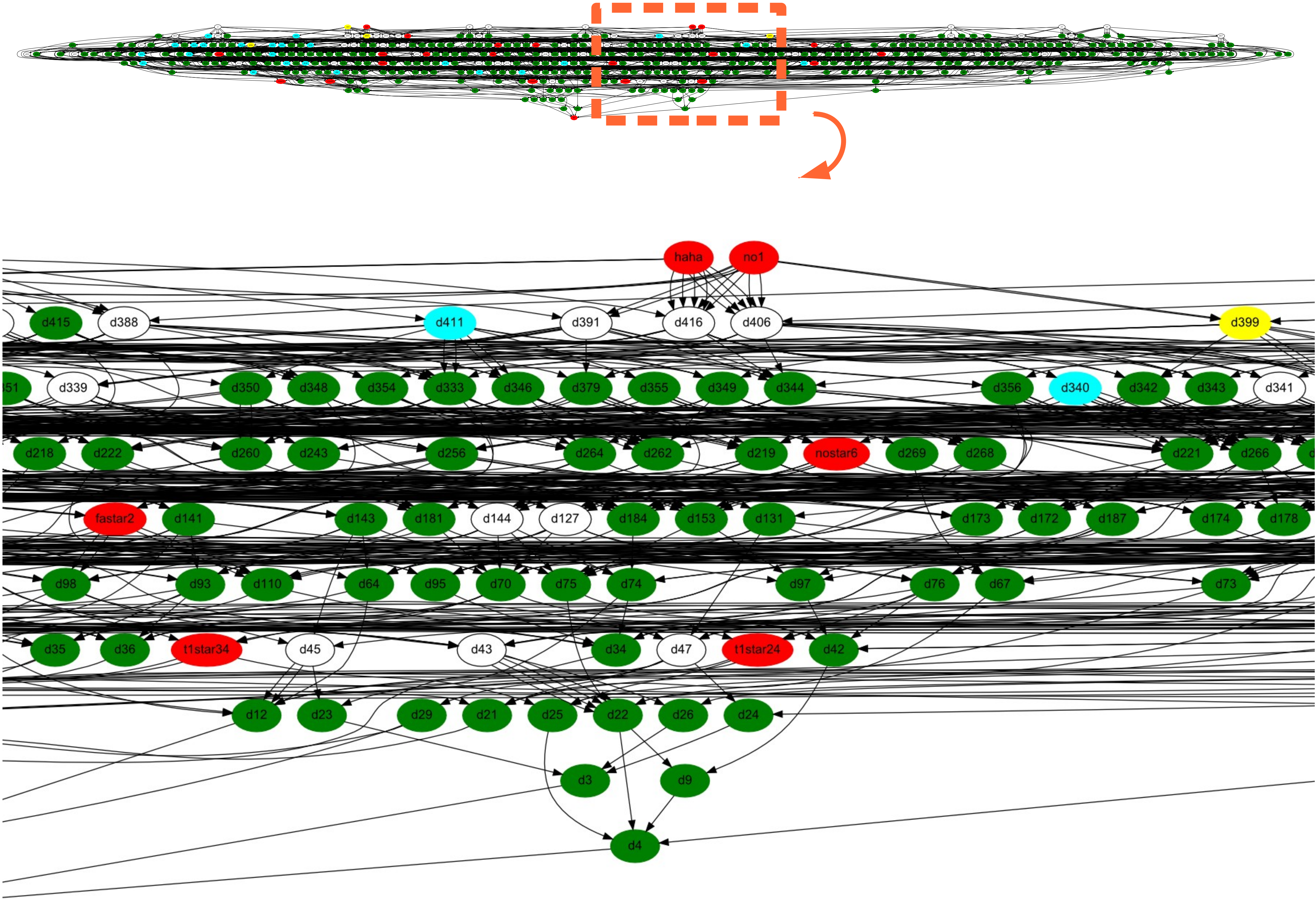


One loop-insertion	(335)	
Carpet integral	(24)	
Triangle rule	(53)	
Diamond rule	(4)	
Special rules	(21)	(pure 4-loop: 9)

Total: 437

- Enormous number of cases!!
- Coding such a reduction by hand is impractical

  ⇨ Automatization


haha
no1
d415
d388
d411
d391
d416
d406
d399
d339
d350
d348
d354
d333
d346
d379
d355
d349
d344
d356
d340
d342
d343
d341
d218
d222
d260
d243
d256
d264
d262
d219
nostar6
d269
d268
d221
d266
fastar2
d141
d143
d181
d144
d127
d184
d153
d131
d173
d172
d187
d174
d178
d98
d93
d110
d64
d95
d70
d75
d74
d97
d76
d67
d73
d35
d36
t1star34
d45
d43
d34
d47
t1star24
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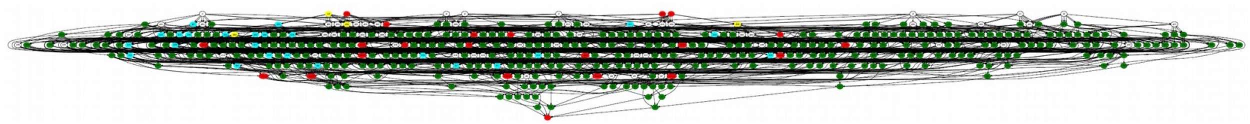
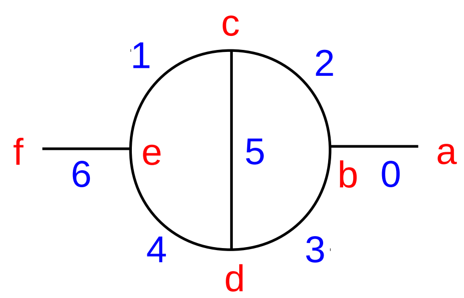
d3
d9
d4

How to handle 

- Each topology as an “undirected graph” in graph theory

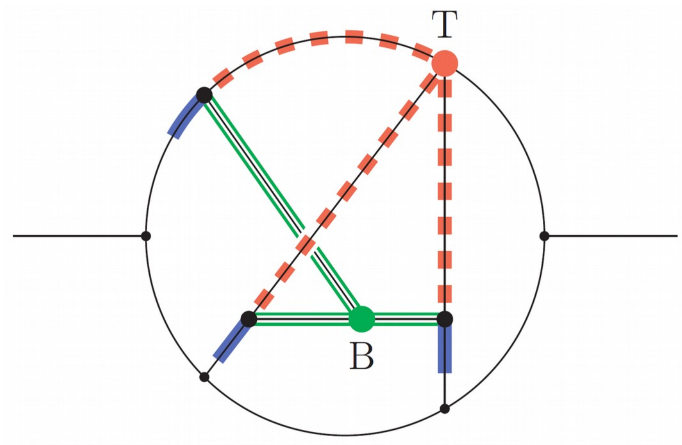


vertices and edges are labeled

- Implementation: Python3 with a graph library “igraph”

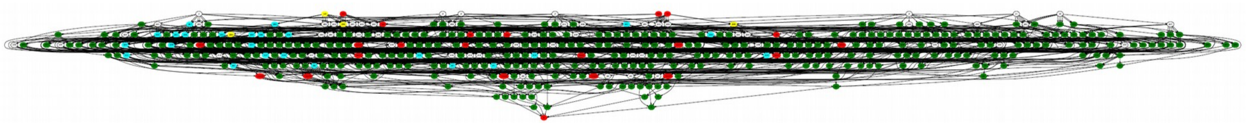
http://igraph.org

- Easy to detect one-loop insertion, carpet, triangles, diamonds and tadpoles

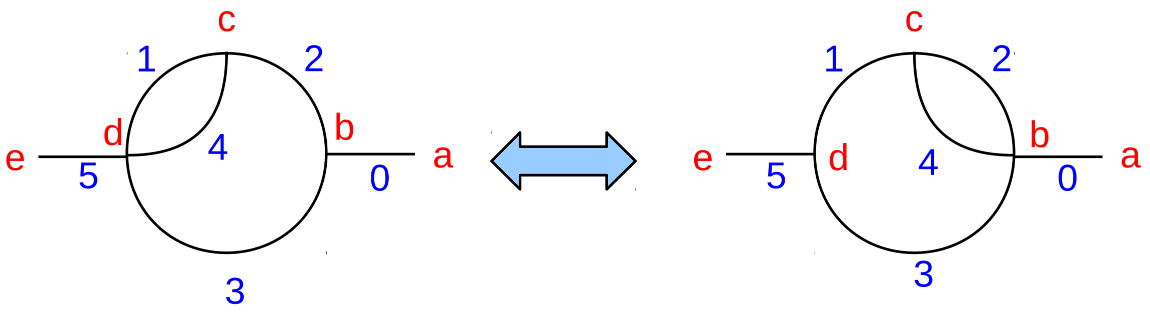


could be difficult by human eyes (diamond example)

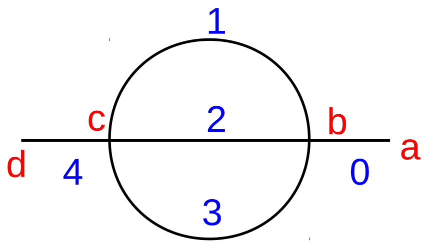
How to handle



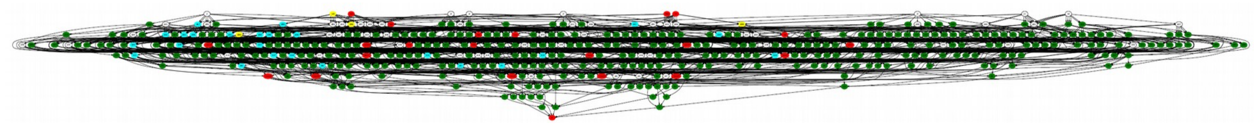
- Graph isomorphism
  - Detects equivalent graphs and finds mappings among them



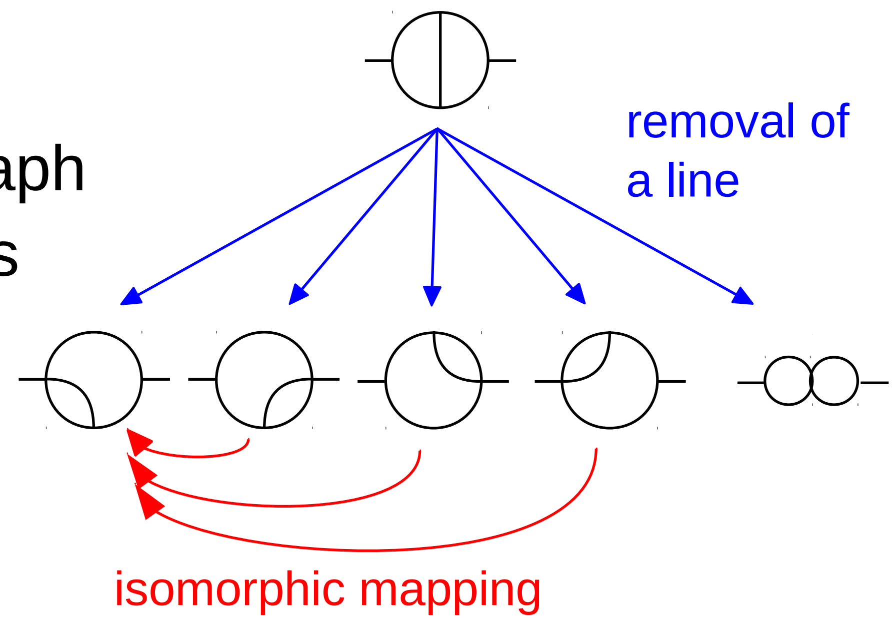
- Graph automorphism
  - Finds symmetry / mappings in each graph



$$1 \leftrightarrow 2 \leftrightarrow 3$$

How to handle 

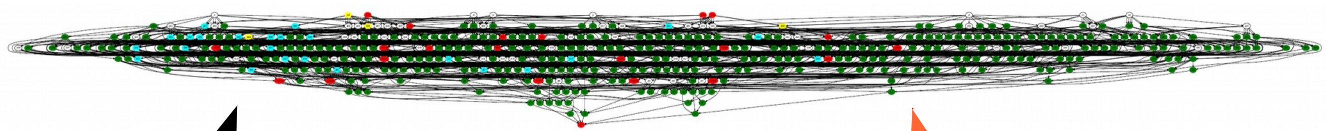
- Input : top-level topologies (11 for 4-loops)
- From each topology, remove a line in all possible ways while taking graph isomorphism and dropping tadpoles into account



- For each topology, the next action is decided (one-loop, carpet, triangle, diamond, otherwise special rule needed)
  - Irreducible numerators (dot products) are chosen such as they do not interfere with the next action

Code generation

- In the end, we get



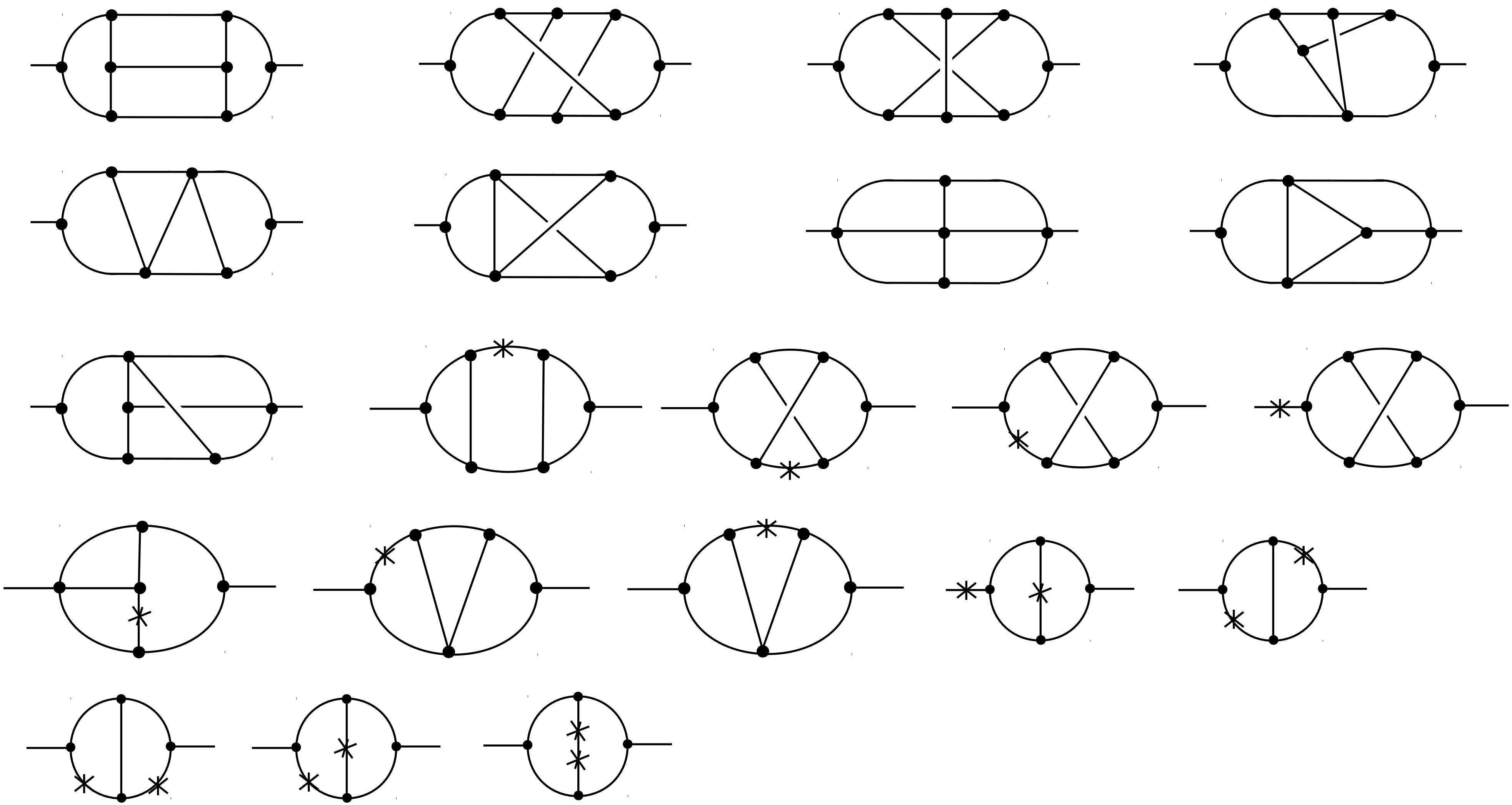
- Code generation from

visualized via Graphviz

  - Adequate subroutines are called at each topology
  - Symmetries from graph automorphism
  - Rewriting propagators and irreducible numerators at all transitions from a topology to another
  - Generates 41240 lines of FORM code for 4-loops
- Works even for 5-loops
  (64 top-topologies, 6570 in total, but 284 special rules)

Manual reduction rules

- 21 topologies require special rules, manually constructed



Finding manual reduction rules

- Shift an index by 1 in IBPs in all possible ways

$$n_1 \to n_1 + 1,\ n_2 \to n_2 + 1,\ n_3 \to n_3 + 1,\ \ldots$$

  - At the 4-loop, 20 IBPs and 14 indices

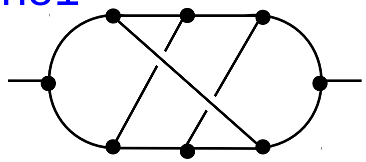
    ⇨ 20 $\times$ 14 = 280 equations

- Eliminate “complicated” integrals that increase indices from the system of equations as possible $F(n_1 + 1, \ldots)$

- This is not complete, but helps a lot. Human eyes for finding good rules for reducing integrals

- Everything should be done by using computer algebra systems (even for just rational arithmetic or simple substitutions)
  A rule can easily be over 1000 lines

Result looks like...



one of the top-level topology
11 propagators
3 irreducible numerators

```
[no1(2,2,2,2,2,2,2,2,2,2,2,-1,-1,-1)] =

-10/9*num(1+2*ep)^2*num(2+5*ep)*num(3+2*ep)^2*num(3+5*ep)*num(4+5*ep)*num(6+5*ep)*num(7+5*ep)*num(8+5*ep
)*num(9+5*ep)*num(36141384+167650024*ep+369157793*ep^2+504389598*ep^3+470560515*ep^4+312347786*ep^5+1497
70838*ep^6+51734214*ep^7+12600912*ep^8+2060632*ep^9+203648*ep^10+9216*ep^11)*den(1+ep)^2*den(2+ep)^2*den
(2+3*ep)*den(3+ep)^2*den(4+3*ep)*den(5+3*ep)*den(7+3*ep)*den(8+3*ep)*Master(no1)

+2/9*num(1+2*ep)*num(1+4*ep)^2*num(3+2*ep)*num(39337381531008+422506983834144*ep+2126828256064272*ep^2+6
682923999248124*ep^3+14715479582570820*ep^4+24142592323497543*ep^5+30606097414180459*ep^6+30663886432898
386*ep^7+24613240685251374*ep^8+15944111435166437*ep^9+8353498138352567*ep^10+3530865125153640*ep^11+119
5052526807120*ep^12+319518770241334*ep^13+66015276933500*ep^14+10173244132808*ep^15+1101566031376*ep^16+
74820000000*ep^17+2400000000*ep^18)*den(1+ep)^2*den(2+ep)^2*den(2+3*ep)*den(3+ep)^2*den(4+3*ep)*den(5+3*
ep)*den(7+3*ep)*den(8+3*ep)*Master(no6)

-4/9*num(ep)^2*num(1+2*ep)*num(3+2*ep)*num(4970268890523930816+65981799142896214872*ep+41636699974387213
0470*ep^2+1663841711414114631648*ep^3+4730696245772216998455*ep^4+10190116418687033776180*ep^5+172831512
25628577356718*ep^6+23674869565869007631757*ep^7+26649665048245779930527*ep^8+24945440138420091798098*ep
^9+19571616320498135318722*ep^10+12933081950816489701881*ep^11+7214358008133792648788*ep^12+339681000556
8803286172*ep^13+1346680080420400500352*ep^14+447318716827968227680*ep^15+123500240869574621248*ep^16+28
008584917867939712*ep^17+5129371482425778688*ep^18+739826312414941184*ep^19+80910145880457216*ep^20+6306
386119458816*ep^21+312125440000000*ep^22+7372800000000*ep^23)*den(1+ep)^2*den(2+ep)^2*den(3+ep)^2*den(3+
4*ep)*den(4+3*ep)*den(5+3*ep)*den(5+4*ep)*den(7+3*ep)*den(7+4*ep)*den(8+3*ep)*den(9+4*ep)*den(11+4*ep)*d
en(13+4*ep)*Master(lala)
```

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(cont'd)

```
+ ( other 17 terms )

+1/69984*num(-1+2*ep)^3*num(7241916201944976216509644800000+3199339702731261014302808309760000*ep+6756458
7046942832244281149906944OO*ep^2+9135045918439165567094477439873024O*ep^3+891289886775817303600596179144
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2804388788688381310O8*ep^9+860795277941095152241349409431372879328O*ep^10+2221477733684447301324246670O1
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972*ep^13+1620920042433367369484228940869152451887 10*ep^14+23601102551449473839547785232652438024 6208*ep
^15+2941166251221279521369966873815880454668 97*ep^16+30622594561109334098894115787780494656 5622*ep^17+25
03955371388627852231881556232455349645 32*ep^18+1278490968230930766213197239347627354187 74*ep^19-32311698
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1003341832521121415433402777 6*ep^45+89282384893529888417960130969 6*ep^46+5238433612681661107047392870 4*e
p^47+22389335378653391404841041 92*ep^48+6202902113698185216000000 0*ep^49+8362641768579072000000000*ep^50)
*den(ep)^6*den(-1+ep)*den(1+ep)^6*den(1+2*ep)^2*den(1+3*ep)^2*den(-2+ep)*den(2+ep)^6*den(2+3*ep)^2*den(3
+ep)^3*den(3+2*ep)^2*den(3+4*ep)*den(4+3*ep)^2*den(5+2*ep)^2*den(5+3*ep)^2*den(5+4*ep)*den(7+3*ep)^2*den
(7+4*ep)*den(8+3*ep)*den(9+4*ep)*den(11+4*ep)*den(13+4*ep)*G10*G20*G30
```

Optionally: ϵ-expansions in intermediate steps

Checks and status

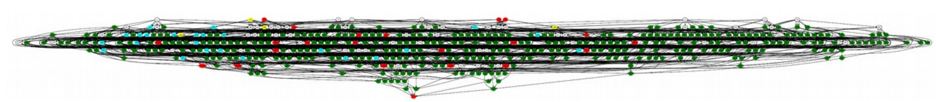
Collaboration with Andreas Vogt

- Recomputing known results – strong non-trivial checks
  - Reproduced the 4-loop QCD β-function via massless propagators [Ritbergen, Vermaseren, Larin '97; Czakon '04]
  - Checked the gauge invariance
  - Also with using background field method
  - Reproduced N=2 and N=4 non-singlet splitting functions
    [Baikov, Chetyrkin '06] [Velizhanin '11] [Velizhanin '14]
- Statistics of these computations tell us bottlenecks
- Trying to optimize the program before going for N=6 (unknown)

Summary

Summary

- Evaluating Massless propagator-type Feynman integrals is one of key components of pQCD
- “Mincer” up to 3-loops
- We have been developing “Forcer” for 4-loops
  - Highly complicated structure of the program / equations

    ⇨ Automatization: write a program for generating a program for 

    Manual rules are derived with the aid of computers
- Status:
  - Checked for known results
  - More optimizations

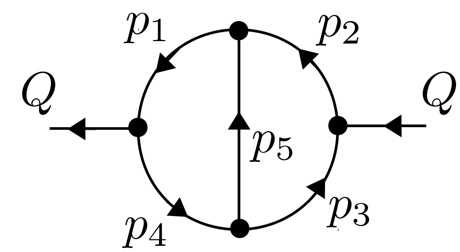
Backup

Integration-by-parts identities (IBPs)

[Chetyrkin, Tkachov '81]

- The Gauss theorem in D-dimension (the surface term vanishes)

$$\int d^D p \frac{\partial}{\partial p^\mu} X^\mu = 0$$



$$p_3 = p_2 - Q$$
$$p_4 = p_1 - Q$$
$$p_5 = p_1 - p_2$$

- Example:

$$F(n_1, n_2, n_3, n_4, n_5) = \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{1}{(p_1^2)^{n_1} (p_2^2)^{n_2} (p_3^2)^{n_3} (p_4^2)^{n_4} (p_5^2)^{n_5}}$$

$$\int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{\partial}{\partial p_1^\mu} \frac{p_2^\mu}{(p_1^2)^{n_1} (p_2^2)^{n_2} (p_3^2)^{n_3} (p_4^2)^{n_4} (p_5^2)^{n_5}} = 0$$

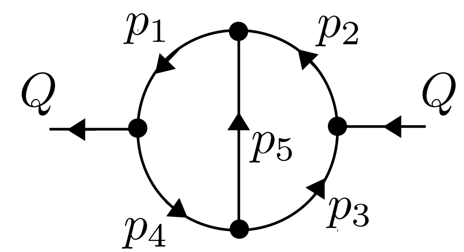
- Performing $\frac{\partial}{\partial p_1^\mu}$ and multiplying p_2^μ give various dot products (e.g., $p_1 \cdot p_2$), which are all decomposed to sums of the propagators ⇒ linear identity among F

Integration-by-parts identities (IBPs)

- $\{p_1, p_2; Q\}$ ⇨ $2 \times 3 = 6$ identities (and their linear combination)

$$\frac{\partial}{\partial p_1} \cdot p_1 \qquad \frac{\partial}{\partial p_1} \cdot p_2 \qquad \frac{\partial}{\partial p_1} \cdot Q$$

$$\frac{\partial}{\partial p_2} \cdot p_1 \qquad \frac{\partial}{\partial p_2} \cdot p_2 \qquad \frac{\partial}{\partial p_2} \cdot Q$$



$$\begin{aligned}
&F(n_1-1, n_2, n_3, n_4+1, n_5) \times (-n_4) \\
&+ F(n_1-1, n_2, n_3, n_4, n_5+1) \times (-n_5) \\
&+ F(n_1, n_2-1, n_3, n_4, n_5+1) \times (n_5) \\
&+ F(n_1, n_2, n_3, n_4+1, n_5) \times (Q^2 n_4) \\
&+ F(n_1, n_2, n_3, n_4, n_5) \times (-2n_1 - n_4 - n_5 + 4 - 2\epsilon) \\
&= 0
\end{aligned}$$

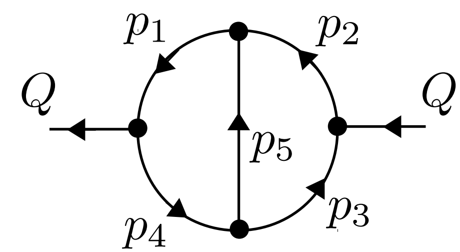
$$\begin{aligned}
&F(n_1-1, n_2, n_3, n_4+1, n_5) \times (-n_4) \\
&+ F(n_1-1, n_2, n_3, n_4, n_5+1) \times (-n_5) \\
&+ F(n_1, n_2-1, n_3, n_4, n_5+1) \times (n_5) \\
&+ F(n_1, n_2, n_3, n_4+1, n_5) \times (Q^2 n_4) \\
&+ F(n_1+1, n_2-1, n_3, n_4, n_5) \times (-n_1) \\
&+ F(n_1+1, n_2, n_3, n_4, n_5-1) \times (n_1) \\
&+ F(n_1, n_2, n_3-1, n_4+1, n_5) \times (-n_4) \\
&+ F(n_1, n_2, n_3, n_4+1, n_5-1) \times (n_4) \\
&+ F(n_1, n_2, n_3, n_4, n_5) \times (-n_1 + n_5) \\
&= 0
\end{aligned}$$

$$\frac{\partial}{\partial p_1} \cdot p_1 - \frac{\partial}{\partial p_1} \cdot p_2 = \text{?}$$

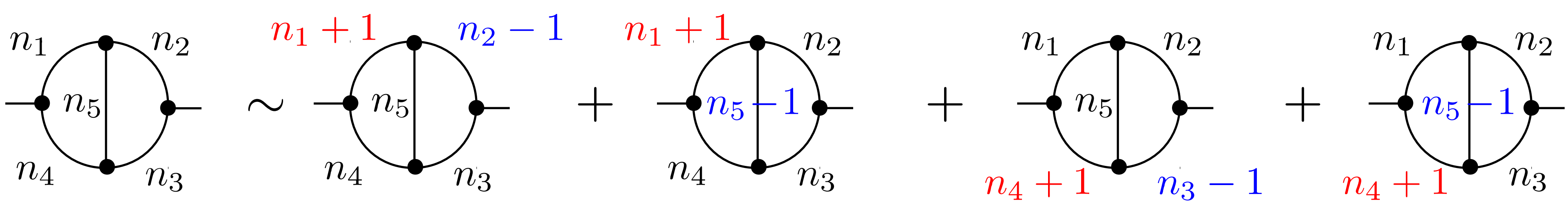
and other 4 identities

Integration-by-parts identities (IBPs)

$$\frac{\partial}{\partial p_1} \cdot p_1 - \frac{\partial}{\partial p_1} \cdot p_2 \Rightarrow$$



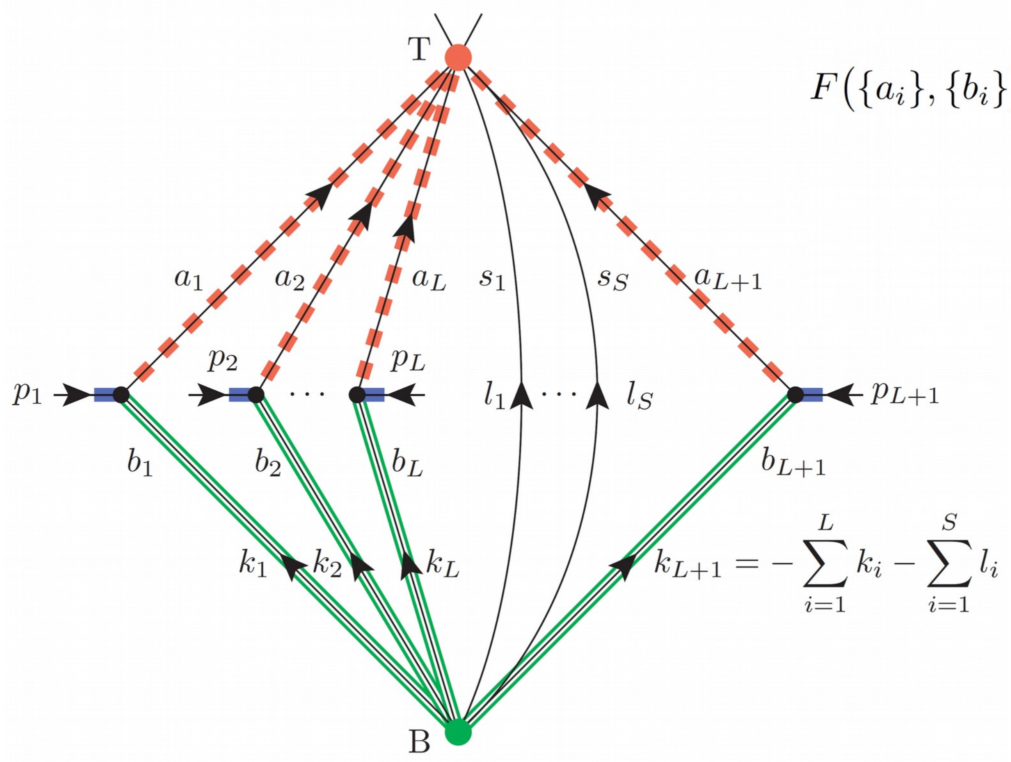
$$F(n_1,n_2,n_3,n_4,n_5) \times (\underbrace{-n_1 - n_4 - 2n_5 + 4 - 2\epsilon}_{\text{never be 0}})$$
$$+ F(n_1+1, n_2-1, n_3, n_4, n_5) \times (n_1)$$
$$+ F(n_1+1, n_2, n_3, n_4, n_5-1) \times (-n_1)$$
$$+ F(n_1, n_2, n_3-1, n_4+1, n_5) \times (n_4)$$
$$+ F(n_1, n_2, n_3, n_4+1, n_5-1) \times (-n_4) = 0$$



- Decreases $(n_2 + n_3 + n_5)$ by 1 at the cost of increasing $(n_1 + n_4)$
- Repeated use of this rule gives integrals with $n_2 = 0$, $n_3 = 0$ or $n_5 = 0$ $\Rightarrow$ simpler topologies with removing one of the lines

Diamond rule

[Ruijl, TU, Vermaseren '15]



$$F(\{a_i\},\{b_i\}) = \left[\prod_{i=1}^{L} \int d^D k_i\right]\left[\prod_{i=1}^{S} \int d^D l_i\right] \times \left[\prod_{i=1}^{L+1} \frac{k_i^{\mu_1^{(i)}} \dots k_i^{\mu_{N_i}^{(i)}}}{\left[(k_i+p_i)^2+m_i^2\right]^{a_i} (k_i^2)^{b_i}}\right]\left[\prod_{i=1}^{S} \frac{l_i^{\nu_1^{(i)}} \dots l_i^{\nu_{R_i}^{(i)}}}{(l_i^2)^{s_i}}\right]$$

$$(L+S)D + \sum_{i=1}^{L+1}(N_i - a_i - 2b_i) + \sum_{i=1}^{S}(R_i - 2s_i) = \sum_{i=1}^{L+1} a_i \boldsymbol{A}_i^+ \left[\boldsymbol{B}_i^- - (p_i^2 + m_i^2)\right]$$

Diamond rule

- Explicit summation formula

$$F(\{a_i\},\{b_i\},\{c_i\}) =$$

$$\sum_{r=1}^{L+1}\Bigg[\Bigg(\prod_{\substack{i=1\\i\neq r}}^{L+1}\sum_{k_i^+=0}^{b_i-1}\Bigg)\Bigg(\prod_{i=1}^{L+1}\sum_{k_i^-=0}^{c_i-1}\Bigg)(-1)^{k^-}\frac{k_r^+(k^++k^--1)!}{\prod_{i=1}^{L+1}k_i^+!k_i^-!}(E+k^+)_{-k^+-k^-}$$

$$\times\Bigg(\prod_{i=1}^{L+1}(a_i)_{k_i^++k_i^-}\Bigg)F\left(\{a_i+k_i^++k_i^-\},\{b_i-k_i^+\},\{c_i-k_i^-\}\right)\Bigg]_{k_r^+=b_r}$$

$$+\sum_{r=1}^{L+1}\Bigg[\Bigg(\prod_{i=1}^{L+1}\sum_{k_i^+=0}^{b_i-1}\Bigg)\Bigg(\prod_{\substack{i=1\\i\neq r}}^{L+1}\sum_{k_i^-=0}^{c_i-1}\Bigg)(-1)^{k^-}\frac{k_r^-(k^++k^--1)!}{\prod_{i=1}^{L+1}k_i^+!k_i^-!}(E+k^++1)_{-k^+-k^-}$$

$$\times\Bigg(\prod_{i=1}^{L+1}(a_i)_{k_i^++k_i^-}\Bigg)F\left(\{a_i+k_i^++k_i^-\},\{b_i-k_i^+\},\{c_i-k_i^-\}\right)\Bigg]_{k_r^-=c_r}$$

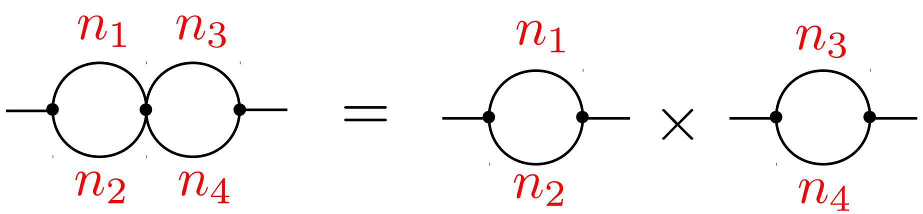
$$E=(L+S)D+\sum_{i=1}^{L+1}(N_i-a_i-2b_i)+\sum_{i=1}^{S}(R_i-2s_i)$$

$$k^+=\sum_{i=1}^{L+1}k_i^+\qquad k^-=\sum_{i=1}^{L+1}k_i^-$$

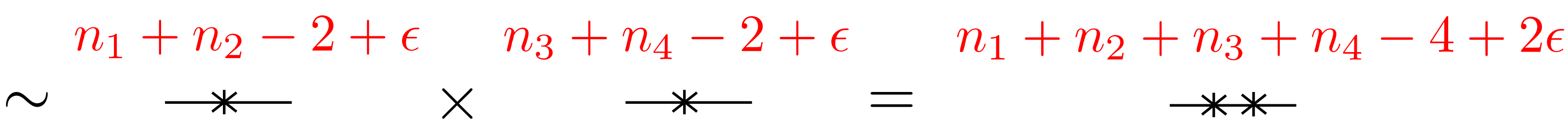
Avoids spurious poles

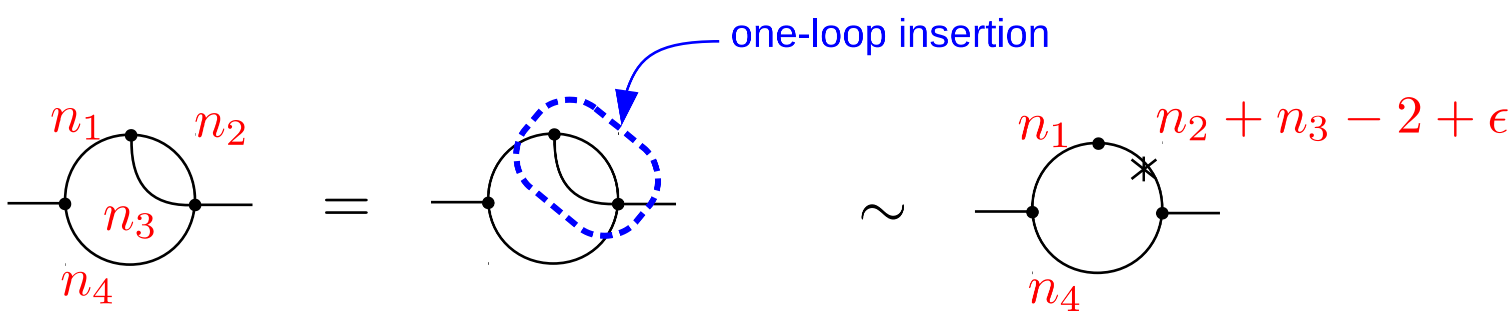
2-loop: easy integrals

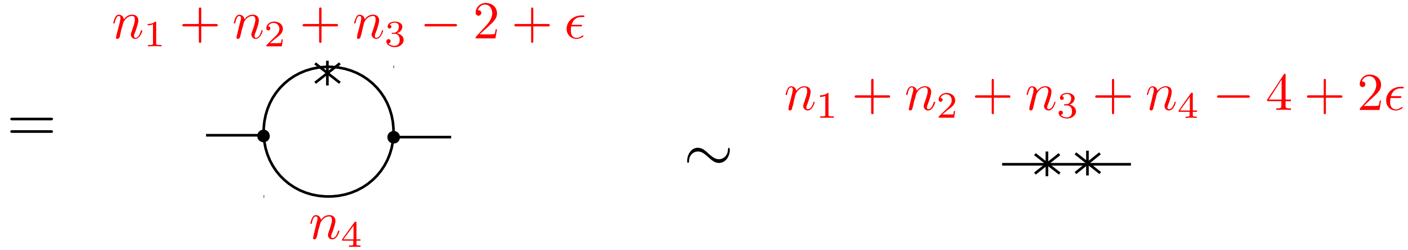
- Products / convolutions of 1-loop integrals



Lines represent denominators
Numerators are no problem

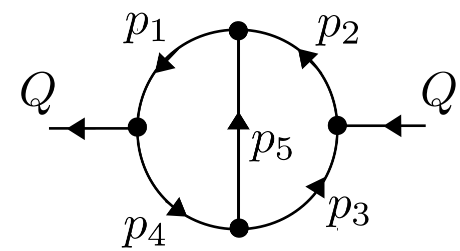






2-loop: generic topology

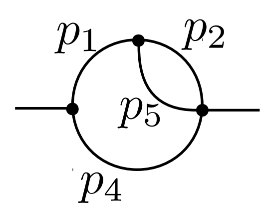
- "Generic" topology with the maximal number of lines at the 2-loop:



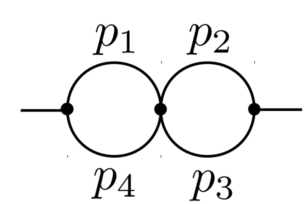
$p_3 = p_2 - Q$
$p_4 = p_1 - Q$
$p_5 = p_1 - p_2$

$$F(n_1, n_2, n_3, n_4, n_5) = \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{1}{(p_1^2)^{n_1} (p_2^2)^{n_2} (p_3^2)^{n_3} (p_4^2)^{n_4} (p_5^2)^{n_5}}$$

- The propagators are "complete": e.g., $2p_1 \cdot p_2 = p_1^2 + p_2^2 - p_5^2$
- Easy when one of the indices is ≤ 0



or

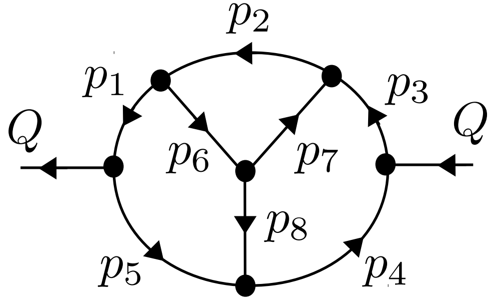


- In the case that all the indices are ≥ 1:
  - In early days: special techniques (e.g., GPXT)
  - Nowadays: IBPs

3-loop level

- Generic topologies have 8 lines, but there are 9 dot products
  Need to introduce another index for an irreducible numerator

$$F(n_1, \ldots, n_9) = \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \times \frac{(p_2 \cdot Q)^{-n_9}}{(p_1^2)^{n_1} \ldots (p_8^2)^{n_8}}$$

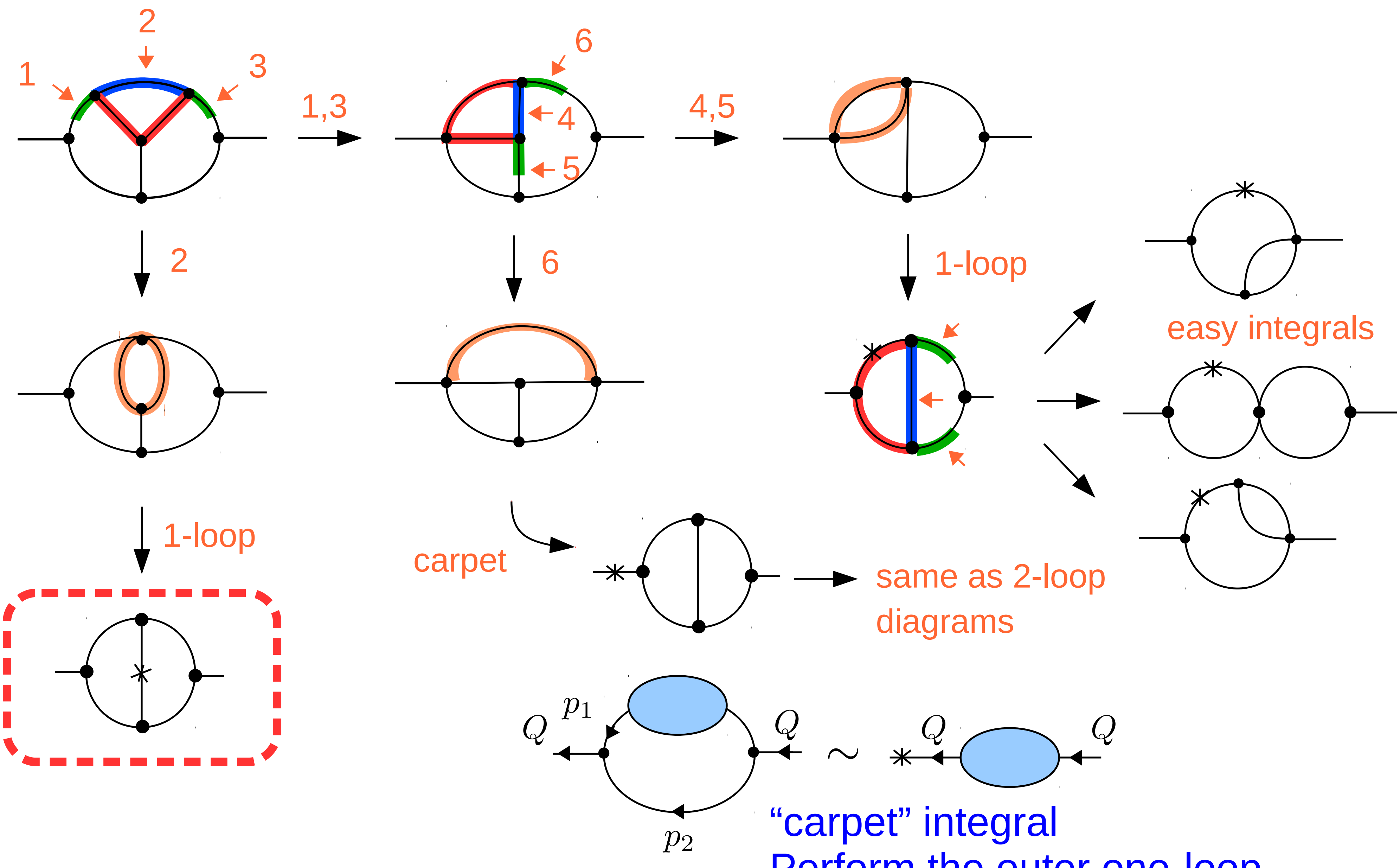


"Benz" topology

  - Our convention
    positive indices : denominators
    negative indices: numerators

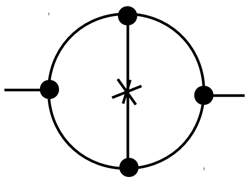
- The irreducible numerator $(p_2 \cdot Q)$ is chosen such that it does not interfere with use of the triangle rule (next page)

Reduction of the Benz topology



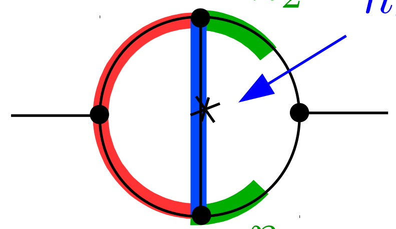
Special topologies at 3-loop

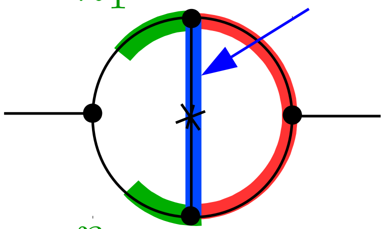
- At the 3-loop level, there are 2 topologies that require special treatment

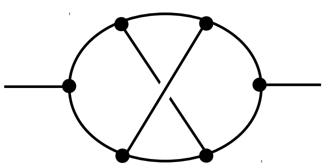


Repeated use of the triangle rule cannot remove the central line because of the non-integer $n_5 + \epsilon$

2-loop topology with ϵ at the central line







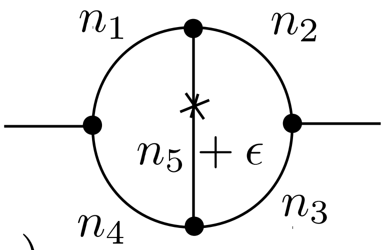
No triangle

generic “non-planer” topology

- Need to “solve” IBPs: solve a system of recurrence relations to reduce these integrals as much as possible

Special topologies at 3-loop

- Fortunately, this is possible:



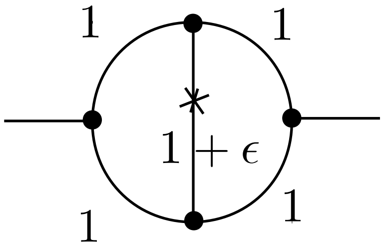
$$F(n_1,n_2,n_3,n_4,n_5+\epsilon)\times\underbrace{(n_1-1)}$$ valid for $n_1 \neq 1$

$$+F(n_1-1,n_2,n_3,n_4,n_5+\epsilon)\times(-1)\times(n_1+2n_4+n_5-5+3\epsilon)$$
$$+F(n_1,n_2,n_3,n_4-1,n_5+\epsilon)\times(-1)\times(n_1-1)$$
$$+F(n_1-1,n_2,n_3-1,n_4,n_5+1+\epsilon)\times(n_5+\epsilon)$$
$$+F(n_1-1,n_2,n_3,n_4-1,n_5+1+\epsilon)\times(-1)\times(n_5+\epsilon)=0$$

Decreases n_1 or n_4 (or both) until $n_1 = 1$. Similar rules for n_2, n_3, n_4

* They can be derived from the triangle rule identity [Ruijl, TU, Vermaseren '15]

Then n_5 can be increased/decreased until $n_5 = 1$ by



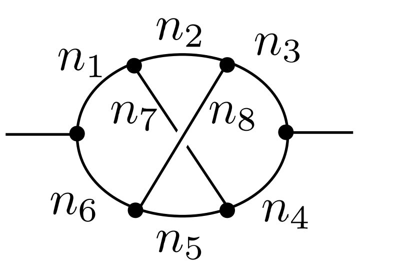
$$F(1,1,1,1,n_5+\epsilon)\times(-1)\times(n_5-1+2\epsilon)$$
$$+F(1,1,1,1,n_5-1+\epsilon)\times(-1)\times(n_5-2+3\epsilon)$$
$$+F(0,1,1,1,n_5+\epsilon)\times(-3+2n_5+5\epsilon)$$
$$+F(1,1,1,0,n_5+\epsilon)\times(-3+2n_5+5\epsilon)=0$$

- "Master" integral $F(1,1,1,1,1+\epsilon)$ cannot be reduced any more

Special topologies at 3-loop

- One can make a reduction scheme also for the non-planer topology

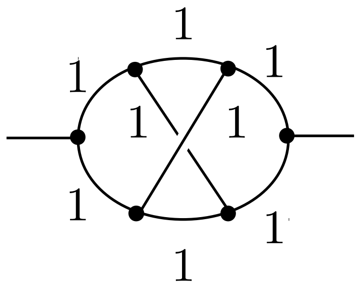
  * Rules for reducing n_1, n_3, n_4, n_6 to 1 can be derived from the diamond rule identity [Ruijl, TU, Vermaseren '15]

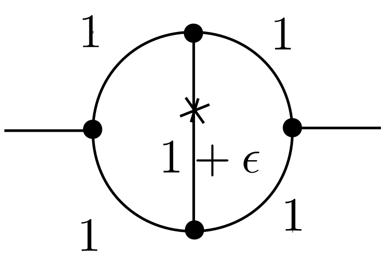


$n_9 : p_2 \cdot Q$

  - Master integral $F(1,1,1,1,1,1,1,1,0)$

- The 2 master integrals (MIs) at the 3-loop level





- They cannot be expressed in terms of Γ-functions
  But their expansions with respect to ϵ are known
  (at least, up to enough orders)