STRONGLY INTENSIVE CUMULANTS

Evan Sangaline
Intoli
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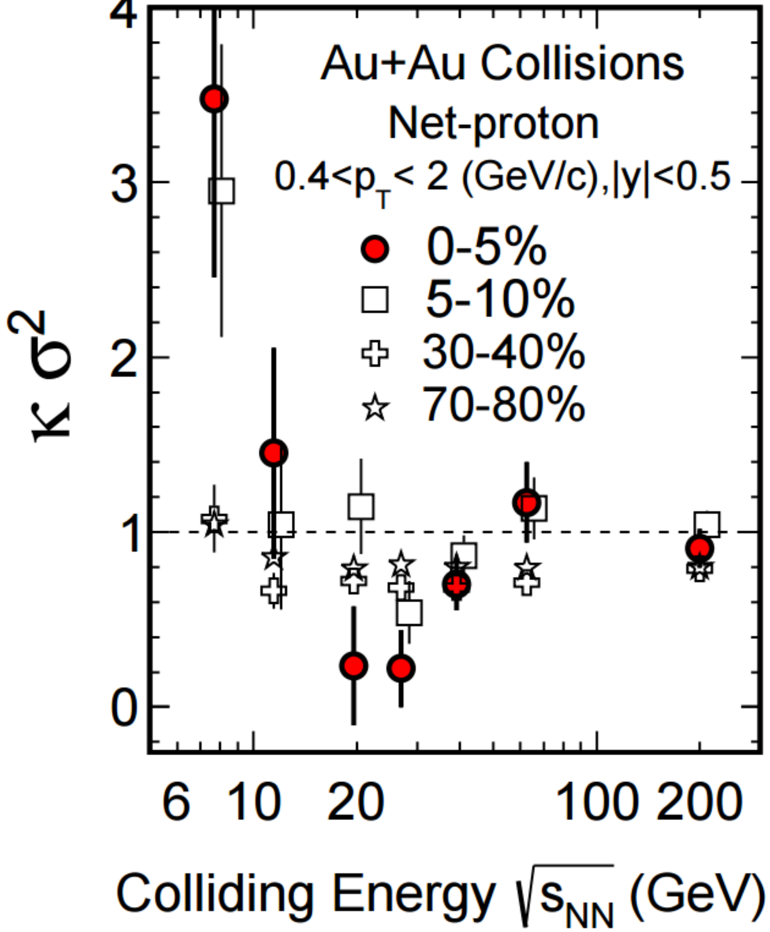
Part I

The Problem

HOW DO WE INTERPRET THIS STRUCTURE?

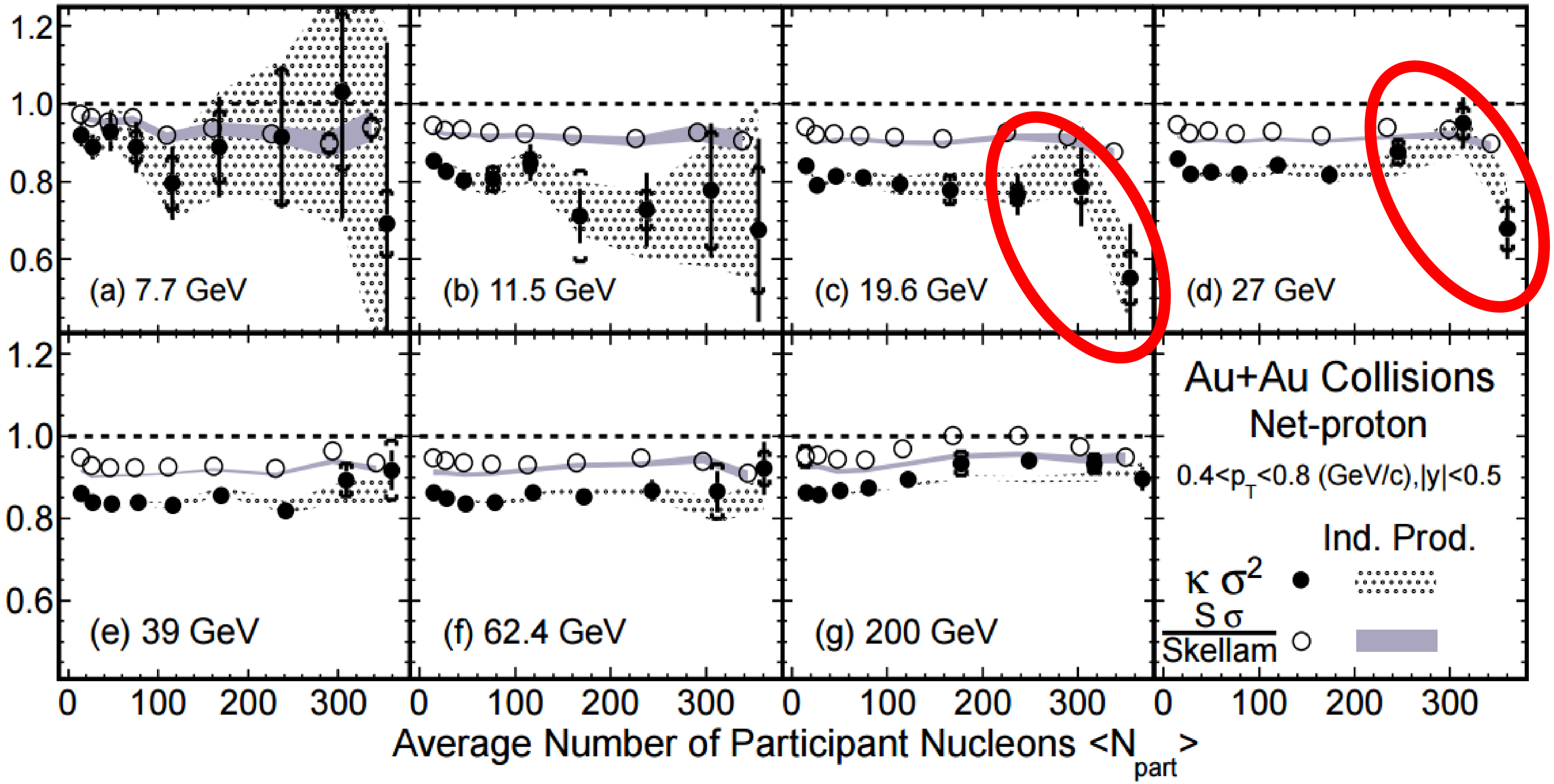
Arxiv:1503.02558



Did STAR find a critical point?

WELL... WHAT IS THE CAUSE OF THIS DIP?

PhysRevLett.112.032302



PICK YOUR FAVORITE EXPLANATION

- Statistical fluctuations
- Critical fluctuations
- Conservation effects
- Pile-up events

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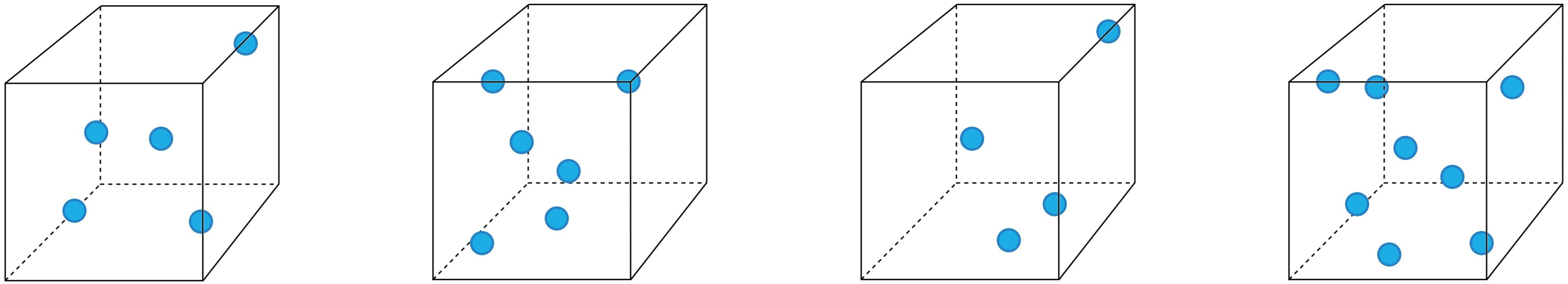
<u>Volume Fluctuations</u>

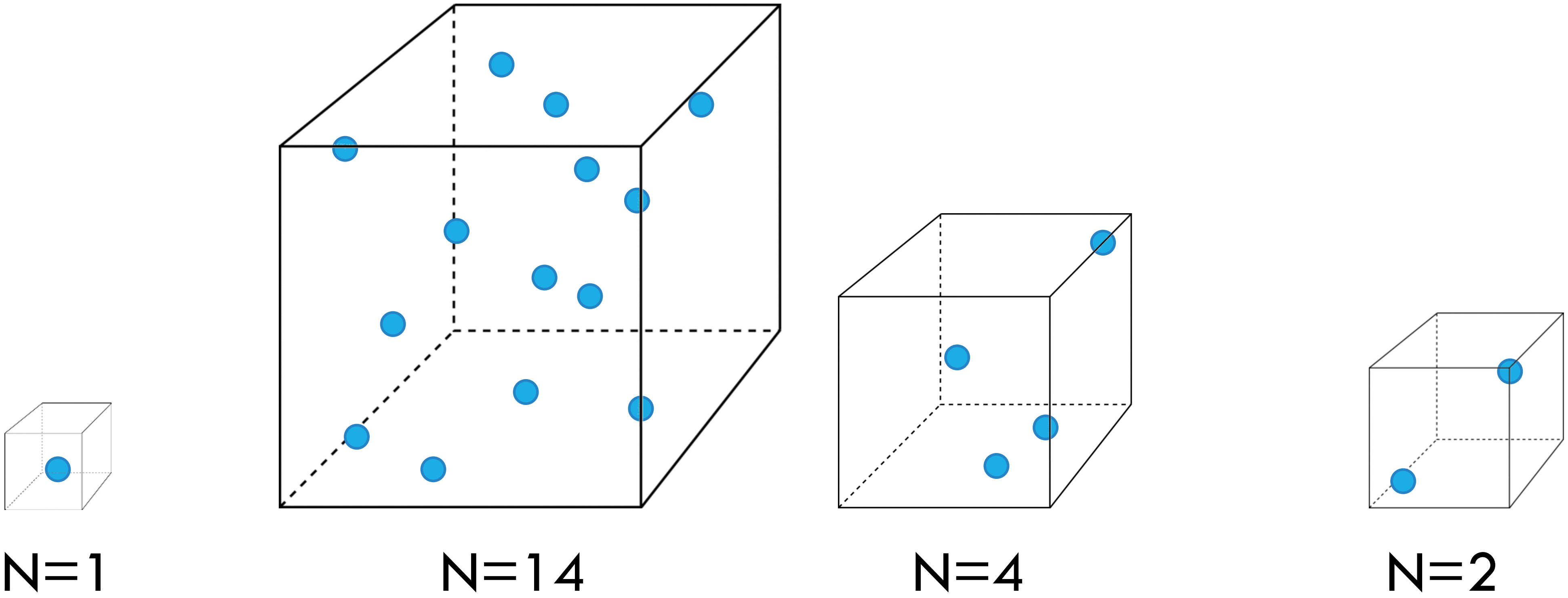
PHYSICS FLUCTUATIONS



N=5 N=6 N=4 N=8

over an ensemble of systems with fixed V, T, and μ

PHYSICS AND VOLUME FLUCTUATIONS

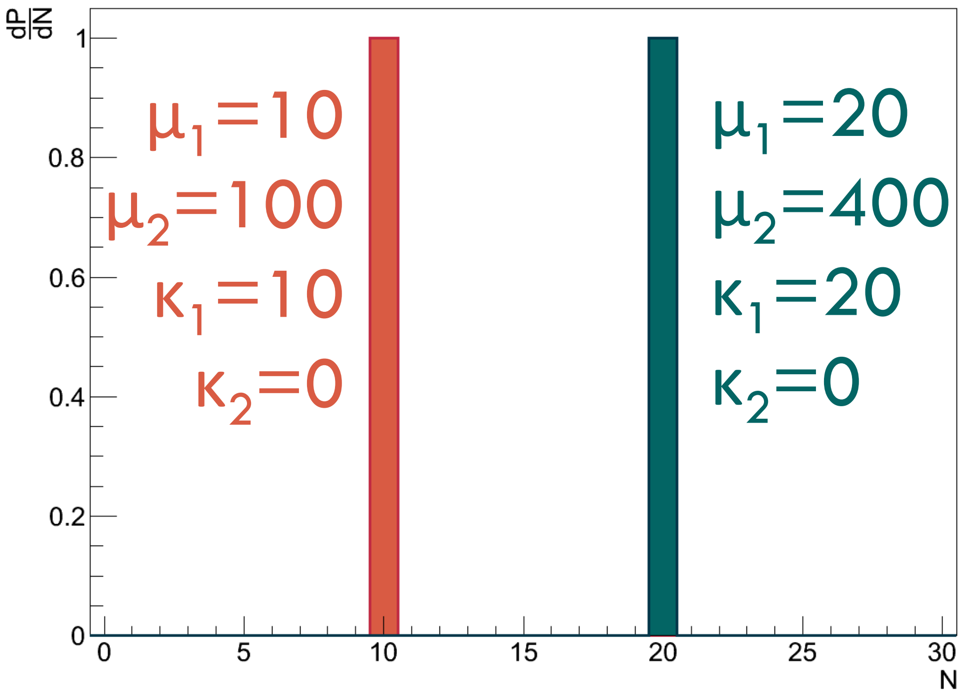


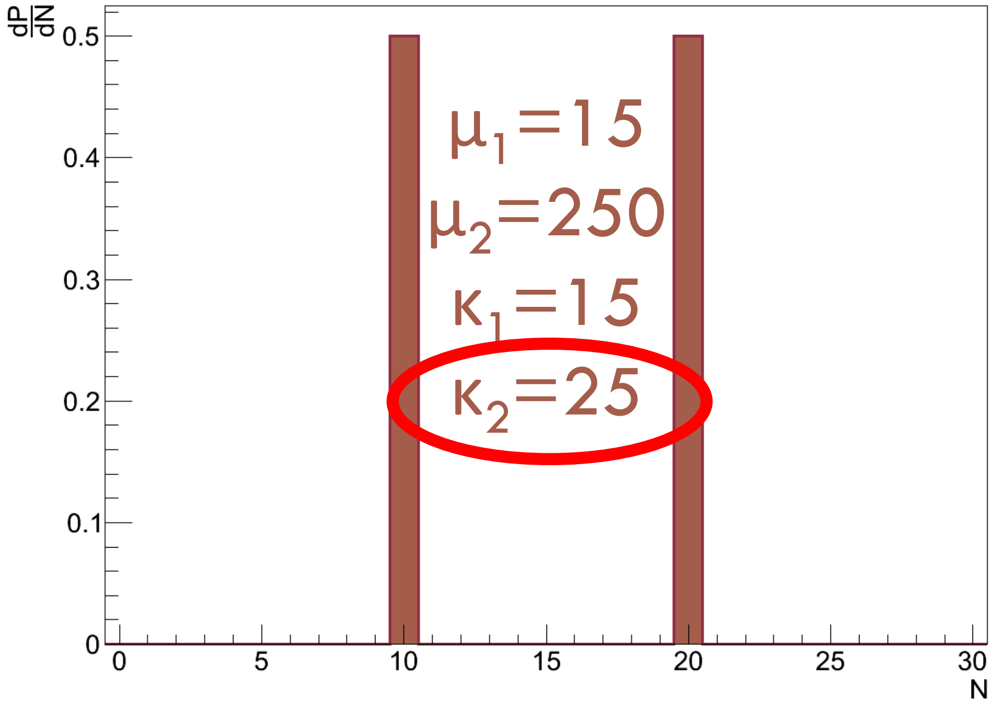
An ensemble with varying volume introduces unwanted fluctuations.

THIS BIASES CUMULANTS AND THEIR RATIOS

Multiple distributions with the same cumulant ratio

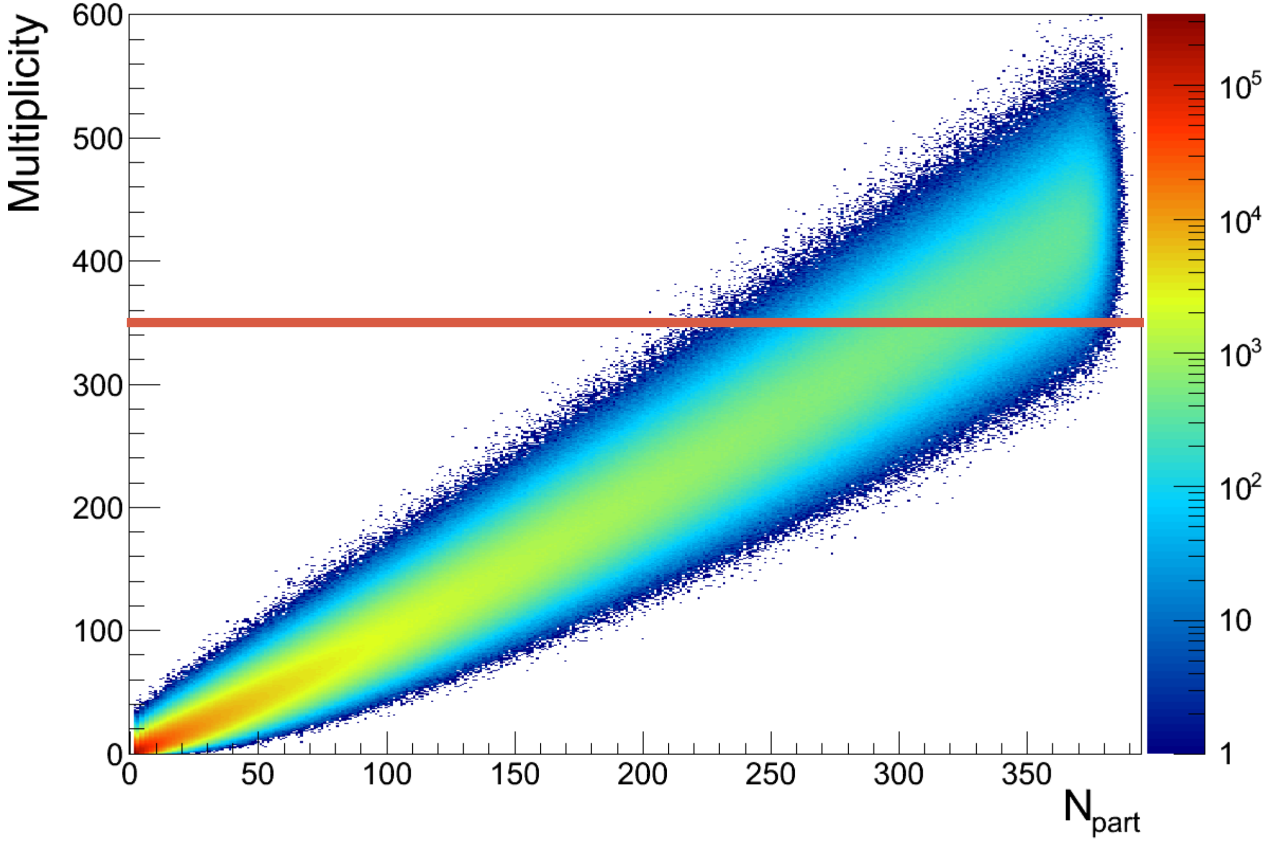
Mixed distribution with different cumulant ratio

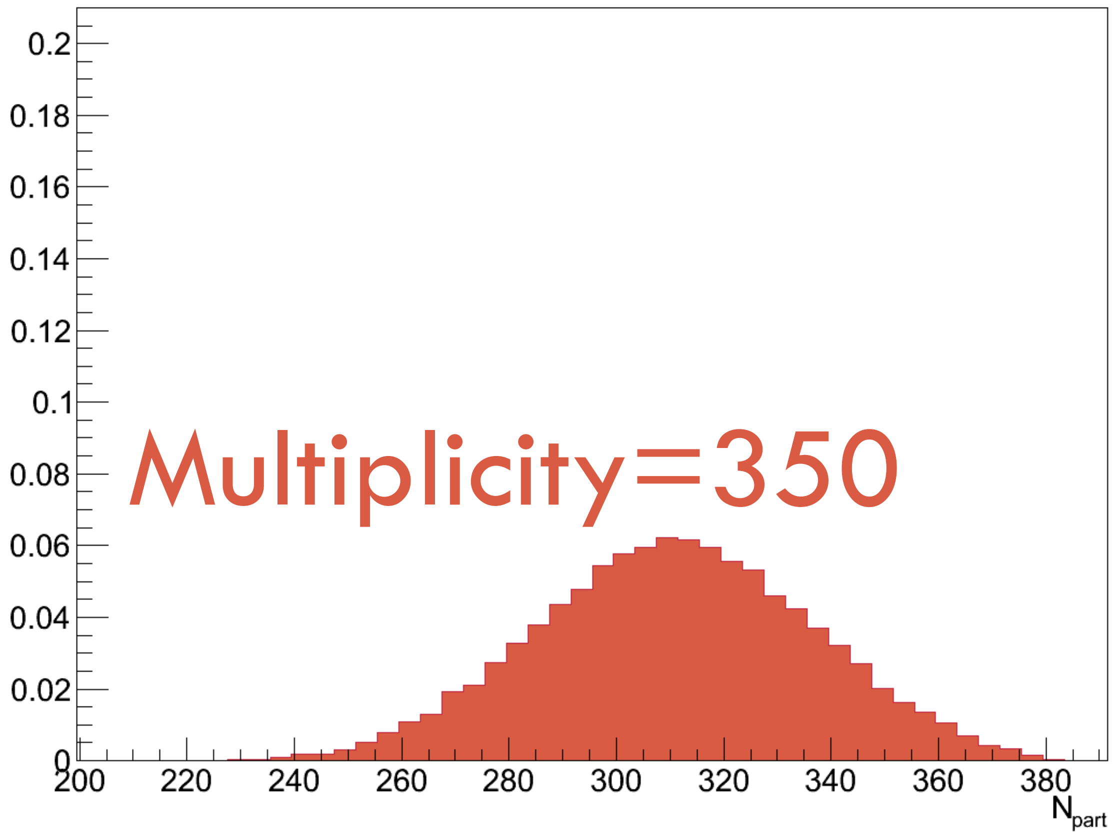




FIXED EVENT MULTIPLICITY ≠ FIXED VOLUME

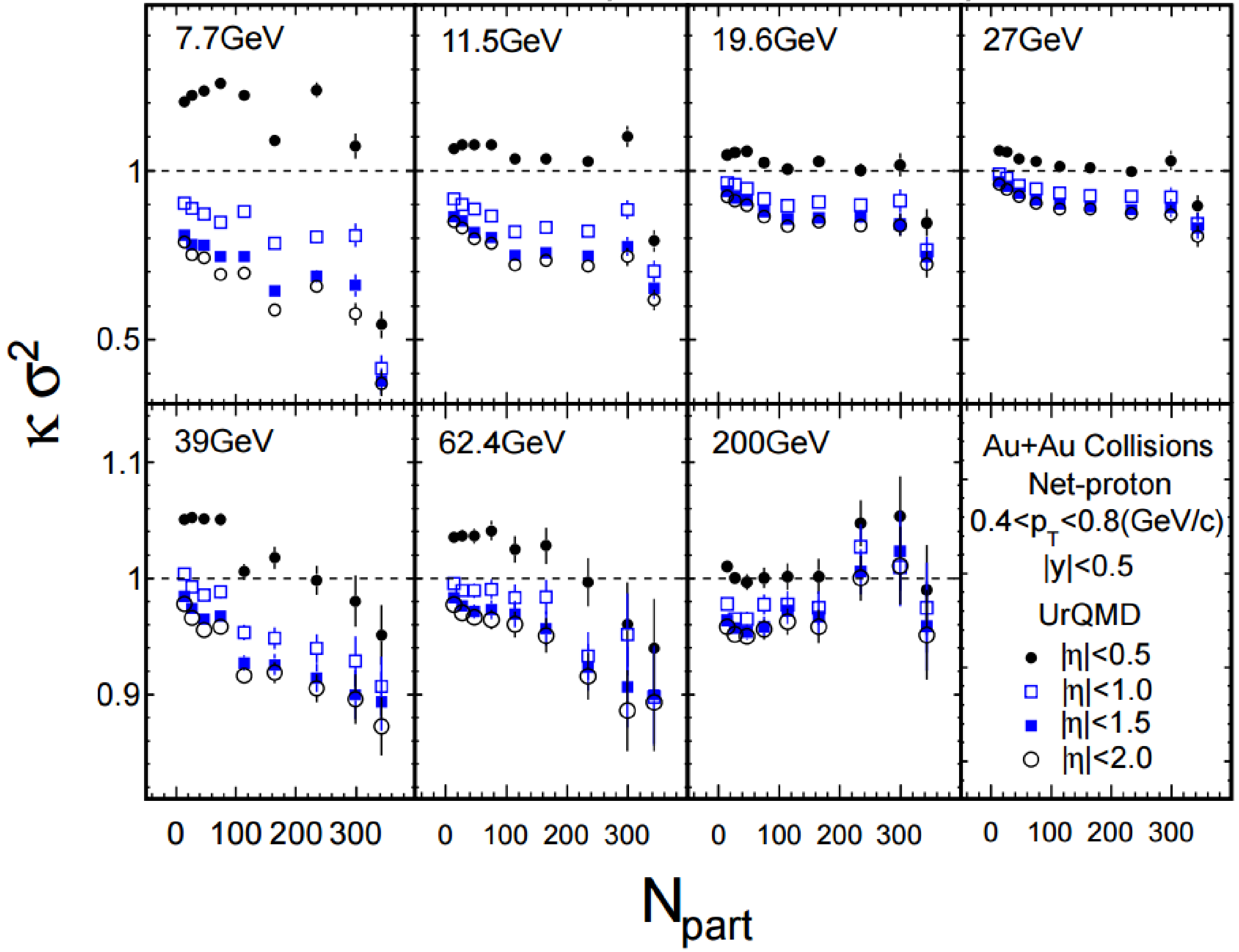
MC Glauber for $\sqrt{s}$=39.6 GeV Au+Au:





K_4/K_2 GETS INFLATED BY THIS EFFECT

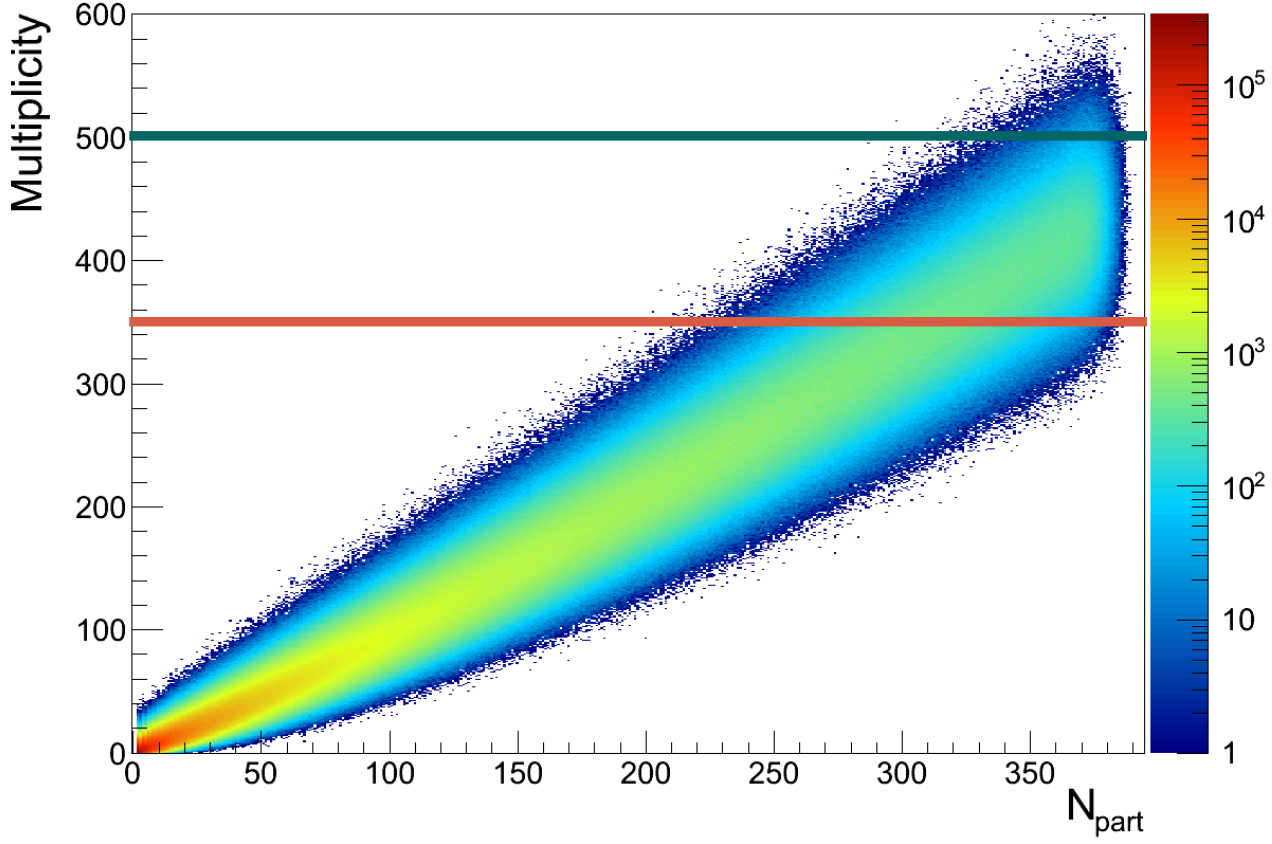
J. Phys. G: Nucl. Part. Phys. 40 105104

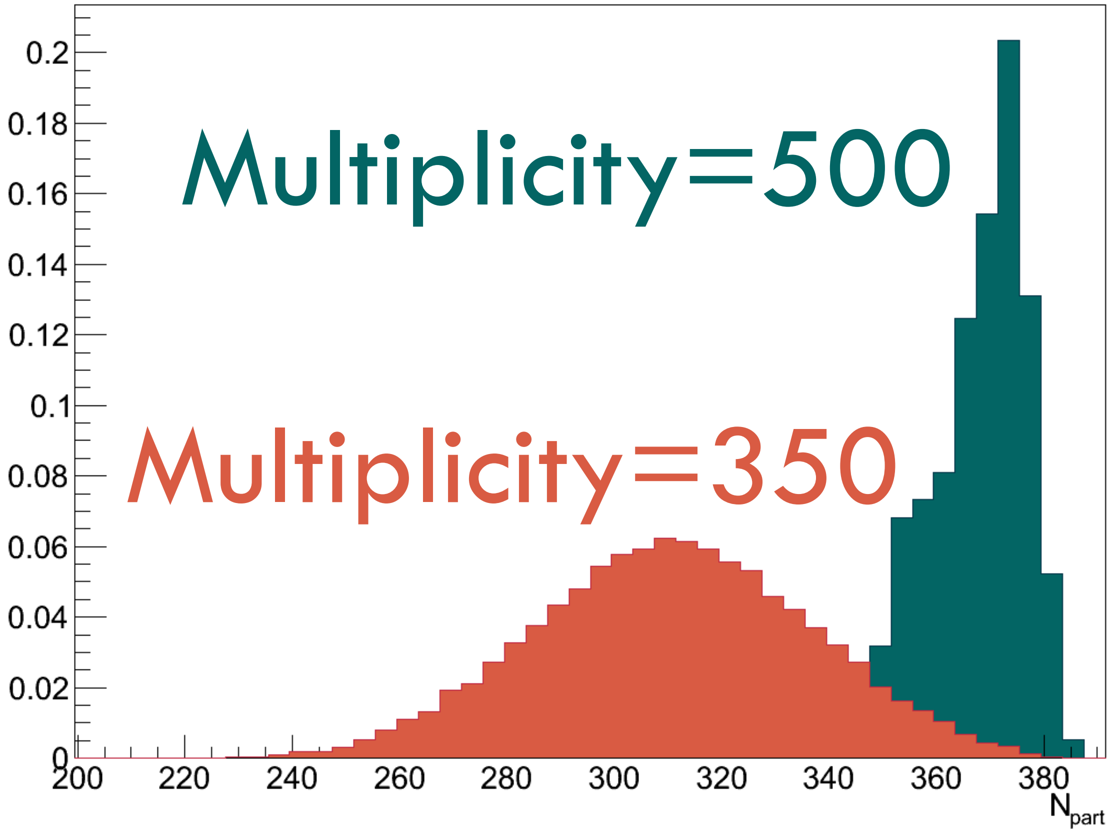


Smaller η windows for centrality mean bigger volume fluctuations mean higher κ_4/κ_2

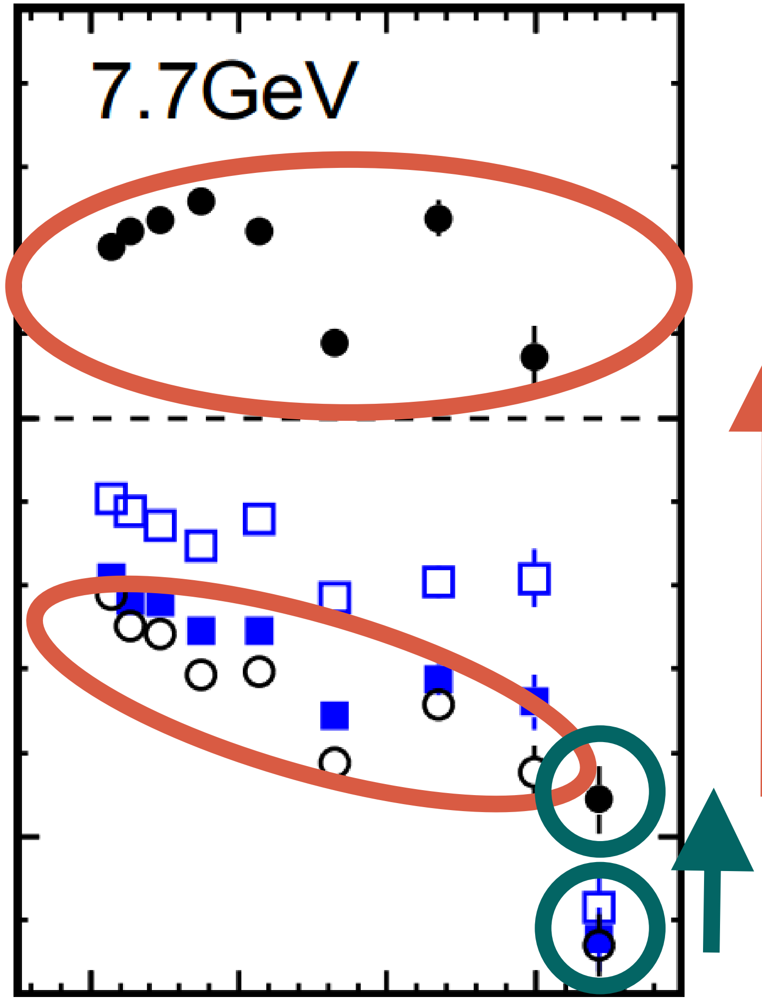
CENTRAL EVENTS HAVE THE BEST RESOLUTION

There's effectively a maximum volume attainable for each collision system.





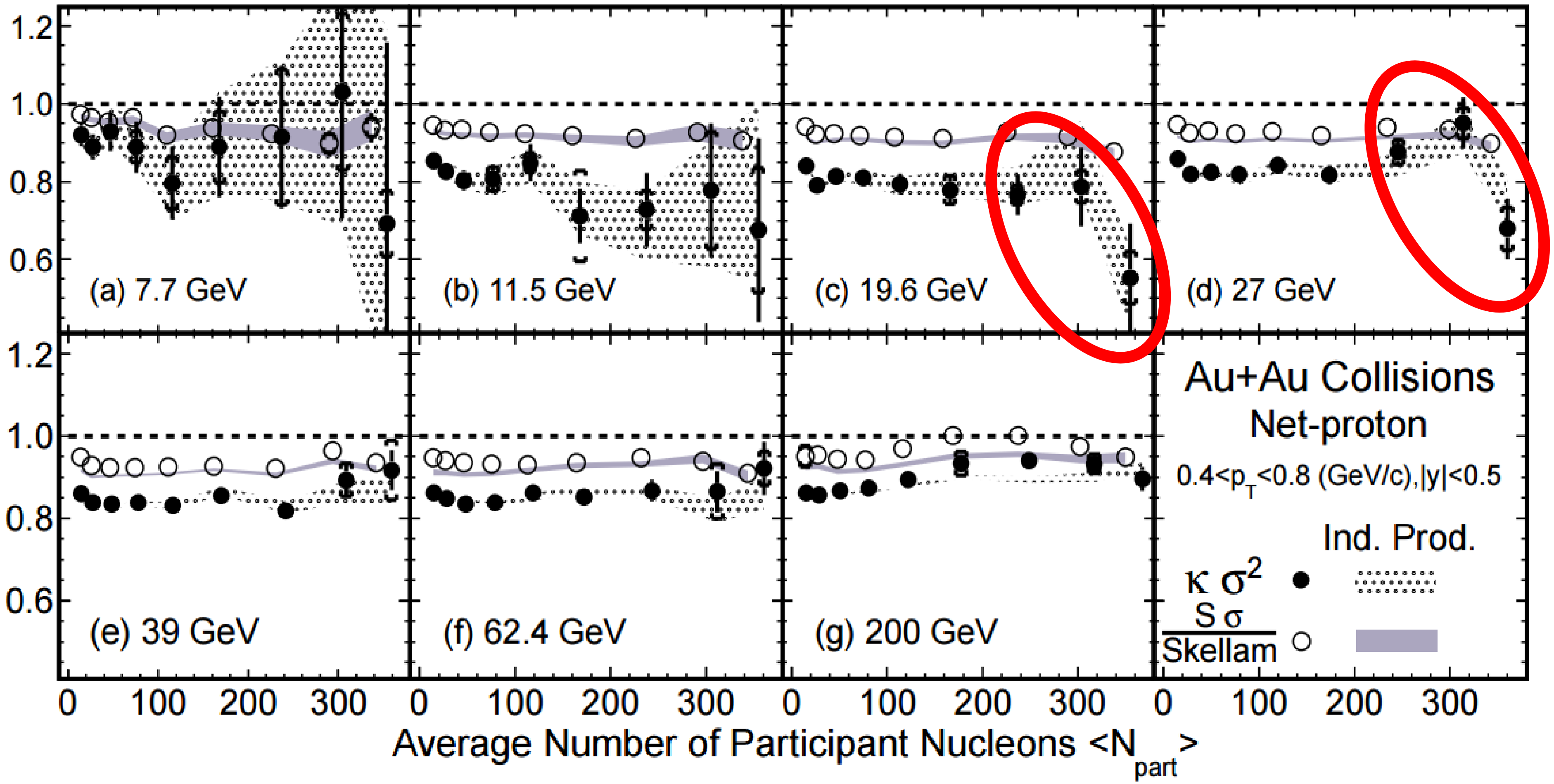
RESULTING IN THE 0-5% DIP



Non-central measurements are inflated a lot

Central measurements are inflated a little

THE DIP IN THE ACTUAL DATA?



SMOKING GUN EVIDENCE…

that <u>**volume fluctuations are not under control**</u> in this analysis.

Part II

Background

A QUICK REFRESHER ON GENERATING FUNCTIONS

Generating functions are simply power series where the coefficients encode information.

This information can be moments or cumulants.

This information can be recovered by taking derivatives.

THE MOMENT GENERATING FUNCTION

For the non-central moments of random variate X:

$$\phi(\xi) \equiv \left\langle e^{\xi_i X_i} \right\rangle_X$$

$$= \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{\mu_{r_1, r_2, \ldots, r_n}}{r_1! r_2! \cdots r_n!} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n}$$

The coefficients of the Taylor series are the moments of X:

$$\mu_{r_1, r_2, \ldots, r_n} = \left[D_1^{r_1} D_2^{r_2} \cdots D_n^{r_n} (\phi) \right]_{\xi=0}$$

$$= \langle X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n} \rangle_X$$

The operator D_n differentiates wrt ξ_n

THE CUMULANT GENERATING FUNCTION

Cumulants are **defined** by their generating function:

$$\psi(\xi) \equiv \ln \phi(\xi) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{\kappa_{r_1, r_2, \ldots, r_n}}{r_1! r_2! \cdots r_n!} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n}$$

Matching Taylor series terms lets us derive recursion relations:

$$\kappa_n(X) = \mu_n(X) - \sum_{i=1}^{n-1} \binom{n-1}{i-1} \kappa_i(X) \mu_{n-i}(X)$$

1D case

WHY DID WE PICK $\psi(\xi) \equiv ln[\phi(\xi)]$???

We think of them as obvious now due to familiarity…

Wasn't done until 1889.

After the initial development of statistical mechanics

BECAUSE IT MAKES THE CUMULANTS ADDITIVE UNDER CONVOLUTION

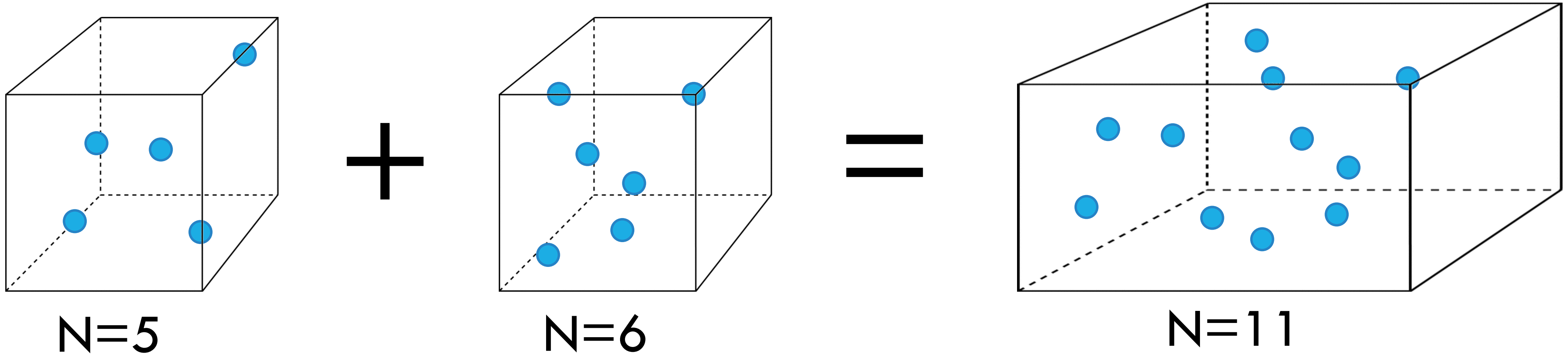
Let X and Y be independent random variates.
What are the cumulants of Z=X+Y?

$\phi_Z(\xi) = \left\langle e^{\xi_i (X+Y)_i} \right\rangle_{X,Y}$ factorizes to… $\phi_Z(\xi) = \left\langle e^{\xi_i X_i} \right\rangle_X \left\langle e^{\xi_i Y_i} \right\rangle_Y$

$$\psi_Z(\xi) = \ln \phi_Z(\xi) = \ln \phi_X(\xi) + \ln \phi_Y(\xi)$$

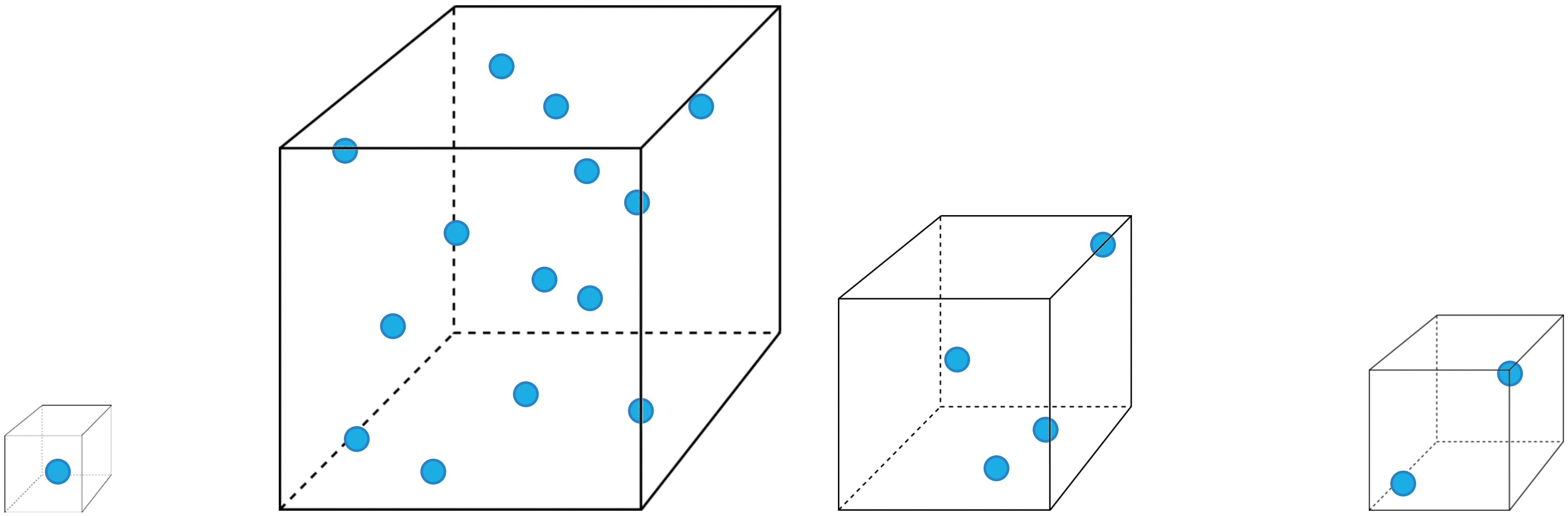
$$\kappa_n(Z) = \kappa_n(X) + \kappa_n(Y)$$

CUMULANTS/VOLUME ARE INHERENTLY INTENSIVE



If you want thermodynamic properties then they're a natural choice…
Even without invoking statistical mechanics at all.

WHAT ABOUT UNDER DISTRIBUTION MIXING?



N=1 N=14 N=4 N=2

This is what a mixed volume ensemble does, formally.

FOR MOMENTS

Simple mixing scenario: $P_Z(Z) = \frac{1}{2}\left(P_X(Z) + P_Y(Z)\right)$

$$\phi_Z(\xi) = \langle e^{\xi_i Z_i} \rangle = \frac{1}{2}\left(\langle e^{\xi_i X_i} \rangle + \langle e^{\xi_i Y_i} \rangle\right) = \frac{1}{2}\left(\phi_X(\xi) + \phi_Y(\xi)\right)$$

The moments are related very simply under distribution mixing.

FOR CUMULANTS

Still: $P_Z(Z) = \frac{1}{2}\left(P_X(Z) + P_Y(Z)\right)$

$$\phi_Z(\xi) = \frac{1}{2}\left(\phi_X(\xi) + \phi_Y(\xi)\right)$$

$$\psi(\xi) = ln\left[\frac{1}{2}\left(\phi_X(\xi) + \phi_Y(\xi)\right)\right]$$
$$= ln\left[\frac{1}{2}\left(e^{\psi_X(\xi)} + e^{\psi_Y(\xi)}\right)\right]$$

Not simple

MIXING AND CONVOLUTION

- Moments combine simply under mixing.
- Cumulants combine simply under convolution.

Is there a general class of statistical quantities that **combine simply under both convolution and mixing?**

No.

No.

Unless we consider a restricted set of distributions…

Part III

A Solution

A SPECIAL CASE OF CONVOLUTION + MIXING

The cumulant generating function of every distribution that we're mixing is related to a single volume independent function by a constant factor.

$$\psi'(\xi) \equiv \psi(\xi)/V$$
$$= \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\kappa'_{r_1,r_2,\ldots,r_n}}{r_1!r_2!\cdots r_n!}\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n}$$

′ denotes volume independence

Translation: Intrinsic thermodynamic variables are independent of volume.

THIS SIMPLIFIES VOLUME FLUCTUATIONS

At a given volume:

$$\phi(\xi) = e^{V\psi'(\xi)}$$
$$= 1 + V\psi'(\xi) + \frac{1}{2!}V^2\psi'(\xi)^2 + \cdots$$

Over a mixed ensemble of volumes:

$$\psi(\xi) = \ln\left\langle e^{V\psi'(\xi)}\right\rangle_V$$
$$= \ln\left(1 + \langle V\rangle_V\,\psi'(\xi) + \frac{1}{2!}\left\langle V^2\right\rangle_V\psi'(\xi)^2 + \cdots\right)$$

WE CAN DETERMINE EXACT EXPRESSIONS FOR BIASES

$$\kappa_2 = \langle V \rangle_V \kappa'_2 + \left(\langle V^2 \rangle_V - \langle V \rangle_V^2 \right) (\kappa'_1)^2$$

Unwanted volume fluctuation term.

Δ and Σ combine second order joint cumulants to cancel these terms

PhysRevC.89.034903

THE STRONGLY INTENSIVE CUMULANTS

The strongly intensive cumulants are **defined** by a differential equation involving their generating function:

$$D_u\left(\psi^*\right) = \frac{D_u\left(\phi\right)}{D_v\left(\phi\right)}$$

The individual SICs are themselves given by the coefficients of their generating function:

$$\psi^*\left(\xi\right) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\kappa^*_{r_1,r_2,\ldots,r_n}}{r_1!r_2!\cdots r_n!}\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n}$$

WHY DID WE PICK THAT???

Reason 1: Because they're entirely independent of the distribution over volume in a mixed ensemble (strongly intensive).

Easy proof:

$$D_1\left(\psi^*\right)=\frac{D_1\left(\left\langle e^{V\psi'(\xi)}\right\rangle_V\right)}{D_n\left(\left\langle e^{V\psi'(\xi)}\right\rangle_V\right)}$$

$$=\frac{\left\langle Ve^{V\psi'(\xi)}\right\rangle_V D_1\left(\psi'\right)}{\left\langle Ve^{V\psi'(\xi)}\right\rangle_V D_n\left(\psi'\right)}=\frac{D_1\left(\psi'\right)}{D_n\left(\psi'\right)}$$

MORE INSIGHTFUL PROOF

Under convolution Z=X+Y:

$$\phi_Z(\xi) = \phi_X(\xi)\,\phi_Y(\xi)$$

$$D_1(\psi_Z^*) = \frac{D_n(\psi_X)\,D_1(\psi_X^*) + D_n(\psi_Y)\,D_1(\psi_Y^*)}{D_n(\psi_X) + D_n(\psi_Y)}$$

If X and Y share the same SICs then $D_1(\psi_X^*) = D_1(\psi_Y^*)$ and:

$$D_1(\psi_Z^*) = \frac{D_n(\psi_X) + D_n(\psi_Y)}{D_n(\psi_X) + D_n(\psi_Y)} D_1(\psi_X^*)$$

$$= D_1(\psi_X^*) = D_1(\psi_Y^*)$$

MORE INSIGHTFUL PROOF

The strongly intensive cumulants of the convolution of two distributions having identical strongly intensive cumulants are equal to those of the individual distributions.

This is equivalent to saying that they're independent of volume (intensive).

MORE INSIGHTFUL PROOF

Under mixing $P_Z(Z) = \frac{1}{2}\left(P_X(Z) + P_Y(Z)\right)$

$$\phi_Z(\xi) = \frac{1}{2}\left(\phi_X(\xi) + \phi_Y(\xi)\right)$$

$$D_1(\psi_Z^*) = \frac{D_n(\phi_X) D_1(\psi_X^*) + D_n(\phi_Y) D_1(\psi_Y^*)}{D_n(\phi_X) + D_n(\phi_Y)}$$

If X and Y share the same SICs then $D_1(\psi_X^*) = D_1(\psi_Y^*)$ and:

$$D_1(\psi_Z^*) = D_1(\psi_X^*) = D_1(\psi_Y^*)$$

MORE INSIGHTFUL PROOF

The strongly intensive cumulants of a mixing of two distributions having identical strongly intensive cumulants are equal to those of the individual distributions.

This, coupled with the convolution property, is equivalent to saying that they're independent of mixing over an ensemble of volumes (strongly intensive).

ANOTHER INSIGHTFUL PROOF

Strongly intensive cumulants have the same algebraic expressions in terms of either cumulants or moments:

$$D_1\left(\psi^*\right) = \frac{D_1\left(\phi\right)}{D_n\left(\phi\right)} = \frac{{}^1\!/\!_\phi D_1\left(\phi\right)}{{}^1\!/\!_\phi D_n\left(\phi\right)} = \frac{D_1\left(\ln\phi\right)}{D_n\left(\ln\phi\right)} = \frac{D_1\left(\psi\right)}{D_n\left(\psi\right)}$$

This is true iff an expression is strongly intensive.

WHY DID WE PICK THAT MESS???

Reason 2: Because they're proportional to the cumulants if used with an uncorrelated noisy volume measure.

$$\left\langle X_1^{r_1} X_2^{r_2} \cdots X_{n-1}^{r_{n-1}} \right\rangle \left\langle X_n^{r_n} \right\rangle = \left\langle X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n} \right\rangle$$

implies

$$\begin{aligned} D_1\left(\psi^*\right) &= \frac{D_1\left(\phi\right)}{D_n\left(\phi\right)} = \frac{D_1\left(\phi\right)}{D_n\left(\left\langle e^{\xi_i X_i}\right\rangle_X\right)} = \frac{D_1\left(\phi\right)}{\left\langle X_n e^{\xi_i X_i}\right\rangle_X} \\ &= \frac{D_1\left(\phi\right)}{\left\langle X_n\right\rangle_X \left\langle e^{\xi_i X_i}\right\rangle_X} = \frac{D_1\left(\phi\right)}{\left\langle X_n\right\rangle_X \phi} \\ &= \frac{D_1\left(\ln\phi\right)}{\left\langle X_n\right\rangle_X} = \frac{D_1\left(\psi\right)}{\left\langle X_n\right\rangle_X} \end{aligned}$$

implies $\kappa^*_{r_1,r_2,\ldots r_n} = \dfrac{\kappa_{r_1,r_2,\ldots r_n}}{\kappa_{0,0,\ldots,0,1}}$

IS THAT ENOUGH REASONS?

- We defined cumulants so that they had desired properties.
- These encompass the same physics as cumulants.
- Without the volume fluctuation contributions.
  - A very desired property.

CALCULATING THEM IN PRACTICE

Recursion equations exist for the general case (arxiv:1505.00261)

Analytical expressions for any order of joint strongly intensive cumulant can be easily calculated

SIMPLEST CASE

- Strongly intensive quantities.
- Same meaning as cumulants.
- Trivial to implement and code.

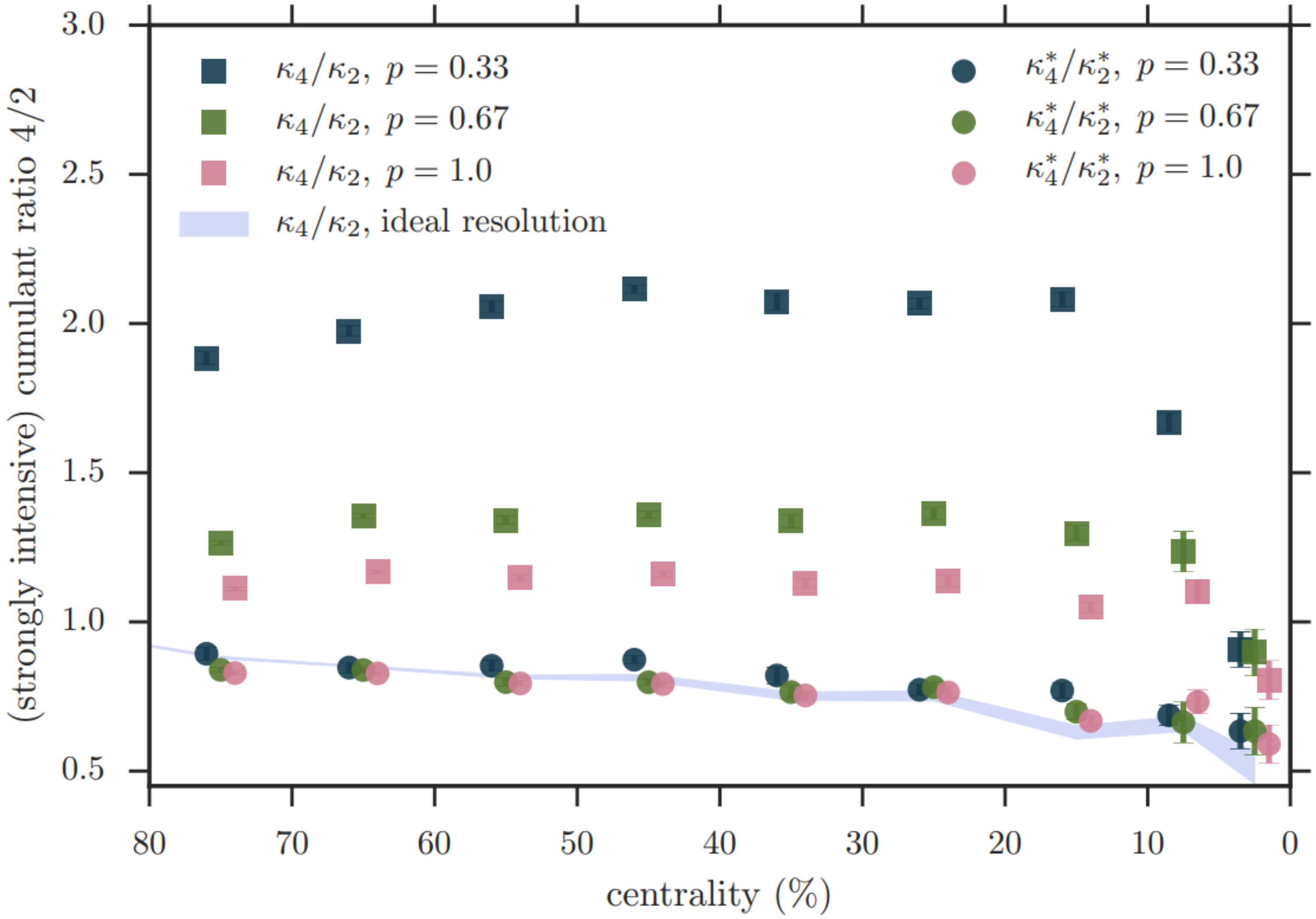
$$\kappa_1^* = \frac{\mu_{1,0}}{\mu_{0,1}}$$

$$\kappa_2^* = \frac{\mu_{2,0}}{\mu_{0,1}} - \frac{\mu_{1,0}\mu_{1,1}}{{\mu_{0,1}}^2}$$

$$\kappa_3^* = \frac{\mu_{3,0}}{\mu_{0,1}} - \frac{2\mu_{2,0}\mu_{1,1} + \mu_{1,0}\mu_{2,1}}{{\mu_{0,1}}^2} + \frac{2\mu_{1,0}{\mu_{1,1}}^2}{{\mu_{0,1}}^3}$$

$$\kappa_4^* = \frac{\mu_{4,0}}{\mu_{0,1}} - \frac{3\mu_{3,0}\mu_{1,1} + \mu_{1,0}\mu_{3,1}}{{\mu_{0,1}}^2} - \frac{3\mu_{2,0}\mu_{2,1}}{{\mu_{0,1}}^2} + \frac{6\mu_{2,0}{\mu_{1,1}}^2 + 6\mu_{1,0}\mu_{2,1}\mu_{1,1}}{{\mu_{0,1}}^3} - \frac{6\mu_{1,0}{\mu_{1,1}}^3}{{\mu_{0,1}}^4}$$

A 7.7 GEV AU+AU URQMD DEMONSTRATION



- SICs agree with the ideal
- Cumulants are inflated
  - As expected
- Tell-tale central dip
  - In the standard cumulants

CONCLUSIONS

- New fluctuation measures have been derived.
  - Independent of volume fluctuations (i.e. strongly intensive)
  - Closely related to standard cumulants and underlying physics.
    - Reduce exactly to cumulants under certain circumstances.
    - Potential drop-in replacement for STAR, ALICE, lattice