

Calculating four-loop massless propagators with Forcer

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Abstract. We present **Forcer**, a new FORM program for the calculation of four-loop massless propagators. The basic framework is similar to that of the **Mincer** program for three-loop massless propagators: the program reduces Feynman integrals to a set of master integrals in a parametric way. To overcome an ineludible complexity of the program structure at the four-loop level, most of the code was automatically generated or made with computer-assisted derivations. Correctness of the program has been checked with the recomputation of some quantities in the literature.

1. Introduction

Calculating massless propagator-type Feynman integrals is one of the basic building blocks in higher-order perturbative calculations in QCD (see Ref. [1] for a recent review). Up to the three-loop level, it turned out that dimensionally regularized [2, 3] massless propagator-type Feynman integrals can be reduced by the so-called triangle rule [4, 5, 6] obtained from integration-by-parts (IBP) identities [4, 5] and successive one-loop integrations with G -functions [7, 5] except for two special topologies, which are again reduced by manually solving IBP identities leading to two master integrals [5]. Together with the results of the master integrals [7], this led to a program **Mincer** [8], implemented in **SCHOONSCHIP** [9] and later reprogrammed [10] in **FORM** [11]. **Mincer** can analytically evaluate arbitrary scalar massless propagator-type Feynman integrals up to the three-loop level as Laurent series expansions in ϵ , where $D = 4 - 2\epsilon$ is the number of space-time dimensions.

On the other hand, at the four-loop level, people have used more systematic or generic ways to perform the IBP reductions: Laporta's algorithm [12] (for public implementations see [13, 14, 15, 16, 17, 18]), Baikov's method [19, 20] and heuristic approaches [21, 22]. The values of the master integrals at the four-loop level are known up to enough orders for practical applications [23, 24]. Then one might think that the **Mincer** program could be extended to the four-loop level, giving specialized (and hopefully optimized) routines for massless propagator-type integrals, and this would be more efficient than general-purpose IBP solvers. This approach has not been pursued, however, to the best of our knowledge. A reason may be that there are too many topologies at the four-loop level, and therefore it is impractical to program treatment of all topologies by hand as it was done for **Mincer**.

The aim of this work is to develop a new FORM program **Forcer** [25], which can be considered as an extension of **Mincer** to the four-loop level. This requires automatized code generation because

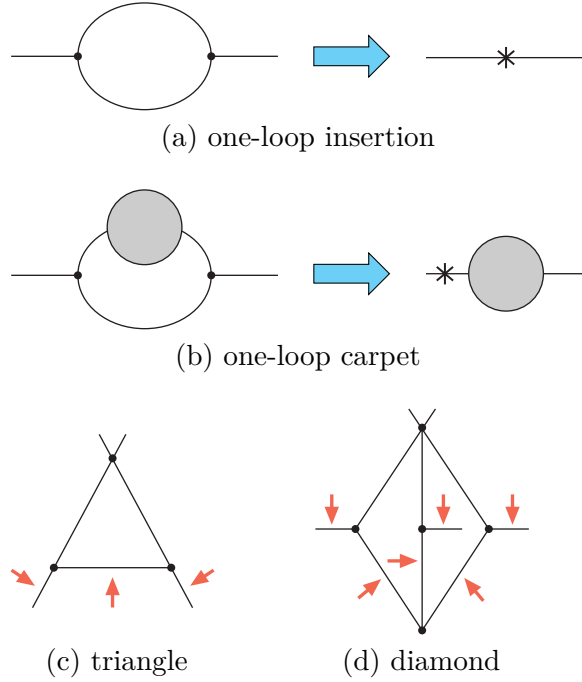


Figure 1. Substructures of topologies that lead to reduction to simpler topologies. (a) One-loop insertion. A line with a non-integer power is indicated with “*”. (b) One-loop carpet. (c) Triangle. The triangle rule removes one of the three lines indicated by arrows. (d) Two-loop diamond. The diamond rule removes one of the six lines indicated by arrows.

the large number of topologies makes the program structure too complicated to program by hand. Such a program is promising when one considers extremely time-consuming computations with existing approaches, for example, higher moments of four-loop splitting functions.

2. Reduction for each topology

Here we briefly summarize how one can reduce integrals belonging to a topology into those in simpler topologies from a diagrammatical point of view. Techniques of 2.1, 2.2 and 2.3 were used in the three-loop *Mincer* program. The technique of 2.4 is new at the four-loop level.

2.1. One-loop insertion integral

When a topology contains a massless one-loop insertion graph as its substructure and both propagators are massless, one can integrate out the corresponding loop momentum and it becomes a line but gets a non-integer power $1/(Q^2)^\epsilon$ in the resultant momentum Q (figure 1a). The powers of the two propagators can be arbitrary numbers. It is also allowed to have polynomial tensorial numerators with respect to the loop momentum.

2.2. One-loop carpet integral

This is another type of massless one-loop integrals that we can perform. When a subgraph is inserted to a one-loop graph, the outer loop can be integrated out first (figure 1b). Similar to the one-loop insertion integrals, the powers of the two propagators of the outer loop can be arbitrary numbers. Numerators are also allowed.

2.3. Triangle rule

When a topology contains a one-loop triangle graph depicted in figure 1c as a subgraph, recursive usage of an IBP identity leads to a sum of integrals corresponding to simpler topologies with one of the three lines indicated by arrows removed. This is possible when the powers of the three propagators are integer; hence this rule cannot be applied if any of the three lines has taken non-integer powers from one-loop integrals explained above. Irreducible numerators in the topology can always be chosen such that they do not interfere with the rule.

2.4. Diamond rule

The triangle rule can be extended for multi-loop diamond-shaped subgraphs [26]. Figure 1d shows the simplest diamond structure, in which one of six lines is removed in the end of the recursion.

2.5. Manual rules

If none of the above substructures is available in a topology, one needs to manually solve IBP identities such that recursive use of rules removes one of the propagators, or at least simplify integrals as much as possible. In the latter case, irreducible integrals are considered as master integrals of the problem. This part has not been fully automatized yet, though we used computer-assisted derivations of reduction rules.

Starting from a set of IBP identities obtained in a normal way, one can generate sets of identities by raising or lowering one of (or combinations of) powers of propagators and irreducible numerators in all possible ways. They are combined into a new set of identities, and then complicated integrals are eliminated by means of Gaussian elimination. Powers of the propagators remain parametric. We construct a reduction scheme out of the resulted identities.

3. Code generation for all topologies

In order to handle many topologies and transitions among them at the four-loop level, we need to automatize code generation for them.

First, we represent a topology as an undirected graph in graph theory. This makes it easy to detect the substructures explained in the previous section by pattern matchings of connections of vertices and edges. We implemented such routines in Python with a graph library `igraph` [27].

Then, starting from the top-level topologies, we consider to remove a line from each topology in all possible ways. If removing a line gives a massless tadpole, then the generated topology is immediately discarded because it is zero. Some of the generated topologies are actually identical and such graph isomorphisms are efficiently detected by the graph library. We keep track of all mappings of momenta between topologies before and after the transition. They are needed to rewrite propagators and irreducible numerators in a topology into those in another topology.

For each topology, the next action is determined from its substructures. Irreducible numerators are chosen such that they do not interfere with the next action. An adequate subroutine must be called for the next action in generated code. Symmetries in each topology are detected as graph automorphisms, which are helpful to reduce the number of terms we need to process in actual calculations.

Repeating this procedure until all topologies are reduced into a trivial Born graph, we obtain a tree of all possible topologies (figure 2). From 11 top-level topologies at the four-loop level, we obtained 437 non-trivial topologies in total. The numbers of topologies that have one-loop insertions, one-loop carpets, triangles and diamonds as their substructures are 335, 24, 53 and 4, respectively, and 21 topologies need construction of manual rules. From the topology tree, we generated FORM code for reductions in all topologies, and rewriting propagators and irreducible numerators at all topology transitions.

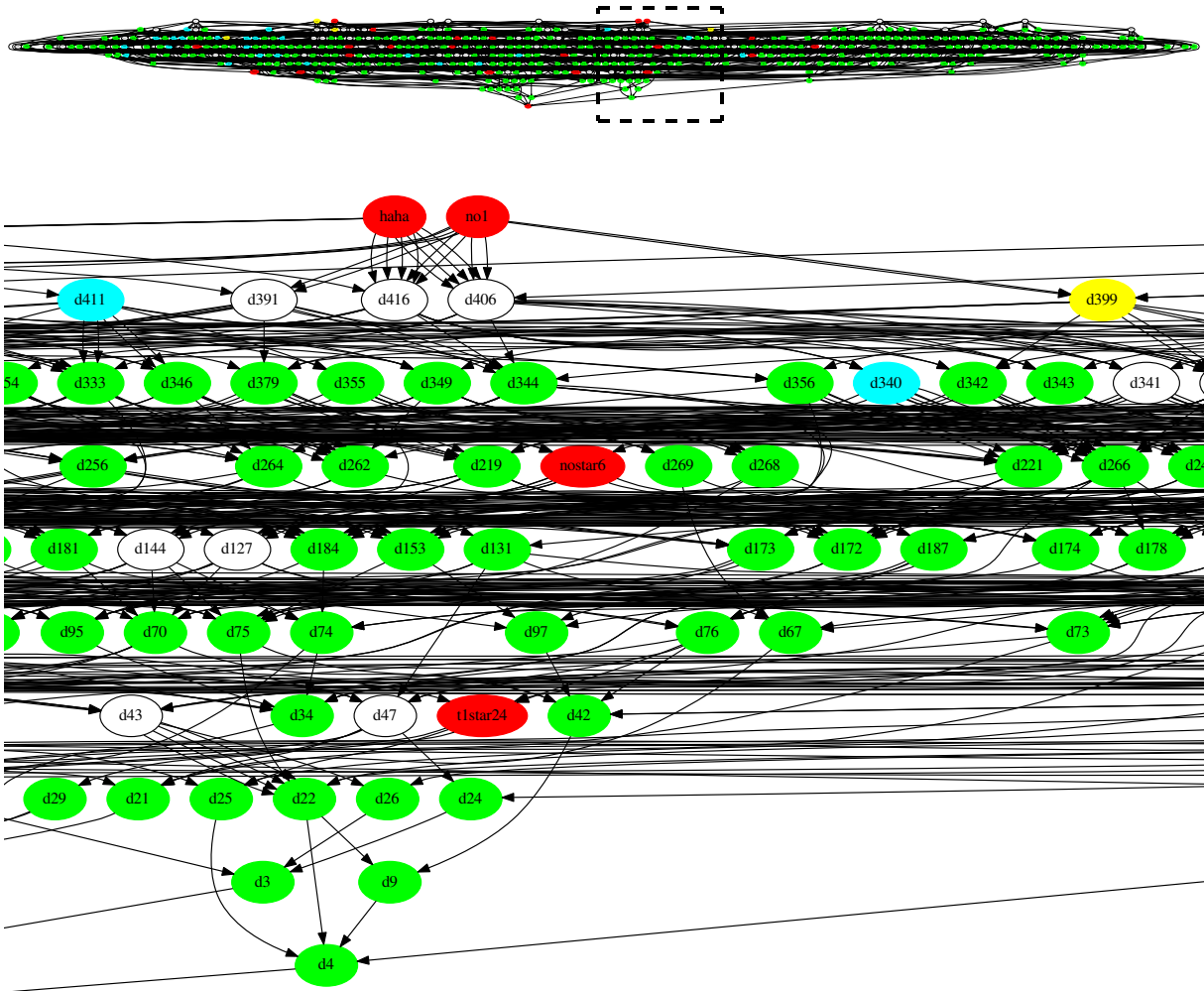


Figure 2. The tree-like graph structure of topology transitions at the four-loop level. The lower figure shows the enlarged view of the boxed region in the upper one. Green, cyan, white, yellow and red circles represent topologies in which one-loop insertion integral, carpet integral, triangle rule, diamond rule and none of them are available, respectively.

4. Conclusion

We have constructed a FORM program `Forcer`, which analytically computes massless propagator-type Feynman integrals up to the four-loop level. The complicated program structure made us opt to go for automatization of code generation. The problem that there are many topologies at this level was solved by representing all topology transitions as a topology tree and generate code in an automatized way. It is difficult to automatize deriving manual rules for topologies where none of known substructures is available. However, we observed that it is helpful to employ computer-assisted derivations and especially Gaussian elimination with a ordering of integrals by their complexities, even for parametric reduction rules.

In order to check correctness of the program, we performed recomputations of several known results in the literature. We recomputed the four-loop QCD β -function [28, 29] and checked the

gauge-invariance of the result. Moreover, we computed the same quantity in the background field method and obtained the same result. Another set of recomputations were done for four-loop anomalous dimensions of fixed N -moments of the non-singlet twist-2 operator. We reproduced $N = 2$ [30, 31], $N = 3$ and $N = 4$ results [32, 33]. More physics results obtained by Forcer will be reported elsewhere [34].

Acknowledgments

We would like to thank Andreas Vogt for discussions and collaboration for physics applications. This work is supported by the ERC Advanced Grant no. 320651, “HEPGAME”.

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