Mass of the bottom quark from Upsilon(1S) at NNNLO

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Introduction

The ($\overline{\rm MS}$) mass of the bottom (b) quark $\overline{m}_b \equiv \overline{m}_b(\overline{m}_b)$, is an important quantity in HEP, free of renormalon ambiguities, and appears in many physical observables. Since it is relatively high, ~ 4 GeV, perturbative QCD (pQCD) methods are suitable for its extraction. The mass of the ground state of the $b\bar{b}$ quarkonium, $\Upsilon(1S)$, $M_{\Upsilon(1S)} = 9.460$ GeV, is one of the best quantities for such an extraction

$$M^{(\rm th)}_{\Upsilon(1S)} = 2 m_b + E_{\Upsilon(1S)} = 9.460 \ {\rm GeV}, \qquad (1)$$

where m_b is the pole mass of the bottom quark ($u = 1/2$ renormalon ambiguity, $\sim \Lambda_{QCD}$), and $E_{\Upsilon(1S)}$ is the binding energy of Upsilon.

The idea is to use the available pQCD expansions of $2m_b/\overline{m}_b$ and of $E_{\Upsilon(1S)}/\overline{m}_b$ in powers of QCD coupling $a(\mu) \equiv \alpha_s(\mu)/\pi$ and thus extract the value of $\overline{m}_b$.
The (leading IR) renormalon ambiguity of $2m_b$ cancels out with that of $E_{\Upsilon(1S)}$ [A.Pineda, PhD Thesis (1998); A.H.Hoang et al., hep-ph/9804227,PRD(1999); M.Beneke, hep-ph/9804241, PLB(1998)].

Introduction

The roadmap of the talk:

1. The correct treatment of m_c mass effects in the perturbation expansion of $m_b/\overline{m}_b$.
2. Asymptotic expressions for the coefficients in the perturbation expansion of $m_b/\overline{m}_b$ and $V(r)$; extraction of the renormalon normalization N_m.
3. The construction of the (modified) renormalon-subtracted mass $m_{b,\mathrm{RS}(')}$, using N_m. Renormalon-free relation between $m_{b,\mathrm{RS}(')}$ and $\overline{m}_b$.
4. Renormalon-free perturbation expansion for $M^{(\mathrm{th})}_{\Upsilon(1S)}$ in terms of $m_{b,\mathrm{RS}(')}$. Extraction, from $M^{(\mathrm{th})}_{\Upsilon(1S)} = 9.460\ \mathrm{GeV}$, of the values of $m_{b,\mathrm{RS}(')}$ ($\Rightarrow \overline{m}_b$).

Charm mass effects in the bottom pole mass

The pole mass m_b and the $\overline{\rm MS}$ mass $\overline{m}_b \equiv \overline{m}_b(\overline{m}_b)$ are related

$$m_b = \overline{m}_b \left(1 + S(N_f)\right) + \delta m_c^{(+)} \ , \tag{2}$$

where

$$\begin{aligned} S(N_f) &= \frac{4}{3} a_+(\mu) \left[1 + r_1^{(+)}(\mu) a_+(\mu) + r_2^{(+)}(\mu) {a_+}^2(\mu) + r_3^{(+)}(\mu) {a_+}^3(\mu) \right. \\ &\left. + \mathcal{O}({a_+}^4) \right] \end{aligned} \tag{3}$$

and the coefficients are for QCD with $N_f = N_l + 1 = 4$ active flavors: $r_j^{(+)}(\mu) \equiv r_j(\mu; N_f)$.
$R_0 = 4/3$: [R.Tarrach, NPB(1981)]; r_1: [N.Gray et al., ZPC(1990)]; r_2: [K.G.Chetyrkin and M.Steinhauser, PRL83(1999); K.Melnikov and T.v.Ritbergen, PLB(2000)]; r_3: [P.Marquard et al., PRL(2015)].

Charm mass effects in the bottom pole mass

These coefficients are

$$\begin{aligned}
r_1(\mu; N_f) &= r_1(N_f) + \beta_0 L_m(\mu) \ , && \text{(4a)}\\
r_2(\mu; N_f) &= r_2(N_f) + \left(2\beta_0 L_m(\mu) r_1 + \beta_0^2 L_m^2(\mu)\right) + \beta_1 L_m(\mu) \ , && \text{(4b)}\\
r_3(\mu; N_f) &= r_3(N_f) + \left(3\beta_0 L_m(\mu) r_2 + 3\beta_0^2 L_m^2(\mu) r_1 + \beta_0^3 L_m^3(\mu)\right) \\
&\quad + \beta_1 \left(2 L_m(\mu) r_1 + \frac{5}{2}\beta_0 L_m^2(\mu)\right) + \beta_2 L_m(\mu) \ , && \text{(4c)}
\end{aligned}$$

where $L_m(\mu) = \ln(\mu^2/\overline{m}_b{}^2)$, and we maintain, for simplicity, the notation $r_j \equiv r_j(\overline{m}_b)$. Note that $\beta_0 = \beta_0(N_f) = (1/4)(11 - 2N_f/3)$ and $\beta_1 = c_1 \beta_0 = (102 - 38N_f/3)/16$ are the first two coefficients of the RGE of $a(\mu)$

$$\frac{d a(Q)}{d \ln Q^2} = -\beta_0 a^2(Q) \left(1 + c_1 a(Q) + c_2 a^2(Q) + c_3 a^3(Q) + \cdots \right) \ . \qquad (5)$$

Charm mass effects in the bottom pole mass

Finite-mass charm effects are incorporated in

$$\delta m_c{}^{(+)} = \delta m_{(c,+)}^{(1)} a_+^2(\overline{m}_b) + \delta m_{(c,+)}^{(2)} a_+^3(\overline{m}_b) + \mathcal{O}(a_+^4)\,, \qquad (6)$$

which vanishes in the $m_c \to 0$ limit.
We have

$$\begin{aligned} \delta m_{(c,+)}^{(1)} &= \frac{4}{3}\overline{m}_b \Delta[\overline{m}_c/\overline{m}_b] = 1.9058 \text{ MeV (Gray et al., 1990)}, \\ \delta m_{(c,+)}^{(2)} &= 48.6793 \text{ MeV (Bekavac et al., 2007)}, \end{aligned} \qquad (7)$$

implying

$$\delta m_{(c,+)}^{(1)} a_+{}^2(\overline{m}_b) = 9.3 \text{ MeV}, \qquad \delta m_{(c,+)}^{(2)} a_+{}^3(\overline{m}_b) = 18.1 \text{ MeV}. \qquad (8)$$

Badly divergent. Why? At loop order n, the natural scale of the loop integral for δm_c is $m_b e^{-n}$ [Ball et al., hep-ph/9502300, PLB(1995)], which for n large enough: $m_b e^{-n} < m_c$. Therefore, for n large c quark appears as very heavy (decoupled), leading to the effective number of flavors being $N_l = 3$ and not $N_f = N_l + 1 = 4$.

Charm mass effects in the bottom pole mass

Therefore, it is convenient to rewrite the relation between the pole and the $\overline{\mathrm{MS}}$ mass in terms of $a_-(\mu) = a(\mu, N_l)$ and $r_j^{(-)}(\mu) \equiv r_j(\mu; N_l)$ $[N_l = 3]$

$$m_b = \overline{m}_b \left(1 + S(N_l)\right) + \delta m_c \ , \tag{9}$$

where

$$S(N_l) = \frac{4}{3} a_-(\mu) \left[1 + r_1^{(-)}(\mu) a_-(\mu) + r_2^{(-)}(\mu) {a_-}^2(\mu) + r_3^{(-)}(\mu) {a_-}^3(\mu) + \mathcal{O}({a_-}^4)\right] \tag{10}$$

$[r_j^{(-)}(\overline{m}_b) = 7.74,\ 87.2,\ 1268.4 \pm 16.1,$ for $j = 1, 2, 3]$
and the effects of the decoupling of S are absorbed in the new δm_c

$$\delta m_c = \left[\delta m^{(1)}_{(c,+)} + \delta m^{(1)}_{(c,\mathrm{dec.})}\right] a_-^2(\overline{m}_b) + \left[\delta m^{(2)}_{(c,+)} + \delta m^{(2)}_{(c,dec.)}\right] a_-^3(\overline{m}_b) + \mathcal{O}({a_-}^4) \ , \tag{11}$$

Charm mass effects in the bottom pole mass

where $\delta m_{(c,\mathrm{dec.})}{}^{(j)}$ are generated by this decoupling and read

$$\delta m_{(c,\mathrm{dec.})}{}^{(1)} = \frac{2}{9} \overline{m}_b \left(\ln \left(\frac{\overline{m}_b{}^2}{\overline{m}_c{}^2} \right) - \frac{71}{32} - \frac{\pi^2}{4} \right) \qquad (12)$$

and $\delta m_{(c,\mathrm{dec.})}{}^{(2)}$ is a similar, but longer expression.
Numerical evaluation gives for $\left[\delta m_{(c,+)}{}^{(1)} + \delta m_{(c,\mathrm{dec.})}{}^{(1)} \right] a_-^2(\overline{m}_b) = -1.6$ MeV and $\left[\delta m_{(c,+)}{}^{(2)} + \delta m_{(c,\mathrm{dec.})}{}^{(2)} \right] a_-^3(\overline{m}_b) = -0.3$ MeV. This means that the previous divergent series (in $\mathrm{QCD}_{N_f=4}$)
$\delta m_c{}^{(+)} = (9.3 + 18.1 + \ldots)$ MeV [Eq. (8)] now tranforms (in $\mathrm{QCD}_{N_l=3}$) to

$$\delta m_c = (-1.6 - 0.3 + \ldots)\ \mathrm{MeV}. \qquad (13)$$

We observe that the series for δm_c in this $QCD_{N_l=3}$ formulation is now convergent, and strong cancellation takes place between $\delta m_{(c,+)}{}^{(j)}$ and $\delta m_{(c,\mathrm{dec.})}{}^{(j)}$, as expected.

Leading renormalon of the pole mass

Determining the pole mass from the $\Upsilon(1S)$ mass has large uncertainties due to the pole mass renormalon ambiguity $\delta m_b \sim \Lambda_{\rm QCD} \sim 0.1$-$1$ GeV [M.Beneke, hep-ph/9402364, PLB(1995)]. To avoid this problem we will work with the renormalon-subtracted (RS, renormalon-free) bottom mass $m_{b,\rm RS}$ instead [A.Pineda, hep-ph/0105008, JHEP(2001)]. Then, $\overline{m}_b$ will be obtained from its stable (renormalon-free) relation with the $m_{\rm RS}$ mass. The use of $m_{\rm RS}$ in the theoretical evaluation of the $\Upsilon(1S)$ mass is convenient because it has no (leading infrared) renormalon ambiguity, and the renormalon cancellation in the quarkonium mass $M_{\Upsilon(1S)} = 2m_b + E_{\Upsilon(1S)}$ is implemented automatically. We could, in principle, use $\overline{m}_b$ mass throughout (instead of $m_{b,\rm RS}$), but then the renormalon cancellation would be less explicit and slower (when number of loops n increases).

Leading renormalon of the pole mass

Overall, the asymptotic behaviour of r_N is determined by the leading IR renormalon $u = 1/2$:

$$\frac{4}{3} r_N^{\rm asym}(\mu) \simeq \pi N_m \frac{\mu}{\overline{m}_b} (2\beta_0)^N \frac{\Gamma(\nu + N + 1)}{\Gamma(\nu + 1)} \times \left[1 + \frac{\nu}{N+\nu} \widetilde{c}_1 + \frac{\nu(\nu-1)}{(N+\nu)(N+\nu-1)} \widetilde{c}_2 + \frac{\nu(\nu-1)(\nu-2)}{(N+\nu)(N+\nu-1)(N+\nu-2)} \widetilde{c}_3 + \mathcal{O}\left(\frac{1}{N^4} \right) \right] . \qquad (14)$$

$$\frac{4}{3} r_N(\mu) = \pi N_m \frac{\mu}{\overline{m}_b} (2\beta_0)^N \sum_{s \geq 0} \widetilde{c}_s \frac{\Gamma(\nu + N + 1 - s)}{\Gamma(\nu + 1 - s)} + h_N(\mu) \ , \qquad (15)$$

where h_N is dominated by the subleading UV renormalon $u = -1$: $h_N / r_N \sim (\overline{m}_b^3 / \mu^3)(-1)^N / 2^N$.

Determination of the renormalon normalization N_m

We can now determine the "strength" N_m of the leading ($u = 1/2$) renormalon by equating the derived asymptotic $r_N^{\rm asym}(\mu)$ with the exact $r_N(\mu)$ ($N = 0, 1, 2$): $r_N^{\rm asym}(\mu) \approx r_N(\mu) \Rightarrow$

$$
\begin{aligned}
N_m \quad \approx \quad & r_N(\mu) \Bigg/ \left\{ \pi \frac{\mu}{\overline{m}_b} (2\beta_0)^N \frac{\Gamma(\nu + N + 1)}{\Gamma(\nu + 1)} \right. \\
& \times \left[1 + \frac{\nu}{N+\nu} \widetilde{c}_1 + \frac{\nu(\nu-1)}{(N+\nu)(N+\nu-1)} \widetilde{c}_2 \right. \\
& \left. \left. + \frac{\nu(\nu-1)(\nu-2)}{(N+\nu)(N+\nu-1)(N+\nu-2)} \widetilde{c}_3 \right] \right\}. \qquad (16)
\end{aligned}
$$

The result for N_m should be the best for the highest available N ($N = 3$), and should also have there reduced spurious μ-dependence.

Determination of the renormalon normalization N_m

In practice, we determined N_m from the asymptotic behavior of the coefficients $v_N(\mu)$ of the static singlet potential $V(r)$ (with $N_l = 3$)

$$V(r) = -\frac{4\pi}{3}\frac{1}{r} a_-(\mu) \left[1 + v_1(\mu) a_-(\mu) + v_2 a_-(\mu)^2 + v_3 a_-(\mu)^3 + \ldots\right] \quad (17)$$

where v_N are known up to $\mathrm{N^3LO}$ (v_3) at present: v_1: Fischler (NPB, 1977); v_2: Y.Schröder (PLB, 1999); v_3: A.V.Smirnov et al. (PLB, 2008); C.Anzai et al. (PRL, 2010); A.V.Smirnov et al. (PRL, 2010)]; cf. also β_3 in V-scheme: A.L.Kataev and V.S.Molokoedov (arXiv:1506.03547, 2015).

We know that in the sum $2m_b + V(r)$ the leading ($u = 1/2$) renormalon gets cancelled. The asymptotic behavior of v_N coefficients can be determined in complete analogy with those of r_N

$$-\frac{4}{3} v_N(\mu) = N_V \mu r (2\beta_0)^N \sum_{s \geq 0} \widetilde{c}_s \frac{\Gamma(\nu + N + 1 - s)}{\Gamma(\nu + 1 - s)} + d_N(\mu) \ . \quad (18)$$

and taking $d_N = 0$ gives

Determination of the renormalon normalization N_m

$$-\frac{4}{3}v_N^{\rm asym}(\mu) \simeq N_V \mu r (2\beta_0)^N \frac{\Gamma(\nu+N+1)}{\Gamma(\nu+1)}$$
$$\times \left[1 + \frac{\nu}{N+\nu}\widetilde{c}_1 + \frac{\nu(\nu-1)}{(N+\nu)(N+\nu-1)}\widetilde{c}_2 \right.$$
$$\left. + \frac{\nu(\nu-1)(\nu-2)}{(N+\nu)(N+\nu-1)(N+\nu-2)}\widetilde{c}_3 + \mathcal{O}\left(\frac{1}{N^4}\right) \right]. \qquad (19)$$

The strengh of the renormalon N_V is then related with N_m by the renormalon cancellation of the sum $2m_b + V(r)$:

$$2N_m + N_V = 0 \ \Rightarrow N_m = -N_V/2. \qquad (20)$$

Determining N_m via N_V gives us a more precise value

$$N_m = -\frac{1}{2}N_V = 0.563(26) \ (N_l = 3). \qquad (21)$$

This is presented in Fig. 1.

Determination of the renormalon normalization N_m

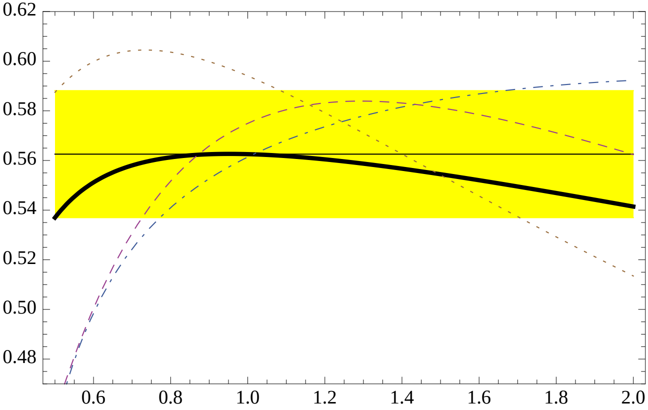

Figure: *Fig. 3: N_m for $N_l = 3$, obtained from $v_N^{\rm asym}/v_N$ ($N = 3$, solid thick curve) and from $r_N^{\rm asym}/r_N$ ($N = 2$, dotted curve). For comparison, the results of the alternative method using the static potential (dashed-dotted curve) and from the pole mass (dashed) are included. All are as functions of the renormalization parameter x, where $x \equiv \mu r$ when using potential, and $x \equiv \mu/\overline{m}_b$ when using the pole mass. The horizontal central line and bands correspond to our final central value and error.*

Renormalon-subtracted (RS) mass of bottom

The RS mass is defined by subtracting the leading renormalon singularity ($u = 1/2$) to the pole mass [A.Pineda, hep-ph/0105008, JHEP(2001)]. Hence

$$m_{b,\mathrm{RS}}(\nu_f) = (m_b - \delta m_c) - N_m \pi \nu_f \sum_{N=0}^{\infty} a_-^{N+1}(\nu_f)(2\beta_0)^N$$
$$\times \left[1 + \frac{\nu}{N+\nu}\widetilde{c}_1 + \frac{\nu(\nu-1)}{(N+\nu)(N+\nu-1)}\widetilde{c}_2 + \frac{\nu(\nu-1)(\nu-2)}{(N+\nu)(N+\nu-1)(N+\nu-2)}\widetilde{c}_3 + \mathcal{O}\left(\frac{1}{N^4}\right) \right] . \qquad (22)$$

Equation (22) is still formal. In practice, one rewrites m in terms of $\overline{m}$ using Eqs. (9)-(10)

$$m_b - \delta m_c = \overline{m}_b(1 + (4/3)a_-(\nu_f) + \ldots), \qquad (23)$$

Renormalon-subtracted (RS) mass of bottom

and reexpands the perturbation series in Eq. (22) around the same coupling $a_-(\mu)$, at fixed but otherwise arbitrary scale μ:

$$m_{b,\mathrm{RS}}(\nu_f) = \overline{m}_b \left[1 + \sum_{N=0}^{\infty} h_N(\nu_f) a_-^{N+1}(\nu_f) \right]$$

$$\Rightarrow \quad m_{b,\mathrm{RS}}(\nu_f) = \overline{m}_b \left[1 + \sum_{N=0}^{\infty} \widetilde{h}_N(\nu_f;\mu) a_-^{N+1}(\mu) \right] , \qquad (24)$$

where $h_N(\nu_f)$ is determined from Eq. (15) (with $\mu = \nu_f$ and with the sum truncated at $\widetilde{c}_3$) for $N = 0, 1, 2$. For $N \geq 4$ we take $h_N(\overline{m}_b) = 0$. The coefficients $\widetilde{h}_N(\nu_f;\mu)$ in Eq. (24) are obtained by expanding $a_-(\nu_f)$ in the expansion (16) in powers of $a_-(\mu)$. Note that $m_{b,\mathrm{RS}}(\nu_f)$ does not depend on μ (it will, but only marginally, when we truncate the infinite sum in Eq. (**??**)). On the other hand the coefficients h_N are functions of ν_f, μ, and $\overline{m}_b$, and are much smaller than $r_N(\mu)$.

Modified renormalon-subtracted (RS') mass of bottom

A variation of the RS mass is the modified renormalon-subtracted (RS') mass $m_{b,\mathrm{RS}'}$, where subtractions start at $\sim a^2$ [i.e., $N=1$ in Eq. (22)] [A.Pineda, hep-ph/0105008, JHEP(2001)

$$m_{b,\mathrm{RS}'}(\nu_f) = (m_b - \delta m_c) - N_m \pi \nu_f \sum_{N=1}^{\infty} a_{-}^{N+1}(\nu_f)(2\beta_0)^N$$
$$\times \left[1 + \frac{\nu}{N+\nu}\widetilde{c}_1 + \frac{\nu(\nu-1)}{(N+\nu)(N+\nu-1)}\widetilde{c}_2 + \frac{\nu(\nu-1)(\nu-2)}{(N+\nu)(N+\nu-1)(N+\nu-2)}\widetilde{c}_3 + \mathcal{O}\left(\frac{1}{N^4}\right) \right] . \qquad (25)$$

Modified renormalon-subtracted (RS') mass of bottom

This gives a relation analogous to Eq. (24)

$$m_{b,\mathrm{RS}'}(\nu_f) = \overline{m}_b \left[1 + \sum_{N=1}^{\infty} h_N(\nu_f) a_-^{N+1}(\nu_f) \right]$$

$$\Rightarrow \quad m_{b,\mathrm{RS}'}(\nu_f) = \overline{m}_b \left[1 + \sum_{N=1}^{\infty} \widetilde{h}'_N(\nu_f;\mu) a_-^{N+1}(\mu) \right] , \qquad (26)$$

where $\widetilde{h}'_N(\nu_f;\mu)$ in Eq. (26) are obtained by expanding $a_-(\nu_f)$ in Eq. (18) in powers of $a_-(\mu)$.

We will take $\mu \sim \nu_f \sim m_b \alpha_s$ ($\sim \mu_{\mathrm{soft}}$), see later.

Bottom mass from heavy quarkonium

We note that $\Upsilon(1S)$ mass is: $M_{\Upsilon(1S)}^{(th)} = 2m_b + \langle \Upsilon(1S) | V(r) | \Upsilon(1S) \rangle$, and that $\langle \Upsilon(1S) | (1/r) | \Upsilon(1S) \rangle \sim m_b \alpha_s \sim \mu_{\rm soft}$. The perturbation expansion of $M_{\Upsilon(1S)}^{(th)}$ is presently known up to $\mathcal{O}(m_b a^5)$ [A.A.Penin, M.Steinhauser, PLB (2002)]:

$$M_{\Upsilon(1S)}^{(th)} = 2m_b - \frac{4\pi^2}{9} m_b a_-{}^2(\mu) \Bigg\{ 1 + a_-(\mu) \left[K_{1,0} + K_{1,1} L_p(\mu) \right] + a_-{}^2(\mu) \sum_{j=0}^{2} K_{2,j} L_p(\mu)^j + a_-{}^3(\mu) \Big[K_{3,0,0} + K_{3,0,1} \ln a_-(\mu) + \sum_{j=1}^{3} K_{3,j} L_p(\mu)^j \Big] + \mathcal{O}(a_-{}^4) \Bigg\} , \qquad (27)$$

μ is the renormalization scale, and

$$L_p(\mu) = \ln \left(\frac{\mu}{(4\pi/3) m_b a_-(\mu)} \right) , \qquad (28)$$

$K_{i,j}(N_f)$ and $K_{3,0,j}$: at the end for reference.

We rewrite m_b in terms of $m_{b,\mathrm{RS}}$ to implement the $u = 1/2$ renormalon cancellation. This gives

$$\frac{M^{(th)}_{\Upsilon(1S)}}{m_{b,\mathrm{RS}}(\nu_f)} = 2 + \left[2\pi N_m b a \mathcal{K}_0 - \frac{4\pi^2}{9} a^2 \right] + \left[2\pi N_m b a^2 \left(\mathcal{K}_1 + z_1 \mathcal{K}_0 \right) - \frac{4\pi^2}{9} a^3 \left(K_{1,0} + K_{1,1} L_{\mathrm{RS}} \right) \right] + \Big[2\pi N_m b a^3 \left(\mathcal{K}_2 + 2 z_1 \mathcal{K}_1 + z_2 \mathcal{K}_0 \right) - \frac{4\pi^2}{9} \left(a^4 \sum_{j=0}^{2} K_{2,j} L^j_{\mathrm{RS}} + b a^3 \pi N_m \mathcal{K}_0 \right) \Big] + \mathcal{O}(b a^4, a^5). \qquad (29)$$

Bottom mass from heavy quarkonium

$$\mathcal{O}(ba^4, a^5) = \Bigg[2\pi N_m b a^4 \left(\mathcal{K}_3 + 3z_1\mathcal{K}_2 + (2z_2 + z_1^2)\mathcal{K}_1 + z_3\mathcal{K}_0 \right)$$
$$- \frac{4\pi^2}{9} \Bigg[a^5 \left(K_{3,0,0} + K_{3,0,1} \ln a + \sum_{j=1}^{3} K_{3,j} L_{\rm RS}^j \right)$$
$$+ b a^4 \pi N_m \left(K_{1,0}\mathcal{K}_0 + (L_{\rm RS} - 1) K_{1,1}\mathcal{K}_0 + \mathcal{K}_1 + z_1\mathcal{K}_0 \right) \Bigg] \Bigg] , \quad (30)$$

where we denoted

$$a \equiv a_{-}(\mu) = a(\mu, N_f = 3) \ , \tag{31a}$$

$$b \equiv b(\nu_f) = \frac{\nu_f}{m_{b,\mathrm{RS}}(\nu_f)} \ , \quad N_m = N_m(N_l = 3) \ , \tag{31b}$$

$$L_{\mathrm{RS}} \equiv L_{\mathrm{RS}}(\mu) = \ln\left(\frac{\mu}{(4\pi/3) m_{b,\mathrm{RS}}(\nu_f) a_{-}(\mu)}\right) \ , \tag{31c}$$

$$\mathcal{K}_N = (2\beta_0)^N \left[1 + \sum_{s=1}^{3} \widetilde{c}_s \frac{\Gamma(\nu + N + 1 - s)}{\Gamma(\nu + 1 - s)}\right] \ . \tag{31d}$$

Bottom mass from heavy quarkonium

In the expression (29)-(30) for $M_{\Upsilon(1S)}$, the terms of the same order $(\nu_f/\overline{m}_b)a^n$ and a^{n+1} were combined in common brackets [. . .], in order to account for the renormalon cancellation.
If using the RS' mass in our approach instead, the above expressions are valid without changes, except that $m_{b,\mathrm{RS}} \mapsto m_{b,\mathrm{RS}'}$ and $\mathcal{K}_0 \mapsto 0$ (and: $h_0(\mu) \mapsto 4/3$).
We note that we take $N_l = 3$ active flavours. The charm quark mass effects in the binding energy $\langle \Upsilon | V(r) | \Upsilon \rangle$ are negligible [N.Brambilla et al., hep-ph/0108084, PRD(2002)].

Bottom mass from heavy quarkonium

We extract the bottom masses from the condition $M^{(th)}_{\Upsilon(1S)} = M^{(exp)}_{\Upsilon(1S)} (= 9.460 \ \mathrm{GeV})$. The error estimates are made assuming $\mu = 2.5^{+1.5}_{-0.7}$ GeV, $\nu_f = 2 \pm 1$ GeV, $\alpha_s(M_z) = 0.1185(6)$ (and decoupling at $\overline{m}_b = 4.2$ GeV and $\overline{m}_c = 1.27$), $N_m = 0.563(26)$, and $(4/3) r_3(\overline{m}_b) = 1691.2 \pm 21.5$.
In RS approach we extract, in MeV:

$$m_{b,\mathrm{RS}}(2\mathrm{GeV}) = 4\,437^{-12}_{+24}(\mu)^{-3}_{+5}(\nu_f)^{-2}_{+2}(\alpha_s)^{-41}_{+41}(N_m)^{-0}_{+0}(r_3); \quad (32a)$$

$$\overline{m}_b(\overline{m}_b) = 4\,216^{-15}_{+39}(\mu)^{-3}_{+5}(\nu_f)^{-4}_{+4}(\alpha_s)^{-11}_{+12}(N_m)^{-4}_{+4}(r_3). \quad (32b)$$

In RS' we extract:

$$m_{b,\mathrm{RS'}}(2 \ \mathrm{GeV}) = 4\,761^{-16}_{+29}(\mu)^{-3}_{+5}(\nu_f)^{+3}_{-3}(\alpha_s)^{-26}_{+26}(N_m)^{-0}_{+0}(r_3); \quad (33a)$$

$$\overline{m}_b(\overline{m}_b) = 4\,221^{-14}_{+48}(\mu)^{-2}_{+4}(\nu_f)^{-4}_{+4}(\alpha_s)^{-1}_{+1}(N_m)^{-4}_{+4}(r_3). \quad (33b)$$

Bottom mass from heavy quarkonium

The renormalon cancellations are reflected numerically in Eqs. (29)-(30) [we take $\mu = 2.5$ GeV]:

$$\mathrm{RS} : M_{\Upsilon(1S)} = (8874 + 432 + 167 + 18 - 31)\ \mathrm{MeV}\,, \quad (34a)$$
$$\mathrm{RS}' : M_{\Upsilon(1S)} = (9522 - 151 + 112 + 8 - 31)\ \mathrm{MeV}\,, \quad (34b)$$

We see that the convergence is good; except for the last (NNNLO) term $\mathcal{O}(a^5, ba^4)$, where the ultrasoft scale $\mu_{\rm us} \sim m_b {\alpha_s}^2$ should be used in part of that term (instead of $\mu = \mu_{\rm soft} \sim m_b \alpha_s$). It is not known at present how to implement this quantitatively.
The relations between RS (RS') mass and $\overline{\mathrm{MS}}$ mass are reasonably convergent:

$$m_{b,\mathrm{RS}}(2\ \mathrm{GeV}) = (4216 + 192 + 36 + 12 - 18)\ \mathrm{MeV}\,, \quad (35a)$$
$$m_{b,\mathrm{RS}'}(2\ \mathrm{GeV}) = (4221 + 479 + 60 + 18 - 17)\ \mathrm{MeV}\,. \quad (35b)$$

Bottom mass from heavy quarkonium

In these extractions, we assumed the relations (22) and (25) between $m_{b,\mathrm{RS}(')}$ and m_b with $\delta m_c = 0$. However, as seen early on, $\delta m_c \approx -1.6 - 0.3$ MeV, i.e., $\delta m_c \approx -2$ MeV. Therefore

$$\begin{aligned} m_{b,\mathrm{RS}(')}(\mathrm{true}) &= m_{b,\mathrm{RS}(')} - \delta m_c \approx m_{b,\mathrm{RS}(')} + 2\ \mathrm{MeV}, \\ \Rightarrow \overline{m}_b(\mathrm{true}) &\approx \overline{m}_b + 2\ \mathrm{MeV}. \end{aligned} \tag{36}$$

Therefore, our final result is (the average of the RS and RS' extractions)

$$\overline{m}_b \equiv \overline{m}_b(\overline{m}_b) = 4.220^{+0.045}_{-0.017}\ \mathrm{GeV}\ . \tag{37}$$

Conclusions

1. We presented strong indications that the charm quark decouples in the relation between m_b and $\overline{m}_b$ ($\Rightarrow N_l = 3$). The charm quark had been known to decouple in the $b\bar{b}$ binding energy $E_{\Upsilon(1S)}$ [N.Brambilla et al, PRD(2002)].
2. An improved determination of the normalization of the leading bottom pole mass (and static potential with $N_l = 3$) was performed: $N_m = 0.563(26)$. This allowed us to obtain a good estimate of the term $\sim a^4$ in the relation between m_b and $\overline{m}_b$.
3. Use of the 3-loop ($\sim a^5 \overline{m}_b$) corection to the $\Upsilon(1S)$ binding energy allowed us to perform extraction of $m_{b,\mathrm{RS}^{(\prime)}}$ and $\overline{m}_b$ to NNNLO.

$$\overline{m}_b \equiv \overline{m}_b(\overline{m}_b) = 4.220^{+0.045}_{-0.017}\ \mathrm{GeV}\ . \tag{38}$$

Expression for $\Delta[r]$

$$\delta m^{(1)}_{(c,+)} = \frac{4}{3}\overline{m}_b \Delta[\overline{m}_c/\overline{m}_b], \tag{39}$$

where

$$\begin{aligned}\Delta[r] \quad = \quad & \frac{1}{4}\Big[\ln^2 r + \frac{\pi^2}{6} - \left(\ln r + \frac{3}{2}\right) r^2 \\ & +(1+r)(1+r^3)\left(\mathrm{Li}_2(-r) - \frac{1}{2}\ln^2 r + \log r \log(1+r) + \frac{\pi^2}{6}\right) \\ & +(1-r)(1-r^3)\left(\mathrm{Li}_2(r) - \frac{1}{2}\ln^2 r + \ln r \ln(1-r) - \frac{\pi^2}{3}\right)\Big]\end{aligned} \tag{40}$$

Expression for $\delta m_{c,\mathrm{dec}}^{(2)}$

$$\delta m_{(c,\mathrm{dec})}^{(2)} = \overline{m}_b \Big[-\frac{2293}{243} - \frac{809}{648}\pi^2 + \frac{61}{1944}\pi^4 - \frac{11}{81}\pi^2 \ln(2) + \frac{2}{81}\pi^2 \ln^2(2)$$
$$+\frac{\ln^4(2)}{81} + \frac{3107}{864}\ln\left(\frac{\overline{m}_b{}^2}{\overline{m}_c{}^2}\right) + \frac{1}{27}\pi^2 \ln\left(\frac{\overline{m}_b{}^2}{\overline{m}_c{}^2}\right)$$
$$+\frac{1}{27}\pi^2 \ln(2) \ln\left(\frac{\overline{m}_b{}^2}{\overline{m}_c{}^2}\right) + \frac{1}{27}\ln^2\left(\frac{\overline{m}_b{}^2}{\overline{m}_c{}^2}\right) + \frac{8}{27}\mathrm{Li}_4\left(\frac{1}{2}\right)$$
$$-\frac{527}{216}\zeta(3) - \frac{1}{18}\zeta(3)\ln\left(\frac{\overline{m}_b{}^2}{\overline{m}_c{}^2}\right) \Big]$$
$$+\frac{1}{3}\ln\left(\frac{\overline{m}_b{}^2}{\overline{m}_c{}^2}\right) \delta m_{(c,+)}{}^{(1)} . \qquad (41)$$

Relations between different running couplings

The relation between $a_+(\nu_f)$ and $a \equiv a_-(\mu)$

$$a_+(\nu_f) = a\left[1 + z_1 a + z_2 a^2 + z_3 a^3 + \mathcal{O}(a^4)\right] , \qquad (42)$$

where the coefficients z_1 account for the $N_f = 4 \mapsto 3$ quark threshold effects and the (subsequent) renormalization group running from ν_f to μ. The threshold effects are taken at the three-loop level according [K.G.Chetyrkin et al., hep-ph/9706430, PRL (1997)] and the renormalization group running at the four-loop level. The resulting coefficients z_j are:

$$\begin{aligned} z_1 &= x_1 + y_1 , \qquad z_2 = x_2 + 2x_1 y_1 + y_2 , \\ z_3 &= x_3 + 3x_2 y_1 + x_1 y_1^2 + 2x_1 y_2 + y_3 . \end{aligned} \qquad (43)$$

Relations between different running couplings

Here, x_j reflect the three-loop quark threshold matching for $N_f = 4 \mapsto 3$ at the chosen threshold scale ν_f,

$$x_1 = -k_1 \ , \quad x_2 = -k_2 + 2k_1^2 \ , \quad x_3 = -k_3 + 5k_1 k_2 - 5k_1^3 \ , \qquad (44)$$

where the expressions for k_j ($j = 1, 2, 3$) are given in [K.G.Chetyrkin et al., hep-ph/9706430, PRL (1997)] ($k_1 = -\ell_h/6$, etc.), with the logarithm there being $\ell_h = \ln(\nu_f^2/{\overline{m}_c}^2)$ and $N_\ell = 3$ (cf. also App.D of [C.Ayala and G.Cvetič,1210.6117 (PRD 2013)]). The coefficients y_j come from the (subsequent) RGE running from ν_f to μ (with $N_f = 3$)

$$y_1 = \beta_0 \ln\left(\frac{\mu^2}{\nu_f^2}\right) \ , \quad y_2 = y_1^2 + c_1 y_1 \ , \quad y_3 = y_1^3 + \frac{5}{2} c_1 y_1^2 + c_2 y_1 \ . \qquad (45)$$

Here, $c_j \equiv \beta_j/\beta_0$.

Coefficients of the binding energy of quarkonium

[W.Fischler(NPB 1977); A.Billoire(PLB 1980); Y.Schröder (PLB 1999); A.Pineda and F.J.Yndurain (PRD 1998); N.Brambilla e tal. (PLB 1999); A.A.Penin et al. (NPB 2002); A.A.Penin and M.Steinhauser (PLB 2002); A.V.Smirnov et al. (PLB 2008); C.Anzai et al. (PRL 2010); A.V.Smirnov et al. (PRL 2010)]
$(\beta_0 = (1/4)(11 - 2N_L/3)$; $\beta_1 = (1/16)(102 - 38N_l/3))$.

$$K_{1,0}(N_l) = \frac{1}{18}(291 - 22N_l) = 16.1667 - 1.22222N_l \ , \quad K_{1,1}(N_l) = 4\beta_0 ; \tag{46}$$

$$\begin{aligned} K_{2,0}(N_l) &= 337.947 - 40.9649N_l + 1.16286N_l^2 \ , \\ K_{2,1}(N_l) &= 231.75 - 32.1667N_l + N_l^2 \ , \\ K_{2,2}(N_l) &= 12\beta_0^2 \ ; \end{aligned} \tag{47a}$$

Coefficients of the binding energy of quarkonium

$$
\begin{aligned}
K_{3,0,0}(N_l) &= 8041.49 - 1318.36 N_l + 75.263 N_l^2 - 1.25761 N_l^3 \ , \\
K_{3,0,1}(N_l) &= \frac{865\pi^2}{18} = 474.289 \ , \qquad (48a)
\end{aligned}
$$

$$
\begin{aligned}
K_{3,1}(N_l) &= 6727.62 - 1212.76 N_l + 69.1066 N_l^2 - 1.21714 N_l^3 \ , \\
K_{3,2}(N_l) &= 2260.5 - 456.458 N_l + 28.5278 N_l^2 - 0.555556 N_l^3 \ , \\
K_{3,3}(N_l) &= 32\beta_0^3 \ . \qquad (49a)
\end{aligned}
$$