The flavour dependence for the four-loop QCD correction to the relation between pole and running heavy quark masses

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based on Phys. Rev. D92 (1995) 5 (discussion of HO corrections to static potential) and arXiV:1511.06898 (derivation of flavour-dependence of 4-loop correction to the relation between pole and running masses)

Plan

- The third order perturbative QCD static potential
- Determining of the heavy quark masses by means of the static QCD potential
- Consideration of the ratio $z_m(\mu^2)$ between the running heavy quark mass in $\overline{\mathrm{MS}}$ scheme and the pole mass at $\mathcal{O}(\alpha_s^4)$ level
- Determination of n_l dependence of $z_m(\mu^2)$ in different normalizations by the least squares method
- Possible evaluation of the inaccuracies of the used method
- Conclusion

The third order perturbative QCD static potential

The static potential in QCD is introduced with using Wilson loop as a potential of interaction between static quark and antiquark at a distance r:

$$\mathrm{V}_{\mathrm{QCD}}(\mu^2, r^2, \alpha_s(\mu^2)) = -\lim_{T\to\infty} \frac{1}{iT} \ln \frac{\langle 0 \mid \mathrm{Tr}\ \mathrm{P}e^{ig \oint_C dx^\mu A_\mu^a T^a} \mid 0\rangle}{\langle 0 \mid \mathrm{Tr}\ 1 \mid 0\rangle} =$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\vec{r}} \mathrm{V}(\vec{q}^{\,2}, \mu^2, \alpha_s(\vec{q}^{\,2})) \quad ,$$

where $\alpha_s = g^2/4\pi$, g is the strong coupling constant of the QCD Lagrangian in $\overline{\mathrm{MS}}$ scheme, μ is the renormalization parameter of dimensional regularization, C is a rectangular loop with time extent T and spatial extent r, T^a is the generator of $SU(N_c)$ group, A_μ^a is the gluon field, P is the ordering operator along the way.

The third order perturbative QCD static potential

In momentum space the QCD static potential can be written into the following form

$$\mathrm{V}(\vec{q}^{\,2},\mu^2,\alpha_s(\vec{q}^{\,2})) = \frac{-4\pi C_F \alpha_s(\vec{q}^{\,2})}{\vec{q}^{\,2}} \Bigg(1 + a_1^{\overline{\mathrm{MS}}} \frac{\alpha_s(\vec{q}^{\,2})}{4\pi} + $$
$$+ a_2^{\overline{\mathrm{MS}}} \left(\frac{\alpha_s(\vec{q}^{\,2})}{4\pi} \right)^2 + \left(a_3^{\overline{\mathrm{MS}}} + 8\pi^2 C_A^3 \ln \frac{\mu^2}{\vec{q}^{\,2}} \right) \left(\frac{\alpha_s(\vec{q}^{\,2})}{4\pi} \right)^3 + \ldots \Bigg) \quad .$$

Here and below it will be used standard notations for colour structures of $SU(N_c)$ group: $[T^a, T^b] = i f^{abc} T^c$, $f^{acd} f^{bcd} = C_A \delta^{ab}$, $(T^a T^a)_{ij} = C_F \delta_{ij}$, C_A and C_F are the Casimir operators, $C_A = N_c$, $C_F = (N_c^2 - 1)/2N_c$.
The additional term $8\pi^2 C_A^3 L$ appears due to the infrared (IR) divergences, which begin to manifest themselves in the the static potential at the three-loop level. We will neglect it in our RG-based analysis.

Colour structures of SU(N_c) group

N_A is the number of the generators of the Lie algebra of the $SU(N_c)$ group, $n_l{=}n_f - 1$, n_f is the number of quark flavours, $d_F^{abcd} = Tr(T^a T^{(b} T^c T^{d)})/6$ and $d_A^{abcd} = Tr(C^a C^{(b} C^c C^{d)})/6$ are the total symmetric tensors, $(C^a)_{bc} = -if^{abc}$, where C^a are the generators of the adjoint representation of the Lie algebra of the $SU(N_c)$ group and

$$N_A = N_c^2 - 1 \ , \frac{d_A^{abcd} d_A^{abcd}}{N_A} = \frac{N_c^2(N_c^2+36)}{24} \ ,$$

$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c(N_c^2+6)}{48} \ , \frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{N_c^4 - 6N_c^2 + 18}{96 N_c^2} \ .$$

High order PT QCD corrections to the static potential

The coefficients $a_i^{\overline{\mathrm{MS}}}$ are calculated from the concrete Feynman diagrams and equal respectively

$$a_1^{\overline{\mathrm{MS}}} = \frac{31}{9} C_A - \frac{20}{9} T_F n_l \ ,$$

(W. Fischler, 1977; A. Billoire, 1980)

$$a_2^{\overline{\mathrm{MS}}} = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3) \right) C_A^2 - \left(\frac{1798}{81} + \frac{56}{3}\zeta(3) \right) C_A T_F n_l$$
$$- \left(\frac{55}{3} - 16\zeta(3) \right) C_F T_F n_l + \left(\frac{20}{9} T_F n_l \right)^2 \ ,$$

(M. Peter, 1997; Y. Schroder, 1999)

High order PT QCD corrections to the static potential

The three-loop constant perturbative contribution to the static potential in the $\overline{\text{MS}}$-scheme can be presented as

$$a_3^{\overline{\text{MS}}} = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)} \ .$$

$$a_3^{(3)} = -\frac{8000}{729} T_F^3 \ ,$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368}{3}\zeta(3) + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416}{3}\zeta(3)\right) C_F T_F^2$$

$$a_3^{(1)} = -709.717 C_A^2 T_F + \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5)\right) C_A C_F T_F$$

$$+\left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5)\right) C_F^2 T_F - 56.83(1) \frac{d_F^{abcd} d_F^{abcd}}{N_A}$$

where the error of numerical calculation of the $C_A^2 T_F$-coefficient is not indicated (A. Smirnov, V. Smirnov, M. Steinhauser, 2008) .

High order PT QCD corrections to the static potential

The numerical expressions of the n_l-independent contributions were obtained by A. Smirnov, V. Smirnov and M. Steinhauser in 2010 and read

$$a_3^{(0)} = 502.24(1)C_A^3 - 136.39(12)\frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

These results should be compared with the independent calculation of C. Anzai, Y. Kiyo and Y. Sumino in 2010

$$a_3^{(0)} = 502.22(12)C_A^3 - 136.8(14)\frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

which have greater inaccuracies. Recent the more accurate result was obtained by Y. Sumino (private communication 2015):

$$a_3^{(0)} = 502.22(12)C_A^3 - 136.6(2)\frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

Intereswting topic- progrms FIESTA and the one used by Y. Sumino NOT YET compared in detail

Subtracted potential

Naturally in addition to the perturbative contributions to the heavy quark potential there are non-perturbative long-distance modifications. They create a relative suppression of the potential of the type $\Lambda^2_{\rm QCD}/\vec{q}^{\,2}$. Such confining part of V corresponds to a linear confining term to the potential in coordinate space $\Lambda_{\rm QCD} r$. One can eliminate it by restricting the Fourier transform to $\mid \vec{q} \mid > \mu_f$ for some factorisation scale. Perturbativity hence requires $\mu_f > \Lambda_{\rm QCD}$. The subtraction should affect the potential only at distances larger than the physical scale of the process described by the potential. Then $\mu_f < 1/r \sim m_q v$, where v is the small relative velocity of two quarks in their centre of mass ($\mu_f \approx 2$ GeV for bottom and 20 GeV for top quark). The subtracted potential $\mathrm{V}(r, \mu_f)$ is defined as

$$\mathrm{V}(r, \mu_f) = \mathrm{V}(r) + 2\delta m_q(\mu_f) \ ,$$

Subtracted potential

where residual mass $\delta m_q(\mu_f)$ is determined

$$\begin{aligned}\delta m_q(\mu_f) &= -\frac{1}{2}\int\limits_{|\vec{q}|<\mu_f} \frac{d^3\vec{q}}{(2\pi)^3}\mathrm{V}(\vec{q}^{\,2}) = \\ &= \mu_f\frac{C_F\alpha_s}{\pi}\left(1+\frac{\alpha_s}{4\pi}(a_1^{\overline{\mathrm{MS}}}+\beta_0(2+\ln(\mu^2/\mu_f^2))+\dots\right)\end{aligned}$$

As a consequence the input parameter for threshold calculations in terms of the subtracted potential is not pole mass M but

$$m_{\mathrm{PS}_q}(\mu_f) = M_q - \delta m_q(\mu_f)\ , \quad \text{M. Beneke 1998}$$

where $m_{\mathrm{PS}_q}(\mu_f)$ is the potential subtracted quark mass and it does not involve large loop perturbative corrections, M_q is the pole mass.

Subtracted potential

Introducing pole mass through running mass, namely:

$$M_q = \overline{m}_q(\overline{m}_q^2)(1 + \sum_{i=1}^{4} l_i a_s^i(\overline{m}_q^2))$$

and substituting this result into the definition of $m_{\mathrm{PS}_q}(\mu_f)$, one can see that the dependence on the factorisation scale μ_f cancels in this process. Hence one can find subtracted quark mass:

$$m_{\mathrm{PS}_q}(\mu_f) = \overline{m}_q(\overline{m}_q^2)\left(1 + \frac{4\alpha_s(\overline{m}_q^2)}{3\pi}\left(1 - \frac{\mu_f}{\overline{m}_q(\overline{m}_q^2)}\right) + \right.$$

$$+ \left(\frac{\alpha_s(\overline{m}_q^2)}{\pi}\right)^2 (13.44 - 1.04 n_f -$$

$$\left. - \frac{\mu_f}{3\overline{m}_q(\overline{m}_q^2)}\left(a_1^{\overline{\mathrm{MS}}} - \beta_0\left(\ln\frac{\mu_f^2}{\overline{m}_q^2(\overline{m}_q^2)} - 2\right)\right)\right) + \ldots\right)$$

Pole quark mass

It is known that the total quark bare propagator has following form

$$i\hat{\mathrm{G}}(k) = \frac{i}{\hat{k} - m_{0,q} + \hat{\Sigma}(k)}$$

Here $\hat{\Sigma}(k)$ denotes a single-particle irreducible self-energy operator of a quark and $m_{0,q}$ is a bare quark mass. In consequence of relativistic invariance is performed equality

$$\Sigma(\hat{k}) = m_{0,q}\Sigma_1(k^2) + (\hat{k} - m_{0,q})\Sigma_2(k^2)$$

Connection of the bare mass $m_{0,q}$ with pole M_q is written in a standard way through the renormalization mass constant in the OS-scheme

$$m_{0,q} = Z_m^{\mathrm{OS}} M_q.$$

Pole and running quark mass

The subtraction OS-scheme demands that the heavy quark propagator has a pole on the mass shell. Hence

$$Z_m^{\mathrm{OS}} = m_0/M_q = 1 + \Sigma_1(M_q^2, \alpha_s(\mu^2)).$$

A similar expression can be written in the $\overline{\mathrm{MS}}$-scheme also:

$$m_{0,q} \quad = \quad Z_m^{\overline{\mathrm{MS}}} \overline{m}_q(\mu^2).$$

Renormalization mass factor $Z_m^{\overline{\mathrm{MS}}}$ is computed by perturbation theory and can be represented as a series in powers of the coupling constant, consisting from ultraviolet divergences in the parameter ε:

$$Z_m^{\overline{\mathrm{MS}}}(\alpha_s(\mu^2)) = 1 + \sum_{i=1}^{\infty} \sum_{j=1}^{i} \frac{z_{ij}(n_l^{i-1})}{\varepsilon^j} \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^i$$

where n_l is a number of light quark flavours and $(...^{i-1})$ is the maximal degree of n_l, on which depends the function z_{ij}.

The ratio $z_m(\mu^2)$ between the running heavy quark mass in $\overline{\mathrm{MS}}$ scheme and the pole mass at $\mathcal{O}(\alpha_s^4)$ level

Consequently the ratio of the running quark mass in $\overline{\mathrm{MS}}$ scheme to the pole mass of heavy quark will be expressed as

$$z_m(\mu^2) = \frac{\overline{m}_q(\mu^2)}{M_q} = \frac{Z_m^{\mathrm{OS}}}{Z_m^{\overline{\mathrm{MS}}}} = 1 + \sum_{i=1}^{\infty} z_m^{(i)} \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^i$$

Due to the fact that the masses $\overline{m}_q(\mu^2)$ and M_q are renormalized finite quantities, hence the quantity $z_m(\mu^2)$ must be finite also, and $z_m^{(i)}$ in particular. Moreover the coefficients of expansion $z_m^{(i)}$ can be represented as polynomials $(i-1)$-power of n_l:

$$z_m^{(i)} = \sum_{j=0}^{i-1} z_m^{(i,\, j)} n_l^j \ ,$$

where all terms before $z_m^{(4)}$ are computed analytically.

The ratio $z_m(\mu^2)$ between the running heavy quark mass in $\overline{\mathrm{MS}}$ scheme and the pole mass at $\mathcal{O}(\alpha_s^4)$ level

For case of $SU_c(3)$ group one can obtain at fixed renormalization parameter $\mu^2{=}M_q^2$:

$$z_m^{(10)} = -\frac{4}{3}\,,\ z_m^{(20)} = -\frac{3019}{288} + \frac{\zeta_3}{6} - \frac{\pi^2\ln 2}{9} - \frac{\pi^2}{3}\,,\ z_m^{(21)} = \frac{71}{144} + \frac{\pi^2}{18}\,,$$

$$z_m^{(30)} = -\frac{9478333}{93312} - \frac{61\zeta_3}{27} - \frac{644201\pi^2}{38880} + \frac{587\pi^2\ln 2}{162} + \frac{22\pi^2\ln^2 2}{81} + \frac{1439\pi^2\zeta_3}{432} - \frac{1975\zeta_5}{216} + \frac{695\pi^4}{7776} + \frac{55\ln^4 2}{162} + \frac{220}{27}\mathrm{Li}_4\left(\frac{1}{2}\right)\,,$$

$$z_m^{(31)} = \frac{246643}{23328} + \frac{241\zeta_3}{72} + \frac{967\pi^2}{648} + \frac{11\pi^2\ln 2}{81} - \frac{2\pi^2\ln^2 2}{81} - \frac{61\pi^4}{1944} - \frac{\ln^4 2}{81} - \frac{8}{27}\mathrm{Li}_4\left(\frac{1}{2}\right)\,,$$

$$z_m^{(32)} = -\frac{2353}{23328} - \frac{7\zeta_3}{54} - \frac{13\pi^2}{324}\,,$$

The ratio $z_m(\mu^2)$ between the running heavy quark mass in $\overline{\mathrm{MS}}$ scheme and the pole mass at $\mathcal{O}(\alpha_s^4)$ level

$$z_m^{(43)} = \frac{42979}{1119744} + \frac{317\zeta_3}{2592} + \frac{89\pi^2}{3888} + \frac{71\pi^4}{25920} \ ,$$

$$z_m^{(42)} = -\frac{32420681}{4478976} - \frac{40531\zeta_3}{5184} - \frac{63059\pi^2}{31104} - \frac{103\pi^2\ln 2}{972} + \\ + \frac{11\pi^2\ln^2 2}{243} - \frac{2\pi^2\ln^3 2}{243} - \frac{5\pi^2\zeta_3}{48} + \frac{241\zeta_5}{216} - \frac{30853\pi^4}{466560} - \\ - \frac{31\pi^4\ln 2}{9720} + \frac{11\ln^4 2}{486} - \frac{\ln^5 2}{405} + \frac{44}{81}\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{8}{27}\mathrm{Li}_5\left(\frac{1}{2}\right) \ ,$$

where were used the following values $C_F = 4/3$, $C_A = 3$ and notations $\zeta_n = \sum\limits_{k=1}^{\infty} k^{-n}$ and $\mathrm{Li_n(x)} = \sum\limits_{\mathrm{k=1}}^{\infty} \mathrm{x^k k^{-n}}$.

The ratio $z_m(\mu^2)$ between the running heavy quark mass in $\overline{\text{MS}}$ scheme and the pole mass at $\mathcal{O}(\alpha_s^4)$ level

It should separately write out the cubic in n_l term $z_m^{(4)}$:

$$z_m^{(4)} = z_m^{(40)} + z_m^{(41)} n_l + z_m^{(42)} n_l^2 + z_m^{(43)} n_l^3 \ .$$

As it has been already mentioned in this expression the last two terms are known in analytical form and computed by R. Lee, P. Marquard, A. V. Smirnov, V. A. Smirnov and M. Steinhauser (2013), and the first two terms are not been yet calculated analytically, but expressions of these coefficients have recently been obtained in the work Y. Kiyo, G. Mishima and Y. Sumino (2015) with the help of fitting procedure (NO ERROR BARS were given!). Let us find the numerical estimates for these two factors by the MATHEMATICAL method - the method of least squares.

The estimation of coefficients for n_l flavour dependence and constant term by the least squares method

Using recently calculated results by P. Marquard, A. V. Smirnov, V. A. Smirnov and M. Steinhauser (2015) for coefficient $z_m^{(4)}$ at fixed n_l, performed due to consideration of $m_{\mathrm{PS}}(\mu_f)$, $\delta m(\mu_f)$ and special algorithms such as the Laporta algorithm, Fiesta program:

$$z_m^{(4)}(M_q^2)\Big|_{n_l=3} = -1744.8 \pm 21.5, \quad z_m^{(4)}(M_q^2)\Big|_{n_l=4} = -1267.0 \pm 21.5,$$

$$z_m^{(4)}(M_q^2)\Big|_{n_l=5} = -859.96 \pm 21.5$$

The uncertainty σ=21.5 may be related to the computation of the four-loop diagrams without insertion of fermion loops into gluon propagators. Indeed the inaccuracies σ do not depend on n_l, and this error is almost entirely defined by the constant term $z_m^{(40)}$, while the errors of $z_m^{(41)}$ are negligible (**this our conclusions not yes confirmed by the MSSS-team)**

The estimation of coefficients for n_l flavour dependence and constant term by the least squares method

We use the formulas given above to find the expressions of the first two coefficients $z_m^{(40)}(M_q^2)$ and $z_m^{(41)}(M_q^2)$. In this case we get the following overdetermined system of linear equations:

$$\begin{aligned} z_m^{(40)} + 3z_m^{(41)} &= -1371.77, \\ z_m^{(40)} + 4z_m^{(41)} &= -614.68, \\ z_m^{(40)} + 5z_m^{(41)} &= 142.32. \end{aligned}$$

To solve this system we use the method of least squares. For this we introduce the function which equal to the sum of squared deviations $\Delta_{l_k} = z_m^{(40)} + z_m^{(41)} n_{l_k} - y_{l_k}$, where index k denotes a number of equation, and y_{l_k} is a right part of equations of the system, i.e. $y_{l_k} = z_{m_k}^{(4)} - z_m^{(42)} n_{l_k}^2 - z_m^{(43)} n_{l_k}^3$:

$$\Phi(z_m^{(40)}, z_m^{(41)}) = \sum_{k=1}^{3} \Delta_k^2 = \sum_{k=1}^{3} (z_m^{(40)} + z_m^{(41)} n_{l_k} - y_{l_k})^2$$

The estimation of coefficients for n_l flavour dependence and constant term by the least squares method

Within LSM the solutions of this linear system $z_m^{(40)}, z_m^{(41)}$ are finding the minimums of the function $\Phi(z_m^{(40)}, z_m^{(41)})$, namely :

$$\frac{\partial\Phi}{\partial z_m^{(40)}} = 0, \quad \frac{\partial\Phi}{\partial z_m^{(41)}} = 0.$$

These conditions allow us to find numerical values of $z_m^{(40)}$ and $z_m^{(41)}$:

$$z_m^{(40)}(M^2) = -3642.9, \quad z_m^{(41)}(M^2) = 757.05.$$

Note, that they agree in reasonable agreement with the values

$$z_m^{(40)}(M^2) = -3551.5, \quad z_m^{(41)}(M^2) = 745.42$$

obtained by Kiyo, Mishima, Sumino (2015) using fitting procedure.

Evaluation of the inaccuracies of the results obtained

The uncertainty of least squares method can be found with using known error of $\sigma = \Delta y_{l_k} \equiv \Delta y_l = 21.5$ for each k=1, 2, 3:

$$\Delta z_m^{(40)} = \sqrt{\sum_{k=1}^{3}\left(\frac{\partial z_m^{(40)}}{\partial y_{l_k}}\Delta y_{l_k}\right)^2} = \frac{\sqrt{\sum\limits_{k=1}^{3} n_{l_k}^2}}{\sqrt{3\sum\limits_{k=1}^{3} n_{l_k}^2 - \left(\sum\limits_{k=1}^{3} n_{l_k}\right)^2}}\Delta y_l \,,$$

$$\Delta z_m^{(41)} = \sqrt{\sum_{k=1}^{3}\left(\frac{\partial z_m^{(41)}}{\partial y_{l_k}}\Delta y_{l_k}\right)^2} = \frac{\sqrt{3}\Delta y_l}{\sqrt{3\sum\limits_{k=1}^{3} n_{l_k}^2 - \left(\sum\limits_{k=1}^{3} n_{l_k}\right)^2}} \,.$$

$$z_m^{(40)}(M_q^2) = -3642.9 \pm 62.0, \quad z_m^{(41)}(M_q^2) = 757.05 \pm 15.20.$$

The fitted results of Kiyo, Mishimo, Sumino have error bars $\delta = \pm 21.5$.

Conclusion

Finally we obtain the following results

$$M_q \approx \overline{m}_q(M_q^2)(1 + 1.33333 a_s(M_q^2) + (-1.0414 n_l + 16.110) a_s^2(M_q^2) +$$
$$+(0.6527 n_l^2 - 29.701 n_l + 239.30) a_s^3(M_q^2) +$$
$$+(-0.6781 n_l^3 + 46.310 n_l^2 - (864.25 \pm 15.2) n_l + 4457.7 \pm 62.0) a_s^4(M_q^2))$$

$$M_q \approx \overline{m}_q(\overline{m}_q^2)(1 + 1.33333 a_s(\overline{m}_q^2) + (-1.0414 n_l + 13.443) a_s^2(\overline{m}_q^2) +$$
$$+(0.6527 n_l^2 - 26.655 n_l + 190.59) a_s^3(\overline{m}_q^2) +$$
$$+(-0.6781 n_l^3 + 43.396 n_l^2 - (745.83 \pm 15.2) n_l + 3556.4 \pm 62.0) a_s^4(\overline{m}_q^2))$$

The last formula should be compared with expression recently estimated by the fitting method (Y. Kiyo, G. Mishima and Y. Sumino in 2015) (with our understanding of errors)

$$M_q \approx \overline{m}_q(\overline{m}_q^2)(1 + 1.33333 a_s(\overline{m}_q^2) + (-1.0414 n_l + 13.443) a_s^2(\overline{m}_q^2) +$$
$$+(0.6527 n_l^2 - 26.655 n_l + 190.59) a_s^3(\overline{m}_q^2) +$$
$$+(-0.6781 n_l^3 + 43.396 n_l^2 - 745.85 n_l + 3556.5 \pm 21.5) a_s^4(\overline{m}_q^2)) \,.$$

Conclusion

n_l	expression for $\overline{m}_q$ mass normalized on M_q^2
3	$\overline{m}_c(M_c^2) \approx M_c(1 - 1.3333a_s(M_c^2) - 11.207a_s^2(M_c^2) - 123.81a_s^3(M_c^2) + (-1744.7 \pm 76.9)a_s^4(M_c^2))$
4	$\overline{m}_b(M_b^2) \approx M_b(1 - 1.3333a_s(M_b^2) - 10.166a_s^2(M_b^2) - 101.45a_s^3(M_b^2) + (-1267.0 \pm 86.8)a_s^4(M_b^2))$
5	$\overline{m}_t(M_t^2) \approx M_t(1 - 1.3333a_s(M_t^2) - 9.125a_s^2(M_t^2) - 80.40a_s^3(M_t^2) + (-859.9 \pm 98.0)a_s^4(M_t^2))$
	expression for $\overline{m}_q$ mass normalized on $\overline{m}_q^2$
3	$\overline{m}_c(\overline{m}_c^2) \approx M_c(1 - 1.3333a_s(\overline{m}_c^2) - 8.541a_s^2(\overline{m}_c^2) - 91.36a_s^3(\overline{m}_c^2) + (-1325.8 \pm 76.9)a_s^4(\overline{m}_c^2))$
4	$\overline{m}_b(\overline{m}_b^2) \approx M_b(1 - 1.3333a_s(\overline{m}_b^2) - 7.500a_s^2(\overline{m}_b^2) - 72.05a_s^3(\overline{m}_b^2) + (-932.4 \pm 86.8)a_s^4(\overline{m}_b^2))$
5	$\overline{m}_t(\overline{m}_t^2) \approx M_t(1 - 1.3333a_s(\overline{m}_t^2) - 6.459a_s^2(\overline{m}_t^2) - 54.04a_s^3(\overline{m}_t^2) + (-603.9 \pm 98.0)a_s^4(\overline{m}_t^2))$
	expression for M_q mass in terms of the $\overline{m}_q(M_q^2)$
3	$M_c \approx \overline{m}_c(M_c^2)(1 + 1.3333a_s(M_c^2) + 12.985a_s^2(M_c^2) + 156.07a_s^3(M_c^2) + (2263.4 \pm 76.9)a_s^4(M_c^2))$
4	$M_b \approx \overline{m}_b(M_b^2)(1 + 1.3333a_s(M_b^2) + 11.944a_s^2(M_b^2) + 130.93a_s^3(M_b^2) + (1698.2 \pm 86.8)a_s^4(M_b^2))$
5	$M_t \approx \overline{m}_t(M_t^2)(1 + 1.3333a_s(M_t^2) + 10.903a_s^2(M_t^2) + 107.11a_s^3(M_t^2) + (1209.4 \pm 98.0)a_s^4(M_t^2))$
	expression for M_q mass in terms of the $\overline{m}_q(\overline{m}_q^2)$
3	$M_c \approx \overline{m}_c(\overline{m}_c^2)(1 + 1.3333a_s(\overline{m}_c^2) + 10.318a_s^2(\overline{m}_c^2) + 116.49a_s^3(\overline{m}_c^2) + (1691.1 \pm 76.9)a_s^4(\overline{m}_c^2))$
4	$M_b \approx \overline{m}_b(\overline{m}_b^2)(1 + 1.3333a_s(\overline{m}_b^2) + 9.277a_s^2(\overline{m}_b^2) + 94.41a_s^3(\overline{m}_b^2) + (1224.0 \pm 86.8)a_s^4(\overline{m}_b^2))$
5	$M_t \approx \overline{m}_t(\overline{m}_t^2)(1 + 1.3333a_s(\overline{m}_t^2) + 8.236a_s^2(\overline{m}_t^2) + 73.63a_s^3(\overline{m}_t^2) + (827.3 \pm 98.0)a_s^4(\overline{m}_t^2))$

Conclusion

Sign-constant and significantly growing coefficients of the corresponding PT series clearly demonstrate their general asymptotic structure. Now we present the specific values for various masses of heavy quarks. For this objective we use following average values of running masses from PDG for the initial data: $\overline{m}_c(\overline{m}_c^2)$=1.275 GeV, $\overline{m}_b(\overline{m}_b^2)$=4.180 GeV, $\overline{m}_t(\overline{m}_t^2)$=163.643 GeV for charm, bottom and top quark masses respectively. In order to compute value $a_s(\overline{m}_q^2)$ at N^{3}LO order for c, b and t quarks we use the energy dependence of a_s up to $\mathcal{O}(a_s^5)$ level, defined through logarithmic terms L $= \ln(\overline{m}_q^2(\overline{m}_q^2)/\Lambda_{\overline{\mathrm{MS}}}^{(n_f)2})$ with parameters $\Lambda_{\overline{\mathrm{MS}}}^{(n_f)2}$, which depend on number of active flavours and order of approximations. For b-quark we take the average world value of $\Lambda^{(n_f=5)}_{\overline{\mathrm{MS}},\,\mathrm{N^3LO}} = 215\mathrm{MeV}$.

Conclusion

The values of $\Lambda^{(n_f=4)}_{\overline{\rm MS},\,\rm N^3LO}$ and $\Lambda^{(n_f=6)}_{\overline{\rm MS},\,\rm N^3LO}$ are obtained at the $\rm N^3LO$ using appropriate matching transformation conditions. The computed by us results are:

$$
\begin{aligned}
\Lambda^{(n_f=4)}_{\overline{\rm MS},\,\rm N^3LO} &= 297\rm MeV, \quad a_s^{\rm N^3LO}(\overline{m}_c^2) = 0.1271, \\
\Lambda^{(n_f=5)}_{\overline{\rm MS},\,\rm N^3LO} &= 215\rm MeV, \quad a_s^{\rm N^3LO}(\overline{m}_b^2) = 0.0723, \\
\Lambda^{(n_f=6)}_{\overline{\rm MS},\,\rm N^3LO} &= 91\rm MeV, \quad a_s^{\rm N^3LO}(\overline{m}_t^2) = 0.0346.
\end{aligned}
$$

Conclusion

Our final result read:

$$
\begin{aligned}
\frac{M_c}{1\text{GeV}} &\approx 1.275 + 0.216 + 0.213 + 0.305 + 0.563 \pm 0.026 \ , \\
\frac{M_b}{1\text{GeV}} &\approx 4.180 + 0.403 + 0.202 + 0.149 + 0.140 \pm 0.010 \ , \\
\frac{M_t}{1\text{GeV}} &\approx 163.643 + 7.549 + 1.613 + 0.499 + 0.194 \pm 0.023 \ .
\end{aligned}
$$

$$
\begin{aligned}
M_b &\approx (5.074 \pm 0.010) \text{ GeV} \ , \\
M_t &\approx (173.498 \pm 0.023) \text{ GeV} \ .
\end{aligned}
$$

Summary

- Estimates of pole heavy quark mass through running mass, normalized on the running mass, calculated by the least squares method are in agreement with estimates, performed by the fitting method
- The uncertainties were evaluated and there was shown that it does not exceed 2 % from coefficients $z_m^{(40)}$ and $z_m^{(41)}$
- There were obtained values of pole heavy quark masses
- It is very important to get analytical expressions for $z_m^{(40)}$ and $z_m^{(41)}$ (this project: already started- R.N. Lee and K.T. Mingulov, arXiV:1507.04256)

Thank you for your attention!