

Computation of Resistive Wakefields

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Resistive Wakefields

1. Longitudinal $m = 0$ wakefield
 - Long Range approximation (Chao)
 - Short Range approximation (Bane and Sands)
 - Full treatment
2. Longitudinal $m > 0$ wakefield (higher order modes)
3. Transverse wakes
4. AC conductivity
5. Implementation

This full treatment can be necessary in the design of modern nano-beam accelerators, where the effects of wakefields are becoming increasingly important and require accurate calculation.

Scenario

- Uniform circular pipe of radius b and conductivity σ
- Assume thick pipe
- Interested in short range intrabunch fields
- Solve Maxwell's Equations in vacuum and in metal pipe
- Decompose into angular modes ($\cos(m\theta)$, $m = 0, 1, 2\dots$)
- Match the boundary conditions
- Work in frequency space as differentiation \rightarrow multiplication

The longitudinal wake for $m = 0$

The Fourier Transform of the wake is the impedance:

$$\tilde{\vec{F}}(k) = \int \vec{F}(s) e^{iks} ds \quad \vec{F}(k) = \frac{1}{2\pi} \int \tilde{\vec{F}}(k) e^{-iks} dk$$

$$\tilde{E}_z(k) = \frac{2q}{b} \frac{1}{\frac{ikb}{2} - \left(\frac{\lambda}{k} + \frac{k}{\lambda}\right) \left(1 + \frac{i}{2\lambda b}\right)}$$

where

$$\lambda(k) = \sqrt{\frac{2\pi\sigma |k|}{c}} (i + \text{sgn}(k))$$

Introduce s_0 , the scaling length (20 μm for 1 cm Copper)

$$s_0 = \sqrt[3]{\frac{cb^2}{2\pi\sigma}} \quad K = s_0 k \quad s' = \frac{s}{s_0}$$

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Bane and Sands

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The simplest case: Chao's formula

In the long range limit $\tilde{E}_z(k) = \frac{2q}{b} \frac{1}{\left(-\frac{\lambda}{k}\right)}$

The Fourier Transform is well known to be: $E_z(s) = \frac{q}{2\pi b} \sqrt{\frac{c}{\sigma}} s^{-\frac{3}{2}}$

We do it the hard way by numerical integration of:

$$E_z(s) = \frac{s_0}{2\pi} \int_{-\infty}^{\infty} (\text{Re}[f_{\text{even}}(K)] \cos(Ks') + \text{Im}[f_{\text{odd}}(K)] \sin(Ks')) dK$$

$$f_{\text{even}}(K) = \frac{1}{2} [f(K) + f(-K)] = -\frac{q}{b^2} \sqrt{K}$$

$$f_{\text{odd}}(K) = \frac{1}{2} [f(K) - f(-K)] = \frac{q}{b^2} i \sqrt{K}$$

Solution: First integrate analytically wrt x . Function becomes $-\cos(Kx) \frac{\sqrt{K}}{K}$
Integrate numerically wrt K .
Differentiate numerically wrt x .

A more accurate formula : Bane and Sands

$$\tilde{E}_z(k) = \frac{2q}{b} \frac{1}{\left(\frac{ikb}{2} - \frac{\lambda}{k}\right)}$$

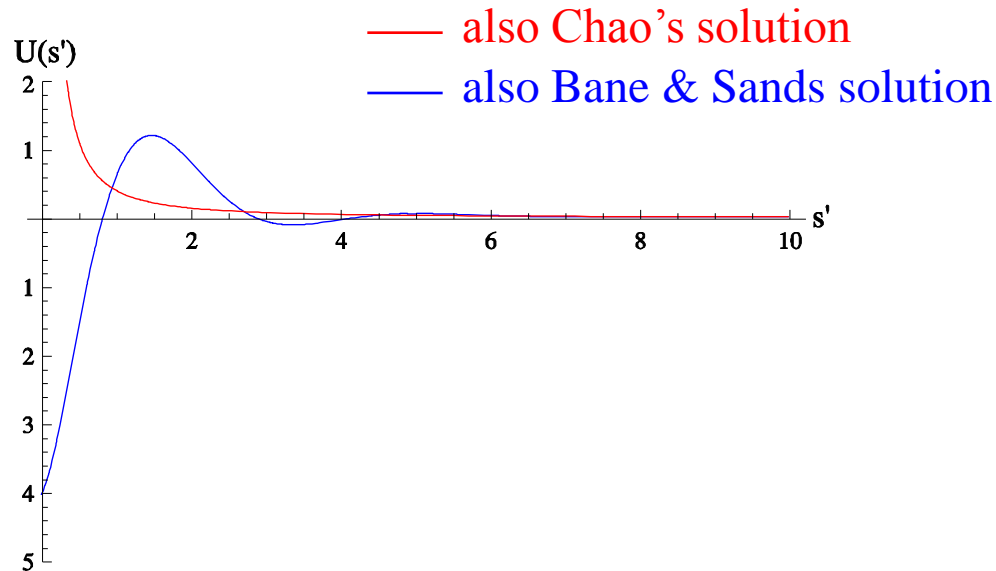
Solution well known to be: $E_z(s) = \frac{4qc}{\pi b^2} \left(\frac{e^{-s'}}{3} \cos(\sqrt{3}s') - \frac{\sqrt{2}}{\pi} \int_0^{\infty} \frac{x^2 e^{-x^2 s'}}{x^6 + 8} dx \right)$

We have:

$$E_z(s) = \frac{s_0}{2\pi} \int_{-\infty}^{\infty} (\operatorname{Re}[f_{\text{even}}(K)] \cos(Ks') + \operatorname{Im}[f_{\text{odd}}(K)] \sin(Ks')) dK$$

$$f_{\text{even}}(K) = \frac{1}{2} [f(K) + f(-K)] = -\frac{q}{b^2} \frac{\frac{2}{\sqrt{K}}}{\left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)^2 + \frac{1}{K}}$$

$$f_{\text{odd}}(K) = \frac{1}{2} [f(K) - f(-K)] = -\frac{q}{b^2} \frac{2i \left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)}{\left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)^2 + \frac{1}{K}}$$



Shows our results. Accurately reproduces Chao and B&S.

Wake is a function of three parameters (s , b and σ), but the use of s_0 enables it to be written as a universal function $U(s')$, where $E_z(s,b) = s_0 q / b^2 U(s/s_0)$.

The full formula

Full version, with $\xi = s_0^2 / b^2$

$$f_{\text{even}}(K) = -\frac{8q}{b^2} \frac{\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}}{4\left[\xi\sqrt{K} - \frac{1}{K}(\xi + 2\sqrt{K}) + K\right]^2 + \left(\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}\right)^2}$$
$$f_{\text{odd}}(K) = -\frac{8q}{b^2} \frac{2i\left[\xi\sqrt{K} - \frac{1}{K}(\xi + 2\sqrt{K}) + K\right]}{4\left[\xi\sqrt{K} - \frac{1}{K}(\xi + 2\sqrt{K}) + K\right]^2 + \left(\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}\right)^2}$$

Although this is no longer a universal curve, it can still be expressed as a function of two variables (s' and ξ) rather than the full set of three.

The B&S approximation corresponds to the function at $\xi = 0$.

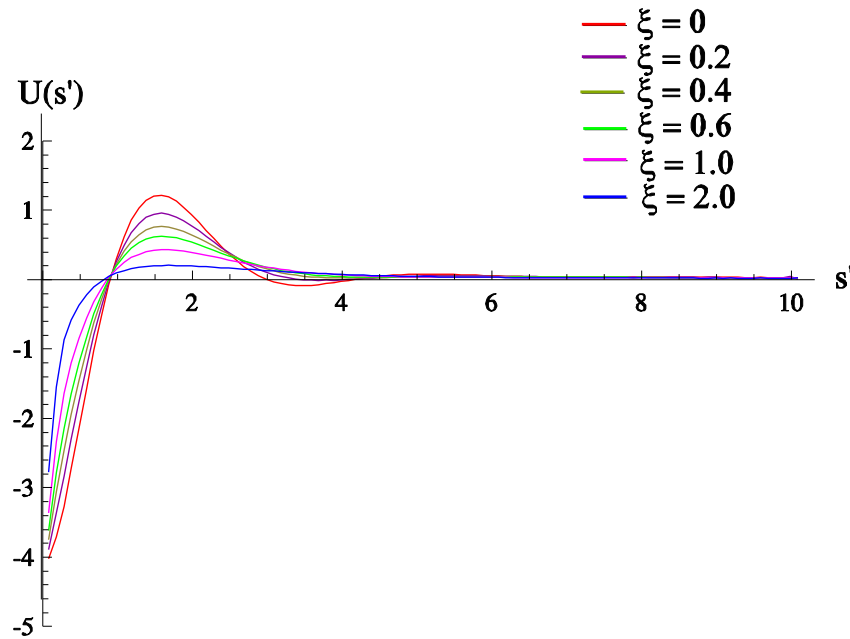


Figure shows how the function changes for different values of ζ .

Longitudinal: Higher order modes

For higher order modes, using the same technique for $m > 0$

$$\tilde{E}_z^m(k) = \frac{4I_m}{b^{2m+1}} \frac{1}{\frac{ikb}{m+1} - \left(\frac{\lambda}{k} + \frac{2k}{\lambda}\right) \left(1 + \frac{i}{2\lambda b}\right) - \frac{im}{kb}}$$

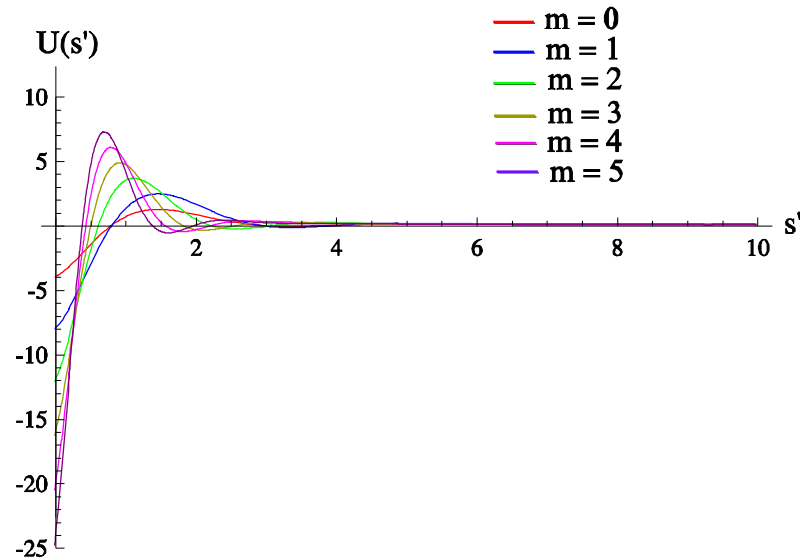
where I_m is the charge moment of order m : $\iint \rho(r, \theta) r^m \cos(m\theta) dr d\theta$
 - Any angular distribution of charges at a particular radius can be described in terms of these moments.

This can be separated into odd and even parts,

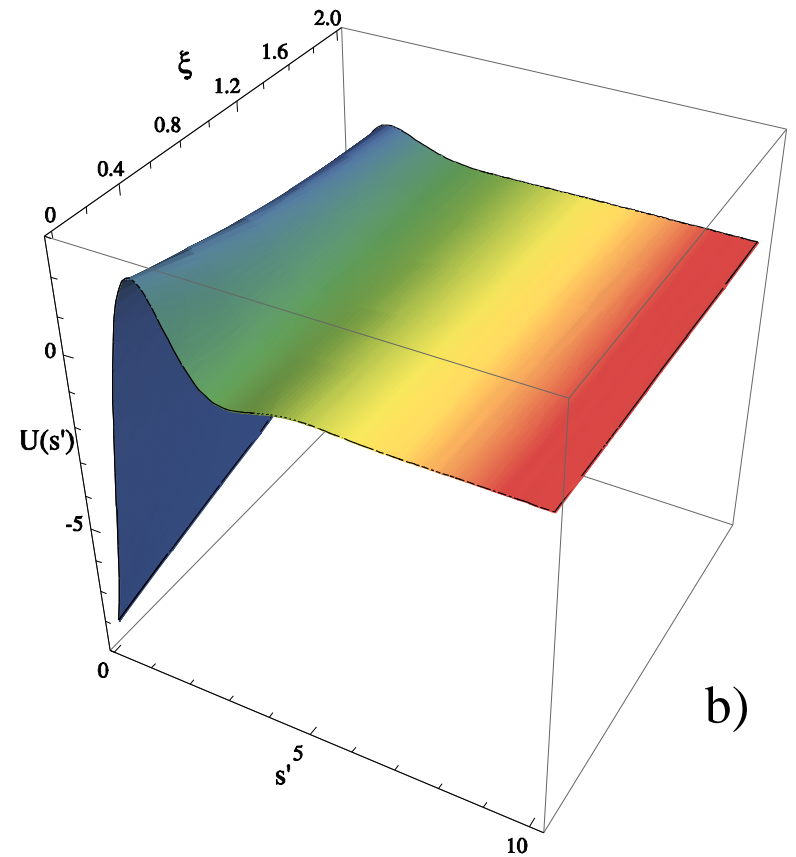
$$f_{\text{even}}(K) = -\frac{8I_m S_0}{b^{2m+2}} \frac{\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}}{4\left[\xi\sqrt{K} - \frac{1}{K}(\xi + 2\sqrt{K}) + 2\left(\frac{K}{m+1} - \xi\frac{m}{K}\right)\right]^2 + \left(\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}\right)^2}$$

$$f_{\text{odd}}(K) = -\frac{8I_m S_0}{b^{2m+2}} \frac{2i\left[\xi\sqrt{K} - \frac{1}{K}(\xi + 2\sqrt{K}) + 2\left(\frac{K}{m+1} - \xi\frac{m}{K}\right)\right]}{4\left[\xi\sqrt{K} - \frac{1}{K}(\xi + 2\sqrt{K}) + 2\left(\frac{K}{m+1} - \xi\frac{m}{K}\right)\right]^2 + \left(\xi^2 + 2\xi\sqrt{K} + \frac{4}{\sqrt{K}}\right)^2}$$

Longitudinal: Higher order modes



a)



b)

Higher order modes. a) various modes with $\xi = 0$;
b) the $m = 1$ wake as a function of ξ and s' .

Transverse wakes

Transverse wake also a sum over angular modes

$$\vec{F}_T(r, \theta, s) = \sum_m r^{m-1} r'^m (\hat{r} \cos(m\theta) - \hat{\theta} \sin(m\theta)) W_T^m(s)$$

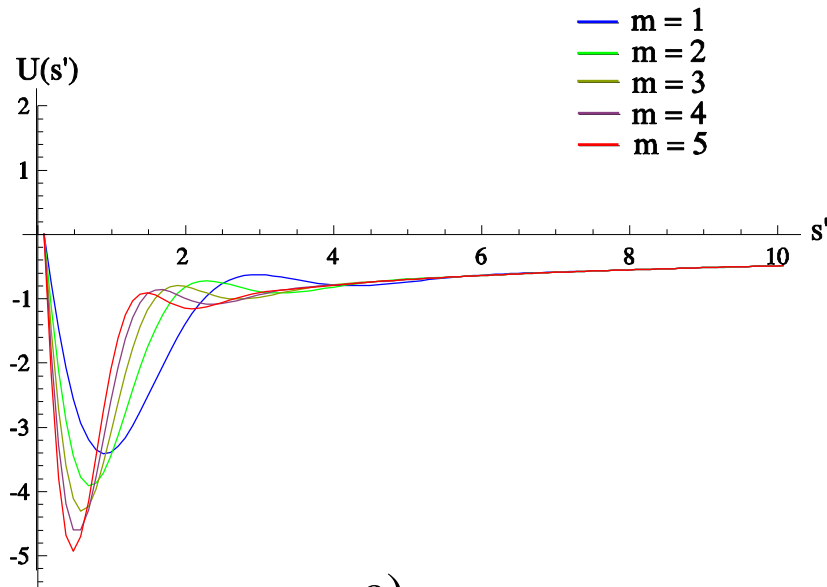
The Panofsky-Wenzel theorem $\nabla_T F = \frac{\partial \vec{F}_T}{\partial z}$

applies term by term giving $W_T^m(s) = E_z^m(s)$

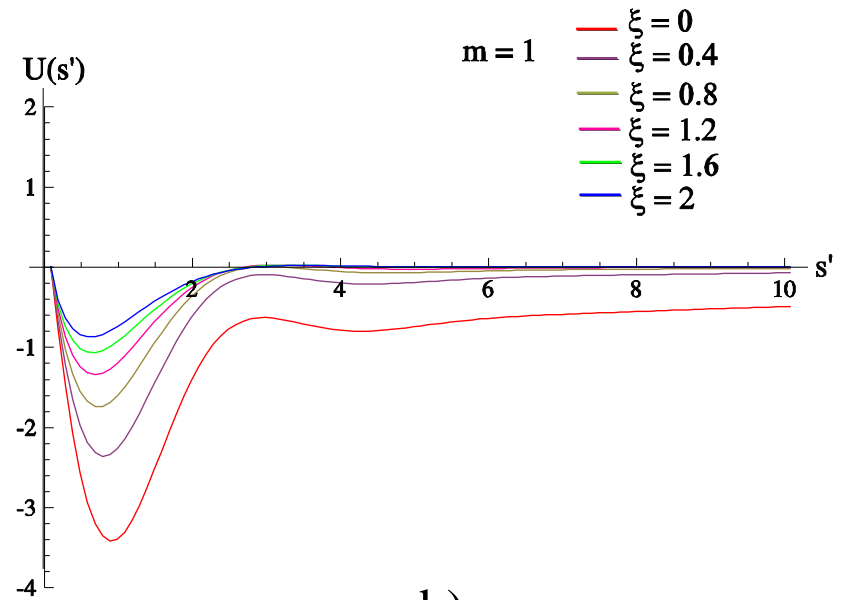
- Thus the transversal wake at any order can be obtained by integrating the longitudinal wake.

Lucky we have that integral already evaluated!

Transverse wakes



a)



b)

Transverse wakes. a) various modes with $\xi = 0$; b) mode $m = 1$ with ξ variable.

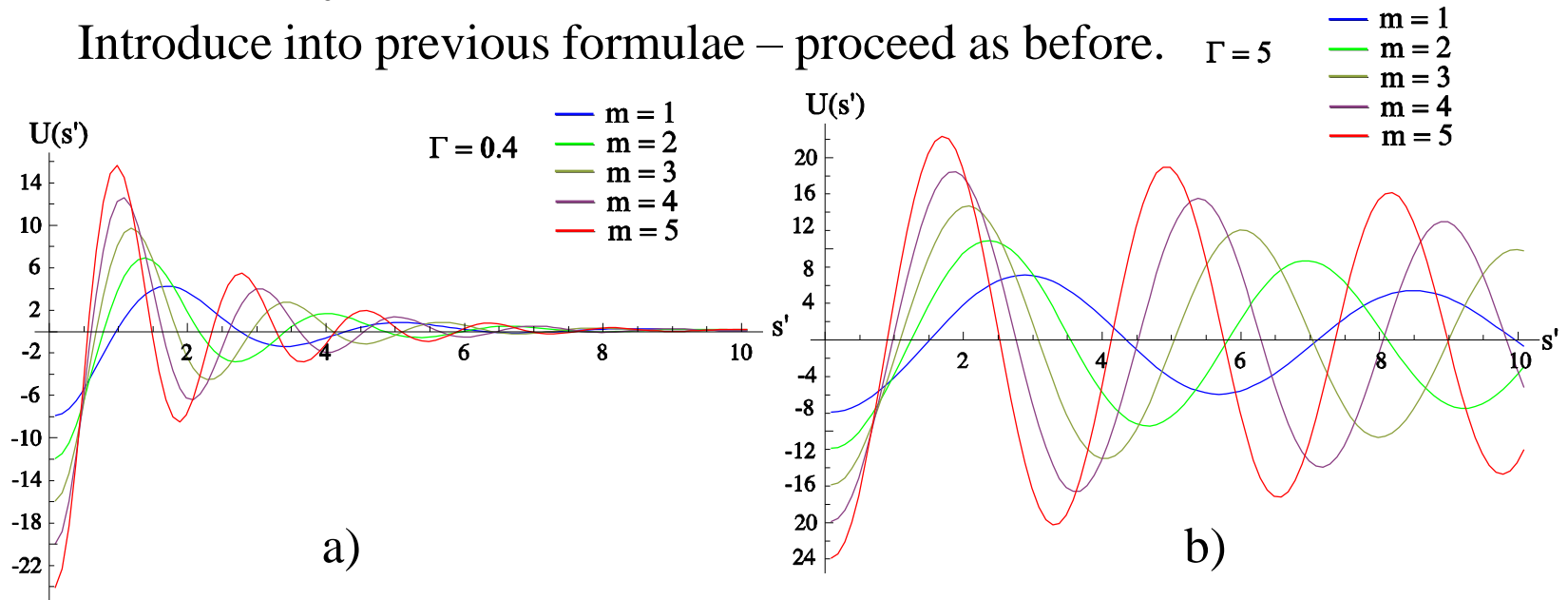
AC Conductivity

In the classical Drude model, in units of normalised wave number $K=s_0k$

$$\tilde{\sigma} = \frac{\sigma}{1 - iK\Gamma}$$

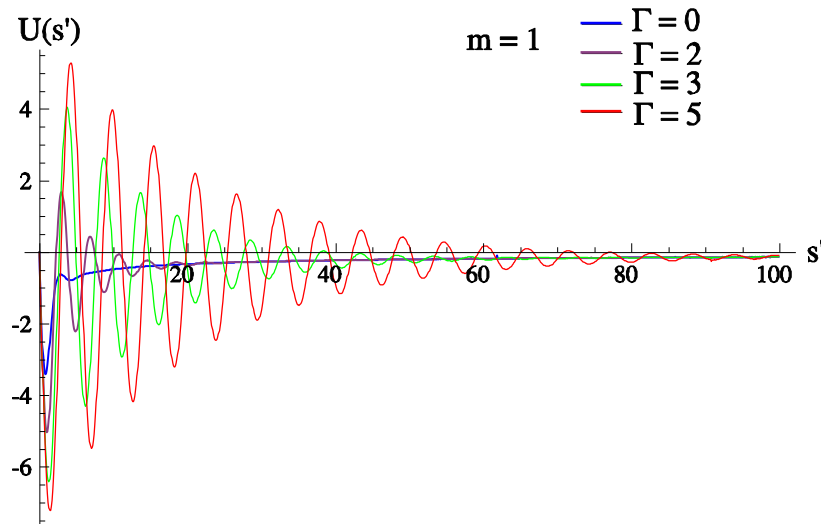
with $\Gamma = c\tau/s_0$ relaxation factor

Introduce into previous formulae – proceed as before. $\Gamma = 5$

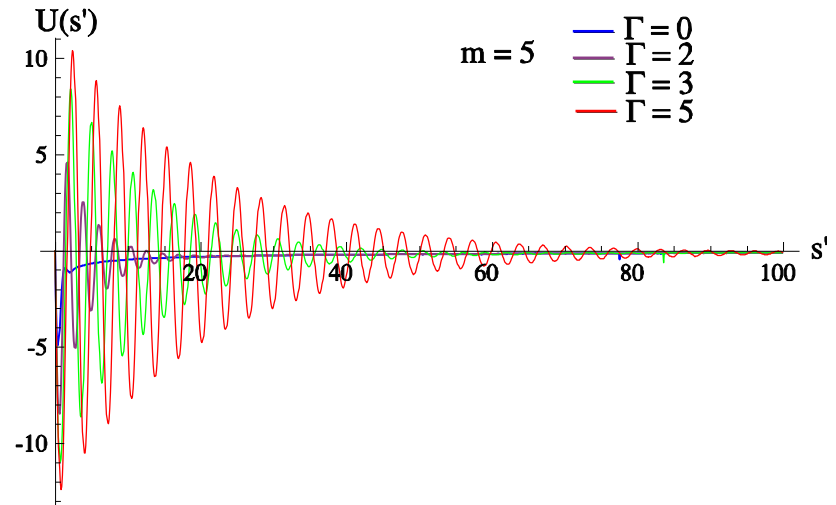


AC Conductivity. Longitudinal wake for various modes ($m > 0$) for
 a) $\Gamma = 0.4$ and b) $\Gamma = 5$.

AC Conductivity



a)



b)

AC Conductivity. Transversal wake as a function of Γ for
a) mode $m = 1$ and b) mode $m = 5$.

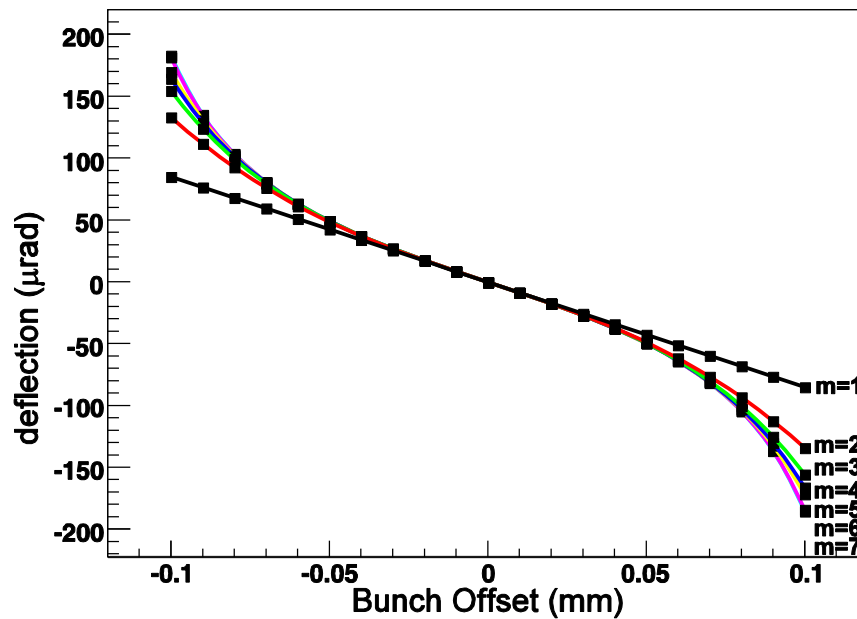
Implementation

- Use Mathematica to integrate even and odd functions and generate tables of values as function of s' , ξ , Γ .
- Only 3 variables - 2 of them don't vary (for a given collimator).
- Write to file C++ object `collimatortable(file, Gamma, xi)` – portable.
- Reads complete table and interpolates to get single table for s'
`collimatortable::interpolate(double sprime)` returns the value.

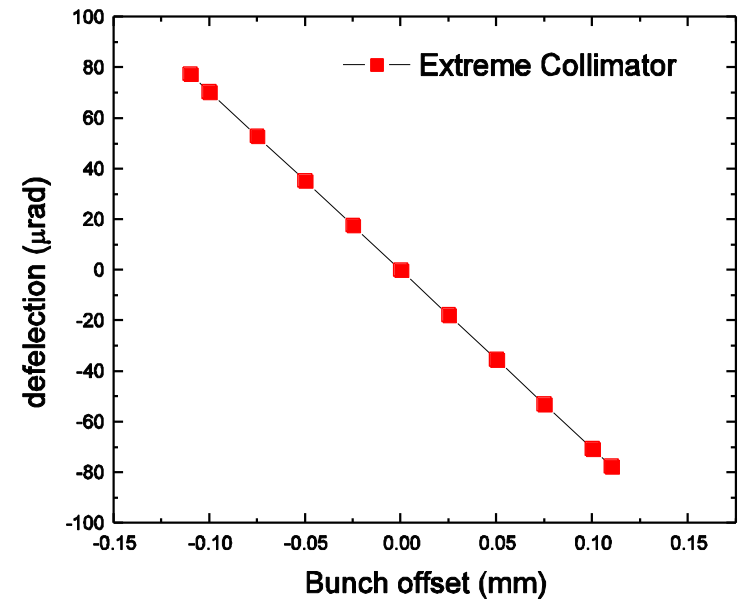
Tables and Documentation at: <http://www.hep.manchester.ac.uk/u/adina>

Examples

Collimator: $b = 0.125$ mm, $\sigma = 7.14 \cdot 10^{14}$ (Ωm)⁻¹ (Graphite), $L = 1$ m
 $\Gamma = 2.85$, $\xi = 0.007$



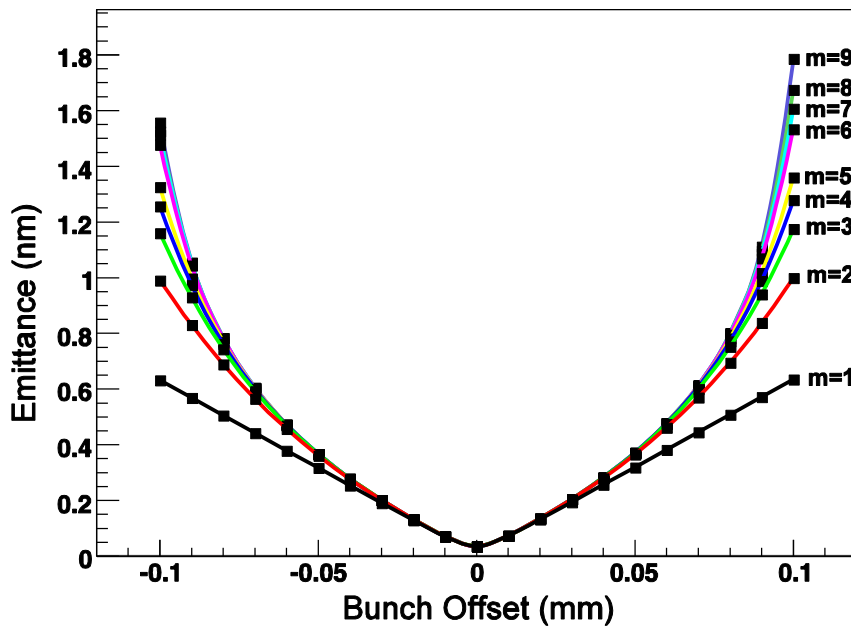
MERLIN



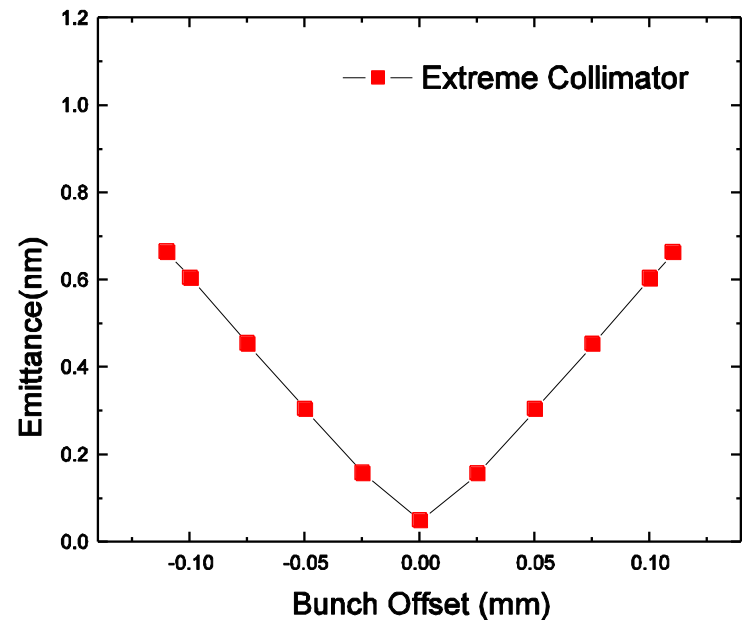
PLACET

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MERLIN



PLACET

Conclusions

- Longitudinal and transverse resistive wakes can be calculated in simulations programs using pretabulated numerical Fourier Transform.
- Higher order angular modes and AC conductivity effects can be easily included.
- We have shown that this can be used by Merlin and PLACET.
- Extend to rectangular apertures.